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ON MODIFIED PROLONGATIONAL LIMIT SETS AND PROLONGATIONS IN DYNAMICAL SYSTEMS ON METRIC SPACES

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The purpose of the present paper is to introduce some modifications of prolongations and prolongational limit sets in dynamical systems on metric spaces. Modified definitions will be equivalent to the classical ones for compact sets, but they will give some essential extensions for sets being not compact. Some theorems similar to classical ones will be proved with respect to notions considered in the paper. Let us notice that formally new are only definitions of modified prolongational limit sets; prolongations in the sense of definitions introduced below in Sec. 2 can be considered as certain special cases of very general notions introduced in [5]. For details see Remark 2.0 below.

1. Preliminaries; classical prolongations and prolongational limit sets.

Let (X,ρ) be a metric space and let $(X,\mathbf{R};\pi)$ be a dynamical system, which means that π is a continuous mapping from $\mathbf{R}\times X$ into X and the two following conditions are satisfied

$$\pi(0,x)=x \quad \text{for} \quad x\in X, \tag{1.1}$$

$$\pi(t,\pi(s,x))=\pi(t+s,x) \quad \text{for} \quad t,s\in\mathbf{R},\ x\in X. \tag{1.2}$$

For $x\in X$ and $M\subset X$ we put

$$\pi_+(x):=\{\pi(t,x)\colon t\geq 0\}, \qquad \pi_-(x):=\{\pi(t,x)\colon t\leq 0\}, \tag{1.3}$$

$$\pi_+(M):=\bigcup\{\pi_+(x)\colon x\in M\}, \quad \pi_-(M):=\bigcup\{\pi_-(x)\colon x\in M\}, \tag{1.4}$$

and

$$\pi(x) := \pi_+(x) \cup \pi_-(x), \quad \pi(M) := \pi_+(M) \cup \pi_-(M); \tag{1.5}$$

the sets $\pi_+(x)$, $\pi_-(x)$ and $\pi(x)$ are called: the positive semitrajectory of x, the negative semitrajectory of x and the trajectory of x, respectively.
For a nonempty subset M of X and a point y of X we put

$$\rho(y, M) := \inf\{\rho(y, x) \colon x \in M\} \tag{1.6}$$

and if $\varepsilon > 0$, then

$$B(M, \varepsilon) := \{y \in X \colon \rho(y, M) < \varepsilon\} \tag{1.7}$$

If $M = \{x\}$ then we write $B(x, \varepsilon)$ instead of $B(\{x\}, \varepsilon)$ and so

$$B(x, \varepsilon) = \{y \in X \colon \rho(x, y) < \varepsilon\} \tag{1.8}$$

is the usual open ball centered at x with the radius ε.
Let us recall now the classical definitions of prolongations, prolongational limit sets and Lyapunov stability in dynamical systems in metric spaces (see for instance [1], [2]).

If $x \in X$ then the (first) positive prolongation of x is the set

$$\begin{aligned} D^+(x) := \Big\{ & y \in X \colon \text{there are sequences } \{t_m\} \text{ of real} \\ & \text{numbers and } \{y_m\} \text{ of elements of } X \text{ such that} \\ & t_m \geq 0,\ y_m \longrightarrow x \text{ and } \pi(t_m, y_m) \longrightarrow y \text{ as } m \longrightarrow \infty \Big\}. \end{aligned} \tag{1.9}$$

The (first) negative prolongations $D^-(x)$ is defined by the formula (1.9) in which the condition "$t_m \geq 0$" is replaced by "$t_m \leq 0$".

The (first) positive prolongational limit sets of x is defined by the formula

$$\begin{aligned} J^+(x) := \Big\{ & y \in X \colon \text{ there are } \{t_m\} \text{ and } \{y_m\} \text{ such that} \\ & t_m \longrightarrow \infty,\ y_m \longrightarrow x \text{ and } \pi(t_m, y_m) \longrightarrow y \\ & \text{as } m \longrightarrow \infty \Big\}. \end{aligned} \tag{1.10}$$

The (first) negative prolongational limit set $J^-(x)$ is defined by the formula (1.10) in which the condition "$t_m \longrightarrow \infty$" is replaced by "$t_m \longrightarrow -\infty$".

For $M \subset X$, $M \neq \emptyset$ we put

$$(1.11)\quad D^+(M) = \bigcup\{D^+(x) \colon x \in M\}, \quad D^-(M) = \bigcup\{D^-(x) \colon x \in M\}$$

$$(1.12)\quad J^+(M) = \bigcup\{J^+(x) \colon x \in M\}, \quad J^-(M) = \bigcup\{J^-(x) \colon x \in M\}$$

We say that M is positively (negatively) Lyapunov stable (see for instance [1], [2]) if and only if: for every $\varepsilon > 0$ and every $x \in M$ there exists $\delta > 0$ such that

$$(1.13)\qquad \pi_+(B(x,\delta)) \subset B(M,\varepsilon)$$

(respectively:

$$(1.14)\qquad \pi_-(B(x,\delta)) \subset B(M,\varepsilon) \quad).$$

The set M is uniformly positively (uniformly negatively) Lyapunov stable if and only if: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(1.15)\qquad \pi_+(B(M,\delta)) \subset B(M,\varepsilon)$$

(respectively:

$$(1.16)\qquad \pi_-(B(M,\delta)) \subset B(M,\varepsilon) \quad).$$

It is known that if M is compact then the positive (negative) stability of M is equivalent to the uniform positive (uniform negative) stability of M. It is known also that the following two classical theorems are true:

THEOREM 1.1. (See for instance [1], [2]). *If M is a nonempty and compact subset of X and M is positively (negatively) Lyapunov stable, then*

$$(1.17)\qquad D^+(M) = M \quad (D^-(M) = M).$$

THEOREM 1.2. (see [1], [2]). *Assume that (X,ρ) is locally compact. If $M \subset X$ is nonempty and compact and* (1.17) *holds true, then M is positively (negatively) uniformly Lyapunov stable.*

COROLLARY 1.1. *If (X, ρ) is locally compact, M is compact and nonempty, then the positive (negative) Lyapunov stability of M is equivalent to* (1.17).

REMARK 1.1. It is clear that $M \subset D^+(M) \cap D^-(M)$ and so the equality (1.17) is in fact equivalent to the inclusion $D^+(M) \subset M$ (respectively: $D^-(M) \subset M$).

REMARK 1.2. It is known that without the compactness of M we cannot obtain the equivalence of the conditions: Stability and (1.17); in particular the closedness of M is not sufficient.

REMARK 1.3. For every compact set M the sets $D^+(M)$, $J^+(M)$, $D^-(M)$ and $J^-(M)$ are closed; if M is not compact then these sets do not need to be closed.

REMARK 1.4. For every $M \subset X$ the set $D^+(M)$ $(D^-(M))$ is positively (negatively) invariant, which means that

$$\pi_+(D^+(M)) \subset D^+(M) \quad (\pi_-(D^-(M)) \subset D^-(M)) \tag{1.18}$$

and the sets $J^+(M)$ and $J^-(M)$ are invariant:

$$\pi(J^+(M)) \subset J^+(M), \quad \pi(J^-(M)) \subset J^-(M). \tag{1.19}$$

Let us recall finally the following classical result (see [1], [2]).

THEOREM 1.3. *For every $x \in X$ the following equalities are true:*

$$D^+(x) = J^+(x) \cup \pi_+(x) \tag{1.20}$$

$$D^-(x) = J^-(x) \cup \pi_-(x). \tag{1.21}$$

As a trivial corollary of this theorem we get corresponding equalities obtained from (1.20) and (1.21) for x replaced by and subset M of X.

2. Generalized prolongations and prolongational limit sets.

We shall propose below some modifications (generalizations and extensions in a certain sense) of the definitions (1.11) and (1.12).

Let M be a nonempty subset of X. We put

$$\begin{aligned} \widehat{D}^+(M) := \Big\{ & y \in X\colon \text{ there are sequences } \{t_m\} \text{ of real numbers} \\ & \text{and } \{x_m\} \text{ of elements of } X \text{ such that } y_m \geq 0, \\ & \rho(x_m, M) \longrightarrow 0, \text{ and } \pi(t_m, x_m) \longrightarrow y \text{ as } m \longrightarrow \infty \Big\}. \end{aligned} \tag{2.1}$$

Replacing in (2.1) the condition "$t_m \geq 0$" by the condition "$t_m \leq 0$" we get the definition of $\widehat{D}^-(M)$.

$$\widehat{J}^+(M) := \Big\{ y \in X\colon \text{ there are sequences } \{t_m\} \text{ of real numbers and } \{x_m\} \text{ of elements of } X \text{ such that } t_m \longrightarrow \infty, \ \rho(x_m, M) \longrightarrow 0 \text{ and } \pi(t_m, x_m) \longrightarrow y \text{ as } m \longrightarrow \infty \Big\}. \tag{2.2}$$

Substituting "$t_m \longrightarrow -\infty$" in the place of "$t_m \longrightarrow \infty$" in (2.1) we get the definition of $\widehat{J}^-(M)$.

