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ON THE HOLOMORPHIC ACTIONS OF A COMPLEX LIE GROUP ON THE GRASSMANNIAN MANIFOLD

BY ZBIGNIEW HAJTO AND WŁADYSŁAW WILK

Abstract. In this note, we are studying holomorphic actions of a complex Lie group M on the Grassmannian manifold $\mathbf{G}_k(X)$. We propose an elementary approach based on Łojasiewicz's version of Chow's Theorem and the theory of constructible sets [5]. The authors are very thankful to the Organizers of the Conference on Functional Equations in Karniowice where the first drafts of this paper were elaborated.

1. Introduction

First we recall the definition and some basic properties of analytically constructible sets which we shall need later. For details we refer to [5].

Let N be a complex analytic manifold. *Analitically constructible sets in* N are the elements of the smallest class of subsets of N that contains all analytic subsets of N and is closed under the following set theoretical operations:

a) locally finite union

b) difference of any two sets.

A nonempty connected analytic submanifold Γ of a complex manifold N is called an *analytically constructible leaf* if it is an analytically constructible subset of N.

Let X be an n-dimensional complex vector space. The Grassmannian space $\mathbf{G}_k(X)$ has the natural structure of holomorphic manifold introduced by the following atlas of inverse charts (cf. [5]):

$$\phi_{E,F} : L(E,F) \ni f \mapsto \widehat{f} = \{u + f(u) : u \in E\} \in \mathbf{G}_k(X),$$

where $E \in \mathbf{G}_k(X), F \in \mathbf{G}_{n-k}(X), E+F=X$.

If $\{e_i\}$, $i=1,\ldots,k$ is a basis of E and $f \in L(E,F)$ then the family $p_i(f)=e_i+f(e_i), i=1,\ldots,k$ is a basis of $\widehat{f}$. Moreover, if $\{e_i\}, i=1,\ldots,n$ is a fixed basis in X we have a finite atlas of inverse charts:

$$(*) \qquad \{\phi_{E_\gamma,E_{\gamma'}}\},$$

where $\gamma=(\gamma_1,\ldots,\gamma_k)$, $\gamma'=(\gamma'_1,\ldots,\gamma'_{n-k})$ are sequences of integers such that $1 \le \gamma_1 < \cdots < \gamma_k \le n$, $1 \le \gamma'_1 < \cdots < \gamma'_k \le n$, $E_\gamma = \sum_1^k \mathbf{C}e_{\gamma_i}$, $E'_\gamma = \sum_1^{n-k} \mathbf{C}e_{\gamma'_i}$ and $E_\gamma + E_{\gamma'} = X$.

PROPOSITION 1. (Lojasiewicz [5], VII 4.3). *Let k,n,l,r be integers such that $0 \le r,k,l \le n$ and $k+l-n \le r \le k,l$. Then the subset*

$$A = A_r^{kl} = \{(U,V) : \dim(U \cap V) = r\} \subset \mathbf{G}_k(X) \times \mathbf{G}_l(X)$$

is an anlytically constructible leaf of dimension

$$r(n-r)+(n-k)(k-r)+(n-l)(l-r)$$

and the mapping

$$A \ni (U,V) \mapsto U \cap V \in \mathbf{G}_r(X)$$

is holomorphic.

PROPOSITION 2. (Lojasiewicz [5] V.1). *If a holomorphic mapping $f\colon M \to N$ of complex manifolds is injective and $\dim N \le \dim M$, then the image $f(M)$ is open in N and $f\colon M \to f(M)$ is biholomorphic.*

2. Theorem of Chow

Let X and Y be two n-dimensional complex vector spaces. Let $L_0(X,Y)$ denote the set of all linear isomorphisms $T : X \to Y$ and $\tau_k, 1 \le k \le n-1$, the following biholomorphism of Grassmannian manifolds

$$\tau_k : \mathbf{G}_k(Y) \ni E \mapsto E^\perp = \{\phi \in Y^* : \phi_E = 0\} \in \mathbf{G}_{n-k}(Y^*).$$

We say that a mapping $f : \mathbf{G}_k(X) \to \mathbf{G}_k(Y)$ is *an isomorphism of Grassmannian spaces* iff there exists $T \in L_0(X,Y)$ such that $f(E)=T(E)$. In this case, we write $f = T_{(k)}$.

