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THE REAL JACOBIAN CONJECTURE FOR CUBIC LINEAR MAPS OF RANK TWO

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Abstract. It is known that in order to verify the Generalized Jacobian Conjecture it is sufficient to check it only for polynomial mappings F of the cubic linear form, i. e.

$$F(x) = \left(x_1 + (\sum_{k=1}^{n} a_1^k x_k)^3, ..., x_n + (\sum_{k=1}^{n} a_n^k x_k)^3\right).$$

Since the Real Jacobian Conjecture is false, it is worth to note the cases when it is still true. We have proved that every polynomial mapping $F : \mathbf{R}^n \to \mathbf{R}^n$ of the cubic linear form with rank $[a_i^j : i, j = 1, ..., n] \leq 2$ and the jacobian Jac $F > 0$ is a polynomial diffeomorphism of $\mathbf{R}^n$ (with an analytic inverse).

1. Introduction

In the sequel $\mathbf{K}$ denotes either $\mathbf{C}$ or $\mathbf{R}$. By $M_n(D)$ we denote the ring of $(n \times n)$ – matrices with entries in a commutative ring D. If $F_j \in \mathbf{K}[X_1, ..., X_n]$, $j = 1, ..., n$, then we put $F = (F_1, ..., F_n)$ and denote

$$\mathrm{J(F)} := \left[\frac{\partial F_i}{\partial x_j} : i, j = 1, ..., n\right] \in M_n(\mathbf{K}[X_1, ..., X_n]),$$
$$\text{Jac } F := \det \mathrm{J(F)} \in \mathbf{K}[X_1, ..., X_n].$$

The famous Generalized Jacobian Conjecture asserts that the following holds:

(JC) For every $n \geq 2$ if $F : \mathbf{K}^n \to \mathbf{K}^n$ is a polynomial mapping such that Jac $F(x) \neq 0$ for any $x \in \mathbf{K}^n$, then F is injective.

Given a matrix $A \in M_n(\mathbf{K})$, we define the polynomial mapping

$$F_A : \mathbf{K}^n \ni x \to x + (Ax)^3 = \left(x_1 + (\sum_{k=1}^{n} a_1^k x_k)^3, ..., x_n + (\sum_{k=1}^{n} a_n^k x_k)^3 \right) \in \mathbf{K}^n,$$

where, for an element $y = (y_1, ..., y_n) \in \mathbf{K}^n$, we write $y^3 = (y_1^3, ..., y_n^3)$. It is well known, cf. e. g. [D1], that (JC) can be equivalently formulated in the following reduced form:

$(JC)_{red}$ For every $n \geq 2$ and every matrix $A \in M_n(\mathbf{K})$, if Jac $F_A(x) \neq 0$ for any $x \in \mathbf{K}^n$, then F_A is injective.

The Complex Jacobian Conjecture is still unsolved. Following some results in [D2]) and [H] the complex version of $(JC)_{red}$ is true for matrices of small rank or corank. Precisely, we have the following

THEOREM 1.1. *For every $n \geq 2$ and for every matrix $A \in M_n(\mathbf{C})$, if Jac $F_A = 1$ and if rank $A \in \{0, 1, 2, 3, 4, n-2, n-1\}$, then F_A is a polynomial automorphism of $\mathbf{C}^n$ (i. e. F is bijective mapping with polynomial inverse).*

Hence, for $n \leq 7$, we have the equivalence:

$$\text{Jac } F_A = 1 \iff F_A \text{ is a polynomial automorphism of } \mathbf{C}^n.$$

It is known that the Real Jacobian Conjecture is false, namely Pinchuk gave a beautiful counterexample, cf. [P]. He presented a polynomial mapping $F = (f, g) : \mathbf{R}^2 \to \mathbf{R}^2$, deg $f = 10$, deg $g = 25$ such that F is not globally injective although Jac $F > 0$ everywhere in $\mathbf{R}^2$. Applying to Pinchuk's mapping the well known reduction procedure we can find $n \in \mathbf{N}$ and a matrix $A \in M_n(\mathbf{R})$ such that Jac $F_A > 0$ everywhere in $\mathbf{R}^n$, but the mapping F_A is not globally injective. Note that the question what is the lowest possible degree of a twodimensional counterexample to the real (JC) remains open. Remember that every quadratic transformation F of $\mathbf{R}^n$ with the nonvanishing jacobian is injective because $F(x) - F(y) = [J(F)(\frac{x+y}{2})](x - y)$. Proposition 2.1 below can be treated as a small step towards the answer to the above question.