REMARK 2.0. M. Sobański introduced in [5] a very general definition of prolongations in generalized (multivalued) semi-systems. In the case of a dynamical system $(X, \mathbf{R}; \sigma)$ where X is a topological space (and obviously $\sigma\colon \mathbf{R} \times X \longrightarrow X$ is a mapping such that $\sigma(0,x) = x$, $\sigma(s, \sigma(t,x)) = \sigma(s+t, x)$) this definition can be written as follows: for $M \subset X$, $M \neq \emptyset$ the positive (negative) prolongation of M with respect to a basis $\mathcal{N}(M)$ of open subsets of X such that $M = \bigcap\{A\colon A \in \mathcal{N}(M)\}$ is the set equal to $\bigcap\{\overline{\sigma_+(A)}\colon A \in \mathcal{N}(M)\}$ ($\bigcap\{\overline{\sigma_-(A)}\colon A \in \mathcal{N}(M)\}$). It is not difficult to observe that in our case, when X is a metric space (and $\pi = \sigma$), the set $\widehat{D}^+(M)$ ($\widehat{D}^-(M)$) is for any closed set M equal to the positive (negative) prolongation of M with respect to $\mathcal{N}(M) := \{B(M,b)\colon b > 0\}$ in the sens of [5]. Observe that $\widehat{D}^+(M) = \widehat{D}^+(\overline{M})$ ($\widehat{D}^-(M) = \widehat{D}^-(\overline{M})$). So all theorems proved with respect to prolongations in the sense of [5] are true for sets of the type $\widehat{D}^+(M)$ and $\widehat{D}^-(M)$. We shall prove however some of them independently below, because of the following reasons:

(1) our proofs seem to be simpler and more elementary, since we may use the natural topology structure induced by the metric,
(2) some results presented below can be formulated in a slightly stronger version, than corresponding results of [5]
(3) we present several results concerning $\widehat{D}^+$ ($\widehat{D}^-$) as well as $\widehat{J}^+$ ($\widehat{J}^-$); the sets $\widehat{J}^+(M)$ and $\widehat{J}^-(M)$ were not considered in [5].

Let us notice finally that one can apply the idea of [5] in order to extend the definitions of prolongational limit sets in such a way that generalized notions will cover $\widehat{J}^+$ and $\widehat{J}^-$ as special cases, similarly as it has been done with respect to $\widehat{D}^+$ and $\widehat{D}^-$. This is however not necessary in our case.

PROPOSITION 2.1. *For every $M \subset X$, $M \neq \emptyset$ we have*

$$D^+(M) \subset \widehat{D}^+(M), \quad D^-(M) \subset \widehat{D}^-(M) \tag{2.3}$$

and

$$J^+(M) \subset \widehat{J}^+(M), \quad J^-(M) \subset \widehat{J}^-(M). \tag{2.4}$$

PROPOSITION 2.2. *If M is compact then in* (2.3) *and* (2.4) *we have equalities instead of the inclusions. In particular for every $x \in M$ we have*

$$D^+(x) = \widehat{D}^+(\{x\}), \quad D^-(x) = \widehat{D}^-(\{x\}), \tag{2.5}$$

and

$$J^+(x) = \widehat{J}^+(\{x\}), \quad J^-(x) = \widehat{J}^-(\{x\}). \tag{2.6}$$

THEOREM 2.1. *For every $M \subset X$, $M \neq \emptyset$ the following statements are true:*
(a) the sets $\widehat{D}^+(M)$, $\widehat{D}^-(M)$, $\widehat{J}^+(M)$ and $\widehat{J}^-(M)$ are closed,
(b) the sets $\widehat{J}^+(M)$ and $\widehat{J}^-(M)$ are invariant,
(c) the set $\widehat{D}^+(M)$ ($\widehat{D}^-(M)$) is positively (negatively) invariant.

PROOF. Let $M \subset X$, $M \neq \emptyset$ be given. Let y be a point belonging to the closure of $\widehat{D}^+(M)$. Let $\{y^m\}$ be a sequence of elements of $\widehat{D}^+(M)$ convergent to y. For every m there are sequences $\{t_k^m\}_{k=1}^{\infty}$ and $\{x_k^m\}_{k=1}^{\infty}$ such that

$$t_k^m \geq 0 \text{ for every } m, k \tag{2.7}$$

$$\rho(x_k^m, M) \longrightarrow 0 \text{ as } k \longrightarrow \infty \tag{2.8}$$

and

$$\rho(\pi(t_k^m, x_k^m), y^m) \longrightarrow 0 \text{ as } k \longrightarrow \infty \tag{2.9}$$

For every fixed m we can find k_m such that

$$\rho(x_{k_m}^m, M) < \tfrac{1}{m} \tag{2.10}$$

and

$$\rho(\pi(t_{k_m}^m, x_{k_m}^m), y^m) < \tfrac{1}{m} \tag{2.11}$$

Applying the triangle inequality we obtain

$$\rho(\pi(t_{k_m}^m, x_{k_m}^m), y) \leq \rho(\pi(t_{k_m}^m, x_{k_m}^m), y^m) + \rho(y^m, y) \tag{2.12}$$

and so, by using (2.11) we get by virtue of the convergence to zero of the sequence $\{\rho(y^m, y)\}$, the relation

$$\pi(t^m_{k_m}, x^m_{k_m}) \longrightarrow y \text{ as } m \longrightarrow \infty. \tag{2.13}$$

Simultaneously we have (compare (2.8))

$$\rho(x^m_{k_m}, M) \longrightarrow 0 \text{ as } m \longrightarrow \infty \tag{2.14}$$

and obviously (see (2.7))

$$t^m_{k_m} \geq 0. \tag{2.15}$$

The conditions (2.13)–(2.15) mean that $y \in \widehat{D}^+(M)$. The set $\widehat{D}^+(M)$ is closed. The proof of the closedness of $J^+(M)$ can be obtained from the above reasoning by a simple modification: if we assume that y belongs to the closure of $\widehat{J}^+(M)$, then we have $y = \lim y^m$ with $y^m \in \widehat{J}^+(M)$ and then there are sequences $\{t^m_k\}$ and $\{x^m_k\}$ for which the relations (2.8) and (2.9) are satisfied and moreover

$$t^m_k \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty; \tag{2.16}$$

we may require that the sequence $\{k_m\}$ chosen above in such a way that the conditions (2.13) and (2.14) hold true, satisfies also the condition

$$t^m_{k_m} \geq m \tag{2.17}$$

which permit us to observe that $y \in \widehat{J}^+(M)$. The proof for $\widehat{D}^-(M)$ ($\widehat{J}^-(M)$) is clearly similar to that given above for $\widehat{D}^+(M)$ ($\widehat{J}^+(M)$). The part (a) of the assertion is proved.

Assume now that $y \in \widehat{J}^+(M)$. So $y = \lim \pi(t_m, y_m)$ where $t_m \longrightarrow \infty$ and $\rho(y_m, M) \longrightarrow 0$ as $m \longrightarrow \infty$. Let $s \in \mathbf{R}$ be arbitrarily fixed. The continuity of π gives:

$$\pi(s, y) = \pi(s, \lim \pi(t_m, y_m)) = \lim \pi(s + t_m, y_m)$$

and so (since $s + t_m \longrightarrow \infty$) we get

$$\pi(s, y) \in \widehat{J}^+(M). \tag{2.18}$$

The relation (2.18) holds true for every $y \in \widehat{J}^+(M)$ and every $s \in \mathbf{R}$; so

$$\pi(\widehat{J}^+(M)) \subset \widehat{J}^+(M)$$

which means that the set $\widehat{J}^+(M)$ is invariant.

Replacing the condition "$t_m \longrightarrow \infty$" by "$t_m \longrightarrow -\infty$" we get for every $s \in \mathbf{R}$ the relations $s + t_m \longrightarrow -\infty$ and repeating the above reasoning we obtain also the inclusion

$$\pi(\widehat{J}^-(M)) \subset \widehat{J}^-(M).$$

The condition (b) has been proved.

It is clear that the same idea proves (c); it is enough to observe that if $t_m \geq 0$ ($t_m < 0$) then $s + t_m \geq 0$ ($s + t_m \leq 0$) for $s \geq 0$ ($s \leq 0$) and using the same argument as previously for $\widehat{J}^+(M)$ ($\widehat{J}^-(M)$) we can show that $\widehat{D}^+(M)$ ($\widehat{D}^-(M)$) is positively (negatively) invariant.

This completes the proof of the theorem.

REMARK 2.1. Theorem 2.1 shows in particular that the sets $\widehat{D}^+(M)$, $\widehat{J}^+(M)$, $\widehat{D}^-(M)$ and $\widehat{J}^-(M)$ have properties similar to that of the set $D^+(M)$, ... etc mentioned in Remark 1.4. The closedness of $\widehat{D}^+(M), \ldots$ etc proved above is not assured in general (without additional assumptions on M) with respect to $D^+(M), \ldots$ etc.

We shall prove now certain extension of Theorem 1.3 mentioned in Sec. 1.

THEOREM 2.2. *Let M be a nonempty subset of X. Then*

$$\widehat{J}^+(M) \cup \pi_+(M) \subset \widehat{D}^+(M) \subset \widehat{J}^+(M) \cup \pi_+(\overline{M}) \tag{2.19}$$

and

$$\widehat{J}^-(M) \cup \pi_-(M) \subset \widehat{D}^-(M) \subset \widehat{J}^-(M) \cup \pi_-(\overline{M}) \tag{2.20}$$

PROOF. Let us discuss (2.19). The first inclusion is trivial, since

$$\widehat{J}^+(M) \subset \widehat{D}^+(M)$$

and (compare (1.20) and (2.3))

$$\pi_+(M) \subset D^+(M) \subset \widehat{D}^+(M).$$

In order to prove the second inclusion appearing in (2.19), assume that y is a given point of $\widehat{D}^+(M)$. Let $\{t_m\}$ and $\{y_m\}$ be such that $t_m \geq 0$, $\rho(y_m, M) \longrightarrow 0$ and $\rho(\pi(t_m, y_m), y) \longrightarrow 0$ as $m \longrightarrow \infty$. If the sequence

$\{t_m\}$ is unbounded, then there exists a subsequence of it tending to infinity; without loss of generality we may assume that $t_m \longrightarrow \infty$, and so $y \in \widehat{J}^+(M)$. If $\{t_m\}$ is bounded, then we may assume that $t_m \longrightarrow t^*$ where t^* is a nonnegative real number. Put $z_m := \pi(t_m, y_m)$. We have

$$z_m \longrightarrow y \tag{2.21}$$

and

$$y_m = \pi(-t_m, z_m) \tag{2.22}$$

Thus, because of the continuity of π we get

$$y_m \longrightarrow \pi(-t^*, y) \quad \text{and} \quad m \longrightarrow \infty. \tag{2.23}$$

The sequence $\{\rho(y_m, M)\}$ tends to zero, and then (2.23) implies the condition

$$\widetilde{y} := \pi(-t^*, y) = \lim y_m \in \overline{M} \tag{2.24}$$

which gives finally

$$y = \pi(t^*, \widetilde{y}) \in \pi_+(\overline{M}). \tag{2.25}$$

We have proved the condition (2.19). The proof of (2.20) is obviously similar.

COROLLARY 2.1. *If M is closed then*

$$\widehat{D}^+(M) = \widehat{J}^+(M) \cup \pi_+(M) \tag{2.26}$$

and

$$\widehat{D}^-(M) = \widehat{J}^-(M) \cup \pi_-(M). \tag{2.27}$$

REMARK 2.2. If M is not closed, then we cannot prove the equalities (2.26) and (2.27) (see for instance the example (ii) in Sec. 6; the equality (2.26) does not hold true for the set M^0 considered there).

REMARK 2.3. It is clear that for every $M \subset X$ we have

$$\overline{M} \subset \widehat{D}^+(M) \cap \widehat{D}^-(M). \tag{2.28}$$

Indeed, if $x \in \overline{M}$, then $x = \lim x_n = \lim \pi(0, x_n)$ where $\{x_n\}$ is a sequence of elements of M and so, x belongs to $\widehat{D}^+(M)$ as well as to the set $\widehat{D}^-(M)$.

Investigating the elementary properties of $\widehat{J}^+(M)$, $\widehat{D}^+(M)$, $\widehat{J}^-(M)$, $\widehat{D}^-(M)$ we have to observe finally, that there are essential qualitative differences between $\widehat{J}^+(\pi(x))$ and $\widehat{J}^+(\pi_+(x))$, as well as between $\widehat{J}^+(\pi(x))$ and $\widehat{J}^+(\pi_-(x))$ (similary for $\widehat{J}^-(\pi(x))$ and $\widehat{J}^-(\pi_-(x))$, for a given point $x \in X$. There are also some relations between $\Lambda^-(x)$ and $\widehat{J}^+(\pi(x))$ ($\Lambda^+(x)$ and $\widehat{J}^-(\pi(x))$ having no analogical ones in the theory of the classical prolongational limit sets.

Namely we have the following

THEOREM 2.3. *For every* $x \in X$

$$\overline{\pi(x)} \subset \widehat{J}^+(\pi(x)) \cap \widehat{J}^-(\pi(x)). \tag{2.29}$$

PROOF. Let us observe first of all that putting $x_n := \pi(-n, x)$, $n = 1, 2, \ldots$ we get clearly:

$$\rho(x_n, \pi(x)) = 0, \quad \pi(n, x_n) = x$$

and so $x = \lim \pi(n, x_n) \in \widehat{J}^+(\pi(x))$. The invariance of $\widehat{J}^+(\pi(x))$ (compare (b) in Theorem 2.1) gives the inclusion $\pi(x) \subset \widehat{J}^+(\pi(x))$. Similarly we prove that $x \in \widehat{J}^-(\pi(x))$ and then $\pi(x) \subset J^-(\pi(x))$.

The closedness of $\widehat{J}^+(\pi(x))$ and $\widehat{J}^-(\pi(x))$ implies the inclusions

$$\overline{\pi(x)} \subset \widehat{J}^+(\pi(x)) \quad \text{and} \quad \overline{\pi(x)} \subset \widehat{J}^-(\pi(x))$$

giving the inclusion (2.29).

REMARK 2.4. It is easy to observe that we have proved in fact some stronger version of the assertion, namely:

$$\pi(x) \subset \widehat{J}^+(\pi_-(x)) \tag{2.30}$$

and

$$\pi(x) \subset \widehat{J}^-(\pi_+(x)) \tag{2.31}$$

as well as

$$\Lambda^+(x) \subset \widehat{J}^+(\pi_-(x)) \tag{2.32}$$

and

$$\Lambda^-(x) \subset \widehat{J}^-(\pi_+(x)). \tag{2.33}$$

REMARK 2.5. The negative semitrajectory $\pi_-(x)$ cannot be replaced in (2.30) by the positive semitrajectory $\pi_+(x)$, as well as $\pi_+(x)$ in (2.31) cannot be replaced – in general – by $\pi_-(x)$. It can be easily observed by discussing for instance the example (ii) in Sec. 6.

As a simple consequence of Theorem 2.2 and 2.3 we get the following

COROLLARY 2.2. *For every $x \in X$ we have*

$$\widehat{D}^+(\pi(x)) = \widehat{J}^+(\pi(x)) \tag{2.34}$$

and

$$\widehat{D}^-(\pi(x)) = \widehat{J}^-(\pi(x)). \tag{2.35}$$

PROOF. Applying Theorem 2.2 for $M = \pi(x)$ we get (see (2.19))

$$\widehat{J}^+(\pi(x)) \cup \pi_+(\pi(x)) \subset \widehat{D}^+(\pi(x)) \subset \widehat{J}^+(\pi(x)) \cup \pi_+(\overline{\pi(x)}). \tag{2.36}$$

Since $\pi(x)$ is positively invariant, as well as $\overline{\pi(x)}$, we have $\pi_+(\pi(x)) = \pi(x)$ and $\pi_+(\overline{\pi(x)}) = \overline{\pi(x)}$ and so by virtue of the inclusion $\pi(x) \subset \overline{\pi(x)} \subset \widehat{J}^+(\pi(x))$ (see (2.29)), (2.36) finishes the proof of (2.34). Similarly, the condition (2.20) together with (2.29) give (2.35).

3. Stability properties of M via some properties of $\widehat{D}^+(M)$ ($\widehat{D}^-(M)$).

Assume as previously that $(X, \mathbf{R}; \pi)$ is a dynamical system with (X, ρ) being a metric space. Let M be a non-empty subset of X.