Our main purpose in this note is to prove some real counterpart of Theorem 1.1, namely we shall prove

THEOREM 1.2. *Assume that a matrix $A \in M_n(\mathbf{R})$ is of rank $A \in \{0, 1, 2, n\}$, $n \geq 2$. If Jac $F_A > 0$ in $\mathbf{R}^n$, then F_A is a polynomial diffeomorphism of $\mathbf{R}^n$. As a consequence, for $n \leq 3$, we have the statement*

$$\text{Jac } F_A > 0 \text{ in } \mathbf{R}^n \iff F_A \text{ is a diffeomorphism of } \mathbf{R}^n.$$

2. Useful results in dimension two

The most important step towards the proof of Theorem 1.2 is the following

PROPOSITION 2.1. *Let* $F : \mathbf{R}^2 \to \mathbf{R}^2$ *be a polynomial mapping of the form* $F = I + H$, *where* I *denotes the identity mapping and* H *a homogeneous polynomial of degree three. Then:* $Jac\, F > 0$ *in* $\mathbf{R}^2 \iff F$ *is a diffeomorphism of* $\mathbf{R}^n$.

Every polynomial mapping $F : \mathbf{K}^n \to \mathbf{K}^n$ of degree d can be uniquely written as the sum $F = F_0 + F_1 + ... + F_d$, where F_j is a homogeneous polynomial mapping of degree j and $F_d \neq 0$; we call F_d the leading homogeneous part of F. Before presenting the proof of Proposition 2.1 we recall the well known criterion of properness of polynomial mappings.

PROPOSITION 2.2 (CF. [R]). *Let* $F : \mathbf{K}^n \to \mathbf{K}^n$ *be a polynomial mapping with the leading homogeneous part* F_d *such that* $F_d^{-1}(0) = \{0\}$. *Then the mapping* F *is proper, i.e. inverse images of compact sets are compact.*

PROOF OF PROPOSITION 2.2. Let $C_d := \min\{|F_d(x)| : |x| = 1\}$, where $|\cdot|$ denotes a fixed norm in $\mathbf{K}^n$. Then $C_d > 0$ and $|F(x)| \geq C_d|x|^d$ for $x \in \mathbf{K}^n$.

If $F = F_0 + F_1 + F_2 + ... + F_d$ is the decomposition of F into homogeneous components, then $\liminf_{|x|\to\infty} \frac{|F_j(x)|}{|x|^d} = 0$ for $j = 0, 1, ..., d-1$ and hence $\liminf_{|x|\to\infty} \frac{|F(x)|}{|x|^d} \geq C_d$. Therefore there exists $C > 0$ such that $|F(x)| \geq C|x|^d$ if $|x|$ is sufficiently large, i.e. the mapping F is proper. □

PROOF OF PROPOSITION 2.1. If $H^{-1}(0) = \{0\}$ then by Proposition 2.2 the mapping F is proper and hence bijective (being a topological covering of simply connected space $\mathbf{R}^2$).

Therefore assume that the homogeneous polynomial mapping H has a non - trivial zero, i.e. $H(x, y) = ((\alpha x + \beta y)P(x, y), (\alpha x + \beta y)Q(x, y))$ for some $\alpha, \beta \in \mathbf{R}$, $\alpha^2 + \beta^2 > 0$ and for some homogeneous polynomial mappings P and Q of degree two. Replacing, if necessary, the mapping F by $L_2 \circ F \circ L_1$ with suitably chosen linear automorphisms L_1, L_2 of $\mathbf{R}^2$, without loss of generality, assume that

$$(*) \qquad F(x, y) = (x + x(ax^2 + bxy + cy^2),\ y + x(Ax^2 + Bxy + Cy^2))$$

for some $a, b, c, A, B, C \in \mathbf{R}$ with $Jac\, F > 0$ on $\mathbf{R}^2$. It is easy to check that $Jac\, F(x, y)$ is equal to the polynomial

$$\Phi(x, y) := 3(aB - bA)x^4 + 2(aC - cB)x^3y + (bC - cB)x^2y^2 + (3a + B)x^2 + 2(b + C)xy + cy^2 + 1$$

We have the following inequalities

(i) $c \geq 0$ (because $\Phi(0,y) > 0$ for $y \in \mathbf{R}$),

(ii) $aB - bA \geq 0$ (because $\Phi(x,0) > 0$ for $x \in \mathbf{R}$),

(iii) $bC - cB \geq 0$ (because $\Phi(x,x^2) > 0$ for $x \in \mathbf{R}$),

and we can assume

(iv) $b \geq 0$ (because Jac $(f(x,-y),-g(x,-y)) = \Phi(x,y) > 0$).

We shall consider four cases

Case 1^0. $c = 0$, $b \neq 0$, $C \neq 0$, i.e. $c = 0$, $b > 0$, $C > 0$. Then

$$\Phi(x,y) = 3(aB - bA)x^4 + 6aCx^3y + 3bc\left(xy + \tfrac{b+C}{3bC}\right)^2 + 3abx^2 + \lambda,$$

where $\lambda := 1 - \frac{(b+C)^2}{3bC} = -\frac{(b^2-bC+C^2)}{3bC} < 0$.

Put $xy + \frac{b+C}{3bC} = 0$, i.e. $y = -\frac{b+C}{3bCx}$. Then

$$\Phi\left(x, -\tfrac{b+C}{3bCx}\right) = 3(aB - bA)x^4 - 2a\frac{b+C}{b}x^2 + (3a+b)x^2 + \lambda.$$

Since $\lim_{x\to 0} \Phi\left(x, -\frac{b+C}{3bCx}\right) = \lambda < 0$, there exists $x \neq 0$ such that $\Phi\left(x, -\frac{b+C}{3bCx}\right) < 0$, a contradiction. Hence Case 1^0 is impossible.

Case 2^0. $c = 0$, $b \neq 0$, $C = 0$, i.e. $c = 0$, $b > 0$, $C = 0$. Then

$$\Phi(x,y) = 3(aB - bA)x^4 + 3abx^2 + 2bxy + 1.$$

Since $\Phi(x, \pm x^5) > 0$ for $x \in \mathbf{R}$, it has to be $b = 0$, a contradiction. Therefore Case 2^0 is impossible too.

Case 3^0. $c = 0$, $b = 0$, C-arbitrary. Then

$$F(x,y) = (x + ax^3, y + Ax^3 + Bx^2y + Cxy^2)$$

and

$$\Phi(x,y) = (1 + 3ax^2)(1 + Bx^2 + 2Cxy) > 0 \text{ for any } x \in \mathbf{R}^2.$$

Hence $a \geq 0$, $B \geq 0$, $C = 0$ and we see that F is bijective with the inverse

$$F^{-1}(u,v) = \left(\phi(u), \tfrac{v - A\phi^3(u)}{1 + B\phi^2(u)}\right), \quad u, v \in \mathbf{R}^2,$$

where ϕ is the inverse of the function $x \to x + ax^3$.

Case 4^0. $C > 0$. Since

$$0 < \Phi\left(x, -\tfrac{b}{2c}x\right) = \left(1 - \left(\tfrac{b^2-4ac}{4c}\right)x^2\right)\left(1 - \frac{bC - cB}{C}x^2\right) \quad \text{for any } x \in \mathbf{R},$$

it has to be $bC - cB \leq 0$ and, from (iii), we get the equality

(1) $bC - cB = 0$,

so $\Phi(x,y) = 3(aB-bA)x^4+6(aC-cA)x^3y+(3a+B)x^2+2(b+C)xy+cy^2+1$.

Since $\Phi(x,\pm x^2) > 0$ for any $x \in \mathbf{R}$, we get

(2) $aC - cA = 0$.

From (1), (2), $c > 0$, we derive the equality

(3) $aB - bA = 0$

By virtue of (1), (2), (3), there exists $\mu \in \mathbf{R}$ such that

(4) $(A,B,C) = \mu\,(a,b,c)$.

Put

$$T(x,y) := (x, y - \mu x), \quad S(x,y) := (x, y + \mu x), \quad G := T \circ F \circ S$$

and observe that, due to (4), we have the following

$$G(x,y) = (G_1(x,y), y),$$

$$\frac{\partial G_1}{\partial x}(x,y) = \text{Jac } G(x,y) = (\text{Jac } F)(x, y + \mu x) > 0 \text{ for } (x,y)\mathbf{R}^2.$$

Hence, we conclude that the function $\mathbf{R} \ni x \to G_1(x,y) \in \mathbf{R}$ is strictly increasing for every fixed $y \in \mathbf{R}$. Therefore the mapping G is a diffeomorphism of $\mathbf{R}^2$ and so is F.