THEOREM 3.1. *If M is closed and positively (negatively) uniformly Lyapunov stable, then*

$$\widehat{D}^+(M) = M \quad (\widehat{D}^-(M) = M). \tag{3.1}$$

PROOF. Assume that M is positively uniformly stable and $\widehat{D}^+(M) \neq M$. Since $M \subset \widehat{D}^+(M)$ we have of course

$$\widehat{D}^+(M) \setminus M \neq 0. \tag{3.2}$$

Let z be a point belonging to $\widehat{D}^+(M) \setminus M$. There are sequences $\{t_m\}$ of real numbers and $\{x_m\}$ of elements of X such that $t_m \geq 0$,

$$\pi(t_m, x_m) \longrightarrow z \quad \text{as} \quad m \longrightarrow \infty \tag{3.3}$$

and

$$\rho(x_m, M) \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty. \tag{3.4}$$

Since $M = M$, there exists $r > 0$ such that

$$B(z, r) \cap M = \emptyset. \tag{3.5}$$

Take now $\varepsilon = r/2$ and choose $\delta > 0$ such that

$$\pi_+(B(M, \delta)) \subset B(M, \varepsilon); \tag{3.6}$$

it is possible since M is positively uniformly stable.

It is clear that for m large enough

$$x_m \in B(M, \delta)$$

(see (3.4)) and so, because of (3.6), we have

$$\pi(t_m, x_m) \in B(M, \varepsilon)$$

for sufficiently large m, This is however impossible by virtue of (3.3) and (3.5). The contradiction shows that (3.2) must be excluded. So $\widehat{D}^+(M) = M$. The proof of the implication: the negative uniform stability gives the equality $\widehat{D}^-(M) = M$ is clearly similar. The theorem has been proved.

REMARK 3.1. The conditions (2.3) and the known inclusions: $M \subset D^+(M)$, $M \subset D^-(M)$ permit us to observe that if M is positively (negatively) uniformly stable, then

$$M = D^+(M) = \widehat{D}^+(M) \quad (M = D^-(M) = \widehat{D}^-(M)).$$

We shall discuss now possible consequences of (3.1)

First we shall consider M such that

$$\widehat{J}^+(M) \setminus M = \emptyset \quad (\widehat{J}^-(M) \setminus M = \emptyset) \tag{3.7}$$

It is obvious that (3.1) implies (3.7)

LEMMA 3.1. *Let M be a nonempty subset of X. Assume* (3.7). *Then for every $y \notin M$ there is $\delta > 0$ such that for every sequence $\{s_m\}$ of real numbers tending to $-\infty$ (to ∞) there is a positive integer m_0 such that*

$$m \geq m_0 \implies \pi(s_m, y) \notin B(M, \delta). \tag{3.8}$$

PROOF. Let M be such that $\widehat{J}^+(M) \setminus M = \emptyset$. Assume that the assertion does not hold true. So there is $y \notin M$ such that for every $\delta > 0$ there exists a sequence $\{s_m\}$ of real numbers such that $s_m \longrightarrow -\infty$ and

$$\forall m_0 \, \exists m \geq m_0 \{\pi(s_m, y) \in B(M, \delta)\}.$$

Taking $\delta = 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots$ we may find sequences $\{s_m^n\}$ of real numbers such that for every n

$$s_m^n \longrightarrow -\infty \quad \text{as} \quad m \longrightarrow \infty$$

and

$$\forall m_0 \, \exists m_n \geq m_0 \{\pi(s_{m_n}^n, y) \in B(M, \tfrac{1}{n})\}. \tag{3.9}$$

For every n we may find k_n such that

$$s_k^n \leq -n \quad \text{for} \quad k \geq k_n. \tag{3.10}$$

Use now (3.9) considering for $m_0 = k_n$ $(n = 1, 2, \ldots)$ suitable $p_n \geq k_n$ such that

$$\pi(s_{p_n}^n, y) \in B(M, \tfrac{1}{n}) \quad \text{for every } n. \tag{3.11}$$

It is clear that putting

$$t_n := s_{p_n}^n \quad \text{and} \quad z_n := \pi(t_n, y)$$

we get (see (3.10) and (3.11) respectcively): $t_n \longrightarrow -\infty$ and $\rho(z_n, M) \longrightarrow 0$ as $n \longrightarrow \infty$. So, because of the obvious fact that

$$-t_n \longrightarrow \infty \quad \text{and} \quad y = \pi(-t_n, z_n) \;\; (= \lim \pi(-t_n, z_n))$$

we have finally

$$y \in \widehat{J}^+(M).$$

This contradicts the first condition in (3.7) assumed at the begining of our reasoning.

The proof in the case $\widehat{J}^-(M) \setminus M = \emptyset$ is clearly similar.

LEMMA 3.2. *Assume* (3.7). *Then for every $y \notin M$ there are $\delta > 0$ and $t^0 \geq 0$ $(t^0 \geq 0)$ such that for each $t < t^0$ $(t > t^0)$*

$$\pi(t, y) \notin B(M, \delta). \tag{3.12}$$

PROOF. Assume the contrary. There is $y \notin M$ such that for every $\delta > 0$ and every $t^0 \leq 0$ $(t^0 \geq 0)$ there is $t < t^0$ $(t > t^0($ for which

$$\pi(t, y) \in B(M, \delta). \tag{3.13}$$

Let $\delta > 0$ be chosen for y in such a way that for every sequence $\{s_m\}$ tending to $-\infty$ (to ∞) there is m_0 for which (3.8) holds true. Consider $t^0 = -1, -2, \ldots$ $(t^0 = 1, 2, \ldots)$ and for each n find $s_n \leq -n$ $(s_n \geq n)$ in such a way that (3.13) is satisfied with $t = s_n$ $(n = 1, 2, \ldots)$. By virtue of Lemma 3.1 we get a contradiction finishing the proof.

THEOREM 3.2. *Assume that M is nonempty, closed and positively (negatively) invariant. Assume also* (3.7). *Then for every $y \notin M$ there is $\eta > 0$ such that*

$$\pi_-(y) \cap B(M,\eta) = \emptyset \quad (\pi_+(y) \cap B(M,\eta) = \emptyset). \tag{3.14}$$

PROOF. Let $y \notin M$ be fixed. Using Lemma 3.2 we can find $\delta > 0$ and $t^0 \leq 0$ ($t^0 \geq 0$) such that (3.12) is satisfied for $t < t^0$ ($t > t_0$).

Consider now the sets

$$I^-(t^0) := \{\pi(s,y) \colon s \in [t^0, 0]\} \text{ if } t^0 \leq 0$$

and

$$I^+(t^0) := \{\pi(s,y) \colon s \in [0, t^0]\} \text{ if } t^0 \geq 0.$$

It is also clear that

$$I^-(t^0) \cap M = \emptyset \quad (I^+(t^0) \cap M = \emptyset)$$

since otherwise positive (negative) invariance of M would imply that $y \in M$. The compactness of the discussed sets and the closedness of M alow us to claim that there is $\lambda > 0$ such that

$$I^-(t^0) \cap B(M,\lambda) = \emptyset \quad (I^+(t^0) \cap B(M,\lambda) = \emptyset)$$

Putting now $\eta = \min(\delta, \lambda)$ we obtain (3.14). The proof is finished.

For $M \subset X$, $M \neq \emptyset$ we define

$$QA^-(M) := \Big\{ y \in X \colon \text{there is a sequence } \{s_m\} \text{ of real numbers such that } s_m \longrightarrow -\infty \text{ and } \rho(\pi(s_m, y), M) \longrightarrow 0 \text{ as } m \longrightarrow \infty \Big\},$$

and we call it the region of negative quasi-atractivity of M.

We define $QA^+(M)$, the region of positive quasi-attractivity of M, by substituting "$s_m \longrightarrow \infty$" in the place "$s_m \longrightarrow -\infty$" in the above definition of $QA^-(M)$.

For $M = \emptyset$ we put $QA^-(\emptyset) = QA^+(\emptyset) = \emptyset$.

We say that M is a negative (positive) quasi-attractor if and only if there exists $\delta > 0$ such that

$$B(M,\delta) \subset QA^-(M) \quad (B(M,\delta) \subset QA^+(M)).$$

We have the following obvious

PROPOSITION 3.1. *For every $M \subset X$*

$$QA^-(M) \subset \widehat{J}^+(M), \quad QA^+(M) \subset \widehat{J}^-(M). \tag{3.15}$$

COROLLARY 3.1. *For every $M \subset X$*

$$QA^-(M) \setminus M \subset \widehat{J}^+(M) \setminus M, \quad QA^+(M) \setminus M \subset \widehat{J}^-(M) \setminus M. \tag{3.16}$$

PROPOSITION 3.2. *If M is closed and*

$$QA^-(M) \setminus M \neq \emptyset \quad (QA^+(M) \setminus M \neq \emptyset) \tag{3.17}$$

then M is not positively (negatively) uniformly stable.