The proof of the Theorem 2.1 is completed. □

3. Cubic linear and cubic homogeneous form

At first we recall the following

LEMMA 3.1. *Let $A \in M_n(\mathbf{K})$ and rank $A = r$. Then:*

(i) There exists a linear automorphism L of $\mathbf{K}^n$ such that

$$F_A^L(x_1,...,x_n) := L^{-1} \circ F_A \circ L(x_1,...,x_n)$$
$$+(x_1 + h_1(x_1,...,x_r), x_2 + h_2(x_1,...,x_r), ..., x_n + h_n(x_1,...,x_r)),$$

where h_j are homogeneous polynomials of degree 3 or h_j are equal to zero, $j = 1,...,n$.

(ii) A polynomial map $F_A : \mathbf{K}^n \to \mathbf{K}^n$ is injective (resp. bijective) if and only if

$$\tilde{F}_A^L(x_1,...,x_r) := (x_1 + h_1(x_1,...,x_r), ..., x_r + h_r(x_1,...,x_r)) : \mathbf{K}^r \to \mathbf{K}^r$$

is injective (resp. bijective).

Also: Jac $F_A > 0 \iff$ Jac $\tilde{F}_A^L > 0$, Jac F_A = const $\iff$ Jac $\tilde{F}_A^L$ = const.

PROOF OF LEMMA 3.1. Since rank $A = r$ we can find invertible matrices L, T such that

$$AL = T \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

where I_r denotes the r – dimensional identity matrix. Obviously, the mapping $L^{-1} \circ F_A \circ L$ has the desired form. Since

$$\text{Jac } F_A(x) = \text{Jac } \tilde{F}_A^L((L^{-1})_1(x), \ldots (L^{-1})_r(x)),$$

(ii) follows for "the injective version". For "the bijective version of (ii)" we use the theorem stating that any injective polynomial transformation of $\mathbf{K}^n$ is bijective, cf. [BR, Ru]. □

Now we are able to prove our main result.

PROOF OF THEOREM 1.2. Let $A \in M_n(\mathbf{R})$ and

$$F_A : \mathbf{R}^n \ni x \longrightarrow (x_1, ..., x_n) + (Ax)^3 \in \mathbf{R}^n.$$

(i) The statement is trivial when rank $A = 0$ and it immediately follows from Proposition 2.1 when rank $A = n$.

(ii) If rank $A = 1$, then, according to Lemma 3.1, we choose a linear automorphism L of $\mathbf{R}^n$ such that the mapping $L^{-1} \circ F_A \circ L(x_1, ..., x_n) = (x_1 + a_1 x_1^3, ..., x_n + a_n x_1^3)$. Since $\tilde{F}_A^L(x_1) = x_1 + a_1 x_1^3$ and Jac $\tilde{F}_A^L(x_1) = 1 + 3a_1 x_1^2 > 0$, the mapping $\tilde{F}_A^L : \mathbf{R} \to \mathbf{R}$ is strongly increasing. Hence, by Lemma 3.1(ii), the mapping F is also injective.

Note that cases (i) and (ii) were proved earlier, cf. [D1].

(iii) Suppose that rank $A = 2$. By Lemma 3.1 there exists a linear automorphism L such that

$$L^{-1} \circ F_A \circ L(x_1, ..., x_n) = (x_1 + h_1(x_1, x_2), x_2 + h_2(x_1, x_2), ..., x_n + h_n(x_1, x_2)),$$

where h_j are homogeneous polynomials of degree 3 or h_j are equal to zero, $j = 1, ..., n$. We know that $F_A : \mathbf{K}^n \to \mathbf{K}^n$ is injective if and only if $\tilde{F}_A^L(x_1, x_2) := (x_1 + h_1(x_1, x_2), x_2 + h_2(x_1, x_2)) : \mathbf{R}^2 \to \mathbf{R}^2$ is injective and Jac $\tilde{F}_A^L > 0$. Due to Theorem 2.1, the mapping $\tilde{F}_A^L$ is a diffeomorphism of $\mathbf{R}^2$.

The proof of Theorem 1.2 is completed. □

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