PROOF. If M is positively (negatively) uniformly stable, then (see Theorem 3.1) $\widehat{D}^+(M) = M$ ($\widehat{D}^+(M) = M$) and so $\widehat{J}^+(M) \setminus M = \emptyset$ ($\widehat{J}^-(M) \setminus M = \emptyset$) and we get a direct contradiction because of (3.1) and (3.17)

COROLLARY 3.2. *If M is a closed negative (positive) quasi-attractor, then M cannot be positively (negatively) uniformly stable.*

REMARK 3.2. It may happen that M is not positively uniformly stable, but $\widehat{J}^+(M) \setminus M = \emptyset$ (and so $QA^-(M) \setminus M = \emptyset$). Such an example (very simple) is shown on the picture (Fig. 1).

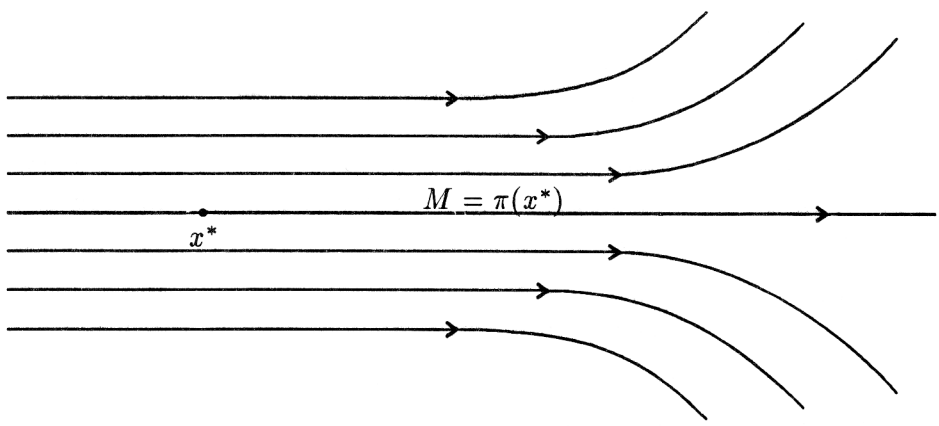

Fig. 1.

Here the set $M = \pi(x^*)$ is clearly not stable, while $\widehat{D}^+(M) = M = D^+(M)$ and so $\widehat{J}^+(M) \setminus M = QA^-(M) \setminus M = \emptyset$; in particular M is not a negative quasi-atractor.
A natural modification of this above example presented on the second picture produces such a system in which M is a negative quasi-attractor, an so it is necessarily not positively uniformly stable (it is not positively stable!); here we have obviously $\widehat{D}^+(M) \setminus M \neq \emptyset$.

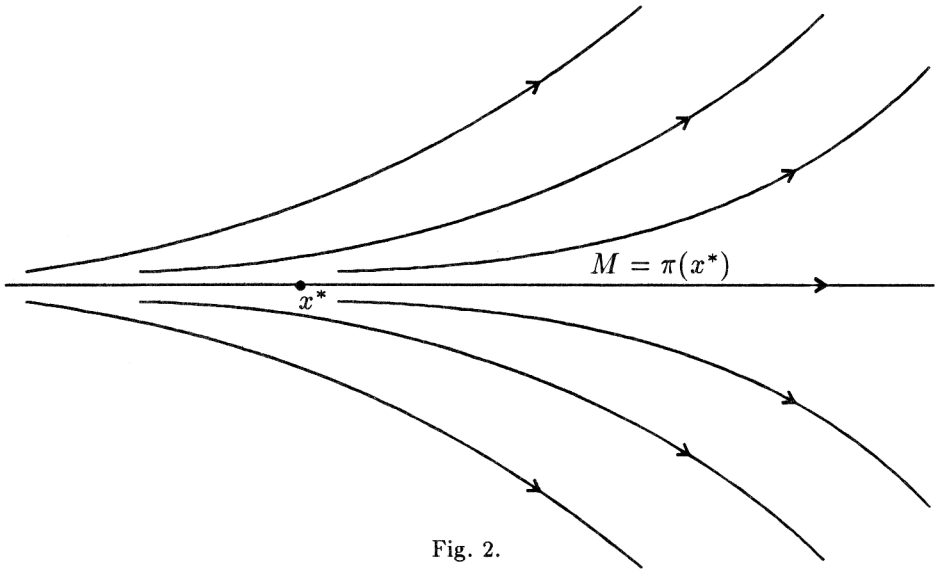

Fig. 2.

We may imagine that the trajectories are – for instance – graphs of some functions of the type $y = ae^{bx-c}$ $a \in \mathbf{R}$, $c \in \mathbf{R}$, $b \geq 0$

We shall say that a nonempty subset M of X satisfies the condition (CB) if and only if for every $\varepsilon > 0$ there is $\eta \in (0, \varepsilon]$ such that the boundary $\partial B(M, \eta)$ of the set $B(M, \eta)$ is compact.

This condition is satisfied for instance if $X = \mathbf{R}^k$, M is bounded or $M = \mathbf{R}^k \setminus N$, where N is bounded. If $X = \mathbf{R}$ then every set M having the form $M = M_1 \cup M_2 \cup M_3$ where $M_1 = (-\infty, a)$ or $M_1 = (-\infty, a]$, M_2 is bounded, $M_3 = (b, \infty)$ or $M_3 = [b, \infty)$ (or some of the sets M_1, M_2, M_3 are empty) satisfies the condition (CB).

THEOREM 3.3. *Assume that M is not empty and satisfies the condition (CB). Then* (3.1) *implies positively (negative) uniform stability of M.*

PROOF. Assume (3.1) and suppose that M is not positively (negatively) uniformly stable. So there is $\varepsilon > 0$ such that for every $\delta > 0$ there are $y \in B(M, \delta)$ and $t \geq 0$ $(t \leq))$ such that $\pi(t, y)$ does not belong to $B(M, \varepsilon)$. Let $\eta \in (0, \varepsilon]$ be such that $\partial B(M, \eta)$ is compact. Taking now $\delta = 1, \frac{1}{2}, \ldots$

we can find sequences: $\{y_n\}$ of elements of X and $\{t_n\}$ of real nonnegative (nonpositive) numbers such that $\pi(t_n, y_n) \notin B(M, \eta)$ for every n and

$$\rho(y_n, M) \longrightarrow \quad \text{as} \quad n \longrightarrow \infty \tag{3.18}$$

It is obvious that there is n^0 such that

$$\pi(s_n, y_n) \in \partial B(M, \eta).$$

Since $\partial B(M, \eta)$ is compact, we may assume without loss of generality, thet the sequence $\{\pi(s_n, y_n)\}$ is convergent to some z belonging to $\partial B(M, \eta)$. Since $s_n \geq 0$ ($s_n \leq 0$) and (3.18) holds true, this element z must belongs to $\widehat{D}^+(M)$ (to $\widehat{D}^-(M)$). So we have a contradiction with (3.1), since $\partial B(M, \eta) \cap M = \emptyset$.

COROLLARY 3.3. *If M is closed, nonempty and satisfies (CB), then M is positively (negatively)uniformly stable if and only if $\widehat{D}^+(M) = M$ ($\widehat{D}^-(M) = M$).*

REMARK 3.3. The above statement extends directly the classical result concerning necessary and sufficient conditions for a compact set to be positively (negatively)uniformly stable in locally compact spaces.

THEOREM 3.4. *Assume that M is nonempty, closed, positively (negatively) invariant and satisfies (CB). If* (3.7) *holds true then M is positively (negatively)uniformly stable.*

PROOF. Positive (negative) invariantness of M gives $\pi_+(M) = M$ ($\pi_-((M) = M$) and so (see Corollary 2.1)

$$\widehat{D}^+(M) = \widehat{J}^+(M) \cup M \quad (\widehat{D}^-(M) = \widehat{J}^-(M) \cup M).$$

Thus, because of (3.7), we have

$$\widehat{D}^+(M) = (\widehat{J}^+(M) \setminus M) \cup M = M \quad (\widehat{D}^-(M) = (\widehat{J}^-(M) \setminus M) \cup M = M).$$

Applying Theorem 3.3 we finish the proof.

4. Regular dependence of $\widehat{D}^+(M)$ and $\widehat{D}^-(M)$ on M.

Theorem on regular dependence of $D^+(x)$, and $D^-(x)$ on x are known in several versions (compare for instance [3], [4]). It is possible to extend that results for the sets $\widehat{D}^+(M)$, and $\widehat{D}^-(M)$. Let us recall first of all that there are possible two definitions of upper semi-continuity of set valued mappings. We

shall use here the notation and terminology used in [4]. Assume that (X, ρ) is a metric space. Let $x \in X$ be given. We say that a mapping

$$F\colon X \ni y \longmapsto F(y) \subset X \tag{4.0}$$

is (H)-usc at a given point x if and only if the following implication holds true: if $\{x_n\}$ and $\{y_n\}$ are sequences of elements of X such that $x_n \longrightarrow x$, $y_n \longrightarrow y$ and $y_n \in F(x_n)$, then $y \in F(x)$.
We say that F is (C)-usc at x if and only if for every $\varepsilon > 0$ then exists $\delta > 0$ such that

$$(y \in X, \rho(x,y) < \delta) \implies F(y) \subset B(F(x), \varepsilon).$$

Some essential differences between these two conditions considered with respect to the mappings $x \longmapsto \Lambda^+(x)$, $x \longmapsto J^+(x)$, $x \longmapsto D^+(x)\ldots$ etc are discussed in [4]. Here we shall consider some extension of the condition (C)-usc.

Let $\widetilde{M}$ be a nonempty subset of X. We say that the mapping (4.0) is uniformly (C)-usc in $\widetilde{M}$ if and only if:
for every $\eta > 0$ there is $\sigma > 0$ such that

$$x \in B(\widetilde{M}, \sigma) \implies F(M) \subset B(F(\widetilde{M}), \eta). \tag{4.2}$$

THEOREM 4.1. *Assume that $\widetilde{M}$ is a nonempty subset of X. If the mapping*

$$\pi_+ : x \longmapsto \pi_+(x) \tag{4.3}$$

(respectively:

$$\pi_- : x \longmapsto \pi_-(x)\ ) \tag{4.4}$$

is uniformly (C)-usc in $\widetilde{M}$, then for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$M \subset B(\widetilde{M}, \delta) \implies \widehat{D}^+(M) \subset B(\pi_+(\widetilde{M}), \varepsilon) \tag{4.5}$$

(respectcively:

$$M \subset B(\widetilde{M}, \delta) \implies \widehat{D}^-(M) \subset B(\pi_-(\widetilde{M}), \varepsilon)\ ). \tag{4.5}$$

PROOF. Assume that the mapping (4.3) is uniformly (C)-usc in $\widetilde{M}$ and suppose that there is $\varepsilon > 0$ such thaat for every $\delta > 0$ there is $M \subset B(\widetilde{M}, \delta)$ for which

$$\widehat{D}^+(M) \setminus B(\pi_+(\widetilde{M}), \varepsilon) \neq \emptyset. \tag{4.7}$$

So there is a sequence $\{M_n\}$ of subsets of X such that

$$M_n \subset B(\widetilde{M}, \tfrac{1}{n}) \tag{4.8}$$

and for some sequnce $\{z_n\}$ of elements of $\widehat{D}^+(M_n)$ we have

$$z_n \notin B(\pi_+(\widetilde{M}), \varepsilon) \tag{4.9}$$

For every n there are sequences $\{t_m^n\}$ and $\{x_m^n\}$ such that

$$t_m^n \geq 0 \tag{4.10}$$

$$\pi(t_m^n, x_m^n) \longrightarrow z_n \quad \text{as} \quad m \longrightarrow \infty \tag{4.11}$$

and

$$\rho(t_m^n, M_n) \longrightarrow z_n \quad \text{as} \quad m \longrightarrow \infty. \tag{4.12}$$

Let $\sigma > 0$ be such that (4.1) holds true for $\eta = \varepsilon/2$ and $F = \pi_+$ (which means that $F(x) = \pi_+(x)$, $F(\widetilde{M}) = \pi_+(\widetilde{M})$). Let now for every n, a positive integer m_n be chosen in such a way that for

$$s_n := t_{m_n}^n \quad \text{and} \quad y_n := x_{m_n}^n$$

we have

$$\rho(y_n, M_n) < \tfrac{1}{n} \tag{4.13}$$

$$\rho(z_n, \pi(s_n, x_n)) < \tfrac{1}{n}; \tag{4.14}$$

Such a choise is possible because of (4.12) and (4.11) respectively. For n sufficiently large we have (see (4.8))

$$M_n \subset B(\widetilde{M}, \sigma/2) \tag{4.15}$$

and (compare (4.13))

$$\rho(y_n, M_n) < \tfrac{\sigma}{2}. \tag{4.16}$$

So, for n large enough, we have

$$\rho(y_n, \widetilde{M}) < \sigma. \tag{4.17}$$

Thus, for n large enough we have

$$\pi(t, y_n) \in B(\pi_+(\widetilde{M}), \tfrac{\varepsilon}{2})$$

for every $t \geq 0$; thus in particular we have

$$\pi(s_n, y_n) \in B(\pi_+(\widetilde{M}), \tfrac{\varepsilon}{2}). \tag{4.18}$$

If n is sufficiently large, then (see (4.14))

$$\rho(z_n, \pi(s_n, x_n)) < \tfrac{\varepsilon}{2}. \tag{4.19}$$

The conditions (4.18) and (4.19), (satisfied for sufficiently large n) imply the condition

$$\rho(z_n, \pi(\widetilde{M})) < \varepsilon \tag{4.20}$$

which contradicts directly (4.9)
The proof of (4.6) if the mapping (4.4) is uniformly (C)-usc is obviously similar.
The theorem has been proved.

COROLLARY 4.1. *If the mapping* (4.3) *(respectively* (4.4)*) is uniformly (C)-usc in* $\widetilde{M}$*, then for every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that:*

$$M \subset B(\widetilde{M}, \delta) \implies \widehat{D}^+(M) \subset B(\widehat{D}^+(\widetilde{M}), \varepsilon) \tag{4.21}$$

(respectively:

$$M \subset B(\widetilde{M}, \delta) \implies \widehat{D}^-(M) \subset B(\widehat{D}^-(\widetilde{M}), \varepsilon). \tag{4.22}$$

COROLLARY 4.2. *If* $x \in X$ *is such that the positive (negative) semi-trajectory* $\pi_+(x)$ *(*$\pi_-(x)$*) is uniformly positively (negatively) Lyapunov stable, then the mapping*

$$y \longmapsto D^+(y) \quad (y \longmapsto D^-(y))$$

is (C)-usc at the point x.

Indeed the uniformly positive stability of $\pi_+(x)$ means that for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$y \in B(\pi_+(x), \delta) \implies \pi_+(y) \subset B(\pi_+(x), \varepsilon)$$

and so in particular

$$y \in B(x,\delta) \implies \pi_+(y) \subset B(\pi_+(x),\varepsilon)$$

which means that the mapping (4.3) is uniformly (C)-usc. So, by using Theorem 4.1 we get in particular: for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\rho(x,y) < \delta \implies \widehat{D}^+(y) \subset B(\pi_+(x),\varepsilon) \subset B(\widehat{D}^+(x),\varepsilon);$$

this gives to the condition of (C)-usc for the mapping $x \longmapsto \widehat{D}^+(x)$. The proof is finished since for the one-element set $\{x\}$ we have $D^+(x) = \widehat{D}^+(\{x\}) = \widehat{D}^+(x)$. The same method proves the assertion for $D^-(x)$.

REMARK 4.1. The assumptions on the uniform stability of $\pi_+(x)$ (resp. $\pi_-(x)$) cannot be replaced by the stability, as it is shown by the example in Sec. 4 of [4] (Fig. 3 in [4]).

5. Some sufficient conditions for the regular dependences of $\widehat{J}^+(M)$ and $\widehat{J}^-(M)$ on M.

Assume that $(X,\mathbf{R},\pi)$ is a dynamical system (with (X,ρ) being a metric space) and take a point $x \in X$. We say that the motion π^x is uniformly positively (negatively) Lyapunov stable if and only if: for every positive number η, there exists a positive number σ such that

$$(5.1) \qquad (t \geq 0, \rho(y,\pi(t,x)) < \sigma) \implies \rho(\pi(t+s,x),\pi(s,y)) < \eta \quad \text{for every } s \geq 0$$

(respectively:

$$(5.1) \qquad (t \leq 0, \rho(y,\pi(t,x)) < \sigma) \implies \rho(\pi(t+s,x),\pi(s,y)) < \eta \quad \text{for every } s \leq 0 \text{).}$$

THEOREM 5.1. *Suppose that $x \in X$ is such that the motion π^x is uniformly positively (negatively) Lyapunov stable and that the closure $\overline{\pi_+(x)}$ of the positive semitrajectory $\pi_+(x)$ (the closure $\overline{\pi_-(x)}$ of the negative semitrajectory $\pi_-(x)$) is compact. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that*
(5.2)

$$\rho(x,y) < \delta \implies \widehat{J}^+(\pi_+(y)) \subset B(\Lambda^+(x),\varepsilon) \ \left(\widehat{J}^-(\pi_-(y)) \subset B\left(\Lambda^-(x),\varepsilon\right)\right)$$

PROOF. Let us discuss the first case of the compactness of $\overline{\pi_+(x)}$ together with the uniform positive stability of π^x. Suppose that the assertion does not hold true. So there is $\varepsilon^0 > 0$ such that for every $\delta > 0$ there is $y \in B(x,\delta)$ for which $\widehat{J}^+(\pi_+(y)) \setminus B(\Lambda^+(x),\varepsilon^0) \neq \emptyset$. Thus there are sequences $\{y_n\}$ and $\{z_n\}$ of elements of X such that for every n.

$$\rho(y_n, x) < \tfrac{1}{n} \tag{5.3}$$

and

$$z_n \in \widehat{J}^+(\pi_+(y_n)) \setminus B(\Lambda^+(x),\varepsilon^0). \tag{5.4}$$

Let now $\{t_m^n\}$ and $\{y_m^n\}$ ($t_m^n \in \mathbf{R}, y_m^n \in X$) be such sequences that for every fixed n:

$$t_m^n \longrightarrow \infty \quad \text{as} \quad m \longrightarrow \infty \tag{5.5}$$

and

$$\rho(y_m^n, \pi_+(y_n)) \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty. \tag{5.6}$$

For every n we may find m_n such that putting $r_n := t_{m_n}^n$ and $w_n := y_{m_n}^n$, we get

$$\rho(\pi(r_n, w_n), z_n) < \tfrac{1}{n} \tag{5.7}$$

$$r_n \geq n \tag{5.8}$$

and

$$\rho(w_n, \pi_+(y_n)) < \tfrac{1}{n}. \tag{5.9}$$

The last inequality means that

$$\rho(w_n, \pi(s_n, y_n)) < \tfrac{1}{n} \tag{5.10}$$

for some $s_n \geq 0$ ($n = 1, 2, \ldots$).
Let now $\kappa > 0$ be arbitrarily fixed and let $\lambda > 0$ be chosen in such a way that the implication (5.1) holds true for $\eta = \frac{1}{2}\kappa$ and $\sigma = \lambda$. Let $n^0 > 0$ be such an integer that

$$\tfrac{1}{n} < \tfrac{\kappa}{2} \quad \text{for } n > n^0. \tag{5.11}$$

Observe that the inequality (5.11) gives, because of (5.7),

$$\rho(\pi(r_n, w_n), z_n) < \tfrac{\kappa}{2} \quad \text{for } n > n^0. \tag{5.12}$$

Let now $\beta > 0$ be chosen in such a way that (5.1) holds true for $\eta = \frac{\lambda}{2}$ and $\sigma = \beta$, and let n^1 be such an integer that $n^1 \geq n^0$ and

$$\tfrac{1}{n} < \tfrac{\lambda}{2} \quad \text{and} \quad \tfrac{1}{n} < \beta \quad \text{for } n > n^1. \tag{5.13}$$

For every n we have obviously

$$\rho(\pi(s_n, x), w_n) \leq \rho(\pi(s_n, x), \pi(s_n, y_n)) + \rho(\pi(s_n, y_n), w_n).$$

Using the condition (5.3) together with (5.1) (by virtue of the second inequality in (5.13)) and (5.10) together with (5.13), we get

$$\rho(\pi(s_n, x), w_n) < \lambda \quad \text{for } n > n^1 \tag{5.14}$$

This implies that for $n > n^1$ we have (compare (5.1) for $\eta = \frac{1}{2}\kappa$, $\sigma = \lambda$) the inequality

$$\rho(\pi(r_n, w_n), \pi(r_n + s_n, x)) < \tfrac{\kappa}{2} \tag{5.15}$$

So for $n > n^1 \geq n^0$ we have (by virtue of (5.12))

$$\rho(\pi(r_n+s_n, x), z_n) \leq \rho(\pi(r_n+s_n, x), \pi(r_n, w_n)) + \rho(\pi(r_n, w_n), z_n) < \kappa. \tag{5.16}$$

Thus we have proved that

$$\rho(\pi(r_n + s_n, x), z_n) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{5.17}$$

The sequence $\{\pi(r_n + s_n, x)\}$ has elements belonging to $\overline{\pi_+(x)}$ which is supposed to be compact. So we may assume that $\pi(r_n+s_n, x) \longrightarrow y^*$ as $n \longrightarrow \infty$, where y^* is a point belonging to $\overline{\pi_+(x)}$. But in fact y^* must be in $\Lambda^+(x)$, since $r_n + s_n \longrightarrow \infty$ (because $s_n \geq 0$ and $r_n \longrightarrow \infty$; compare (5.8)). Thus $z_n \longrightarrow y^* \in \Lambda^+(x)$ as $n \longrightarrow \infty$. This contradicts (5.4).

The proof for π^x being uniformly negatively stable is similar. The theorem has been proved.

COROLLARY 5.1. *If the assumptions of Theorem 5.1 are satisfied, then for every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\rho(x, y) < \delta \implies \widehat{J}^+(\pi_+(y)) \subset B(J^+(x), \varepsilon)$$
$$(\widehat{J}^+(\pi_-(y)) \subset B(J^-(x), \varepsilon))$$

and so, of course, we have also (in two alternative versions):

$$\rho(x, y) < \delta \implies \widehat{J}^+(\pi_+(y)) \subset B(\widehat{J}^+(\pi_+(x)), \varepsilon)$$
$$(\widehat{J}^+(\pi_-(y)) \subset B(\widehat{J}^-(\pi_-(x)), \varepsilon)).$$

6. Some examples.

We shall limit ourselves to examples presented practically only on pictures.

(i) Consider a dynamical system $(\mathbf{R}^2, \mathbf{R}; \pi)$ being a modification of the systems presented by Fig. 1 and Fig. 2, having now trajectories presented below (Fig. 3). Trajectories of points below the first axis of coordinates are stright lines parallel to the axis, the first coordinate axis itself is also a trajectory. Points $(x, y) \in \mathbf{R}^2$ such that $y > 0$ have trajectories having qualitative properties similar to exponential curves (we may assume simply that they are curves of the type $y = ae^{x-b}$ for some positive constants a and b).

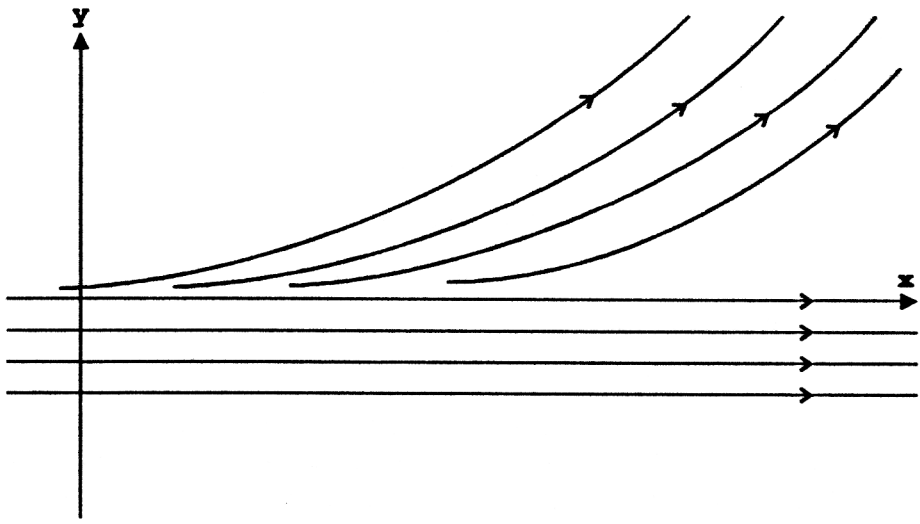

Fig. 3.

Here $M = \mathbf{R} \times \{0\}$ has the following properties:

$$\widehat{J}^+(M) = \mathbf{R} \times [0, \infty) = \widehat{D}^+(M)$$

and

$$D^+(M) = M, \quad J^+(M) = \emptyset.$$

(ii) Let $(\mathbf{R}^2, \mathbf{R}; \pi^1)$ be a dynamical system which has trajectories presented on the next picture (Fig. 4).

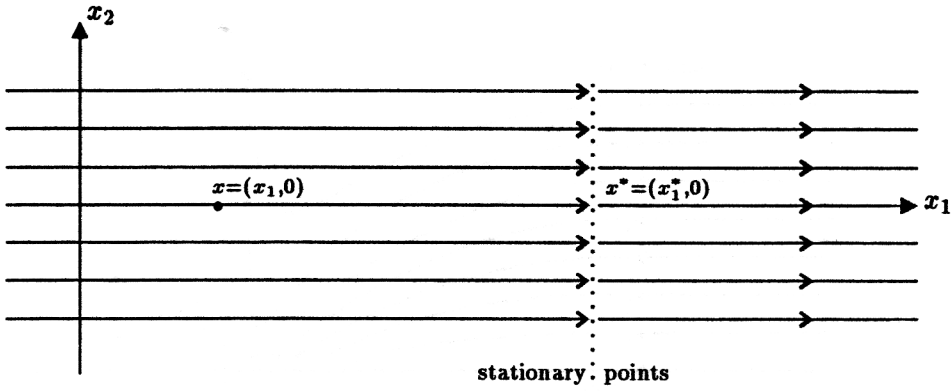

Fig. 4.

Let M_1 be equal to $\pi_+^1(x)$. We have

$$
\begin{aligned}
J^+(M_1) &= \Lambda_+(x) = \{x^*\} \\
D^+(M_1) &= \pi_+^1(M_1) \cup \{x^*\} \\
\widehat{J}^+(M_1) &= [x^*, \infty) \times \{0\} \\
\widehat{D}^+(M_1) &= \pi_+^1(M_1) \cup \widehat{J}^+(M_1) = [x_1, \infty) \times \{0\}.
\end{aligned}
$$

In this case we have in particular $M_1 \setminus J^+(M_1) = M_1$ (compare Remark 2.3). Observe, that taking $M^0 := M_1 \setminus \{x\}$ we get

$$
\widehat{D}^+(M^0) = \widehat{J}^+(M^0) \cup \overline{M^0} = [x, \infty) \times \{0\} \neq \widehat{J}^+(M^0) \cup \pi_+^1(M^0)
$$

(compare Remark 2.2).

(iii) Suppose that $(\mathbf{R}^2, \mathbf{R}; \pi^2)$ is such a dynamical system, that trajectories are given by curves presented on Fig. 5.

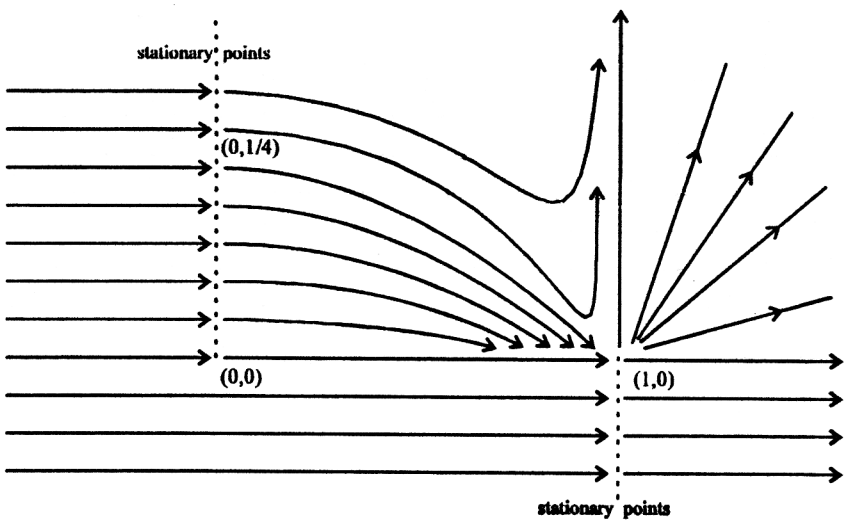

Fig. 5.

Trajectories of the points $(x,y) \in (0,1) \times (0,\infty)$ such that $\Lambda^-((x,y)) = \{(0,\alpha)\}$ with $\alpha > \frac{1}{4}$ are approaching the half line $\{(1,0)\} \times (0,\infty)$, while for $\alpha \leq \frac{1}{4}$ are tending to $(1,0)$ (more precisely $\Lambda^+((x,y)) = \{(1,0)\}$ if $\Lambda^-((x,y)) = \{(0,\alpha)\}$ with $\alpha \leq \frac{1}{4}$). All other essential properties of trajectories are clearly given by the picture.

Let $\widetilde{M}$ be now the interval $\{(x,0)\colon 0 < x < 1\}$; we have

$$J^+(\widetilde{M}) = \{(1,0)\}$$
$$D^+(\widetilde{M}) = \widetilde{M} \cup \{(1,0)\}$$
$$\widehat{J}^+(\widetilde{M}) = \widehat{D}^+(\widetilde{M}) = \{(x,y)\colon 0 \leq x \leq 1, y = 0\} \cup \{(x,y)\colon 1 \leq x, 0 \leq y\}.$$

(iv) Let us modify the above example (iii) in such a way, that after this modification all trajectories from Fig. 5 approaching the half line

$$H := \{(x,y)\colon x = 1, y > 0\}$$

(but having empty positive limit sets) will be replaced by trajectories having one point positive limit sets lying on the half line $\{(x^*,y)\colon y^* \leq y\}$ with some $0 < x^* < 1$ and $0 < y^*$ (see Fig. 6). This line is built by stationary points. All other trajectories are as in the example (iii).

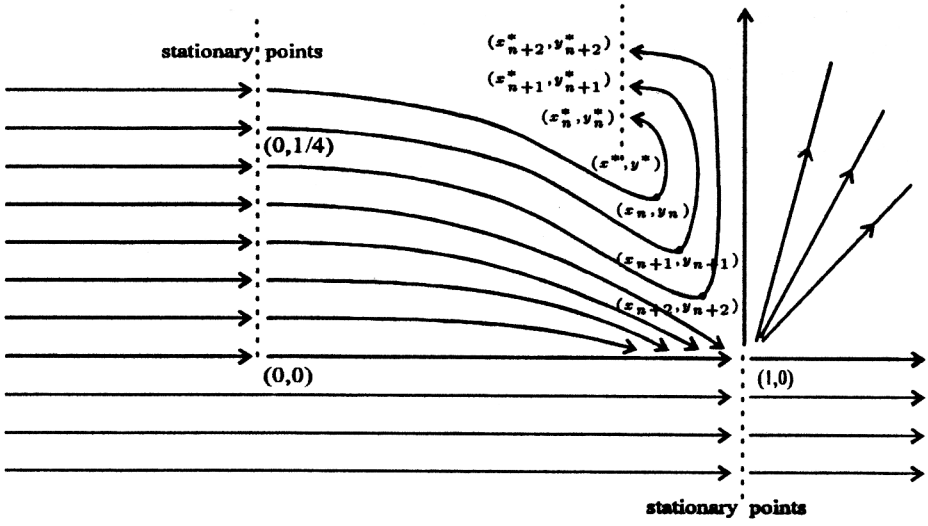

Fig. 6.

Here $J^+(\widetilde{M})$, $D^+(\widetilde{M})$, $\widehat{J}^+(\widetilde{M})$ and $\widehat{D}^+(\widetilde{M})$ are as in the example (iii) but there is a sequence $\{M_n\}$ of sets such that $M_n \subset B(\widehat{M}, \frac{1}{n})$ (it is enough to take M_n being the positive semi-trajectory of (x_n, y_n) where (x_n, y_n) are as on the picture) and $\Lambda_+(M_n) = J^+(M_n) = \{(x_n^*, y_n^*)\}$ (see Fig. 6). Since $\rho((x_n^*, y_n^*), H) = a = \text{const} > 0$, we have in this case the following statement: there is $\varepsilon^0 = a > 0$ such that for every $\delta > 0$ there is $M \subset B(\widetilde{M}, \delta)$ for which

$$\widehat{J}^+(M) \setminus B(\widehat{J}^+(\widetilde{M}), \varepsilon^0) \neq \emptyset$$

as well as

$$\widehat{D}^+(M) \setminus B(\widehat{D}^+(\widetilde{M}), \varepsilon^0) \neq \emptyset.$$

This is possible since the mapping $x \longmapsto \pi_+(x)$ is not uiformly (C)-usc in $\widetilde{M}$.

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