THEOREM OF CHOW. ([2], [5] VII.20.2). *Every biholomorphism $f\colon \mathbf{G}_k(X) \to \mathbf{G}_k(Y)$ of Grassmannian spaces is of the form $f = T_{(k)}$ or in the case when $n=2k$ and $n>2$ it may be of the form $f = \tau_k^{-1} \circ T'_{(k)}$ where $T' \in L_0(X,Y^*)$.*

PROOF. See [5] VII.20.2.

Let $\mathrm{Aut}_{\mathrm{hol}}(\mathbf{G}_k(X))$ denote the group of biholomorphisms of $\mathbf{G}_k(X)$. From the theorem of Chow we have that for $n \in \{1,2\}$, $k = 1$ and for $n > 2$, $1 \leq k \leq n-1$, $n \neq 2k$

$$\mathrm{Aut}_{\mathrm{hol}}(\mathbf{G}_k(X)) = \mathbf{P}GL(n);$$

for $n > 2$ and $n = 2k$

$$\mathrm{Aut}_{\mathrm{hol}}(\mathbf{G}_k(X)) = \mathcal{G}$$

where $\mathcal{G} = G_1 \cup G_2$ is a group such that

$$G_1 = \{f \in \mathrm{Aut}(\mathbf{G}_k(X)) : f = A_{(k)}, A \in L_0(X,X)\}$$

$$G_2 = \{f \in \mathrm{Aut}(\mathbf{G}_k(X)) : f = \tau_k^{-1} \circ A'_{(k)}, A' \in L_0(X,X^*)\}.$$

$\mathbf{P}GL(n)$ has the natural structure of a complex Lie group.

Let $e_1, \ldots e_n$ be a fixed basis of X, and let $I : X \to X^*$ be the natural isomorphism given by $I(e_i) = e_i^*$, for $i = 1, \ldots, n$, where $\{e_i^*\}$ is the basis dual to $\{e_i\}$.

For $A' \in L_0(X,X^*)$ we obviously have $A' = I \circ A$ for $A \in L_0(X,X)$. The structure of complex Lie group on $\mathcal{G}$ is introduced by the following two inverse charts:

$$\Lambda_1 : \mathbf{P}GL(n) \ni \alpha \mapsto A_{(k)} \in G_1 \qquad \text{where } \alpha = \mathbf{C}A$$

$$(**) \qquad \Lambda_2 : \mathbf{P}GL(n) \ni \alpha \mapsto \tau_k^{-1} \circ I_{(k)} \circ A_{(k)} \in G_2 \quad \text{where } \alpha = \mathbf{C}A$$

3. Equation of translation

Let the notations be like in Section 2 and let M denote a complex Lie group. We shall describe all holomorphic mappings $F : M \times \mathbf{G}_k(X) \to \mathbf{G}_k(X)$ with the following properties:

$$(1) \qquad F(\beta\alpha, E) = F(\beta, F(\alpha, E)) \text{ for } \alpha, \beta \in M \text{ and } E \in \mathbf{G}_k(X)$$

$$(2) \qquad F(1, E) = E \qquad \text{for } 1 \in M \text{ and } E \in \mathbf{G}_k(X)$$

In the theory of functional equations, (1) is called the translation equation and (2) the identity condition (cf. [1] or [6]).

THEOREM. *There exists a one-to-one correspondence between holomorphic mappings* $F : M \times \mathbf{G}_k(X) \to \mathbf{G}_k(X)$ *with properties (1), (2) and holomorphic homomorphisms:*

$$\epsilon : M \to \mathrm{Aut}_{\mathrm{hol}}(\mathbf{G}_k(X))$$

given by $F(\alpha, E) = \epsilon(\alpha)(E)$ *for* $\alpha \in M$ *and* $E \in \mathbf{G}_k(X)$.

PROOF. We consider two cases:
I. $\mathrm{Aut}_{\mathrm{hol}}(\mathbf{G}_k(X)) = \mathbf{P}GL(n)$,
II. $\mathrm{Aut}_{\mathrm{hol}}(\mathbf{G}_k(X)) = \mathcal{G}$.

Case I. First let us take a holomorphic F with properties (1), (2): If we fix $\alpha \in M$ then $F_\alpha : \mathbf{G}_k(X) \to \mathbf{G}_k(X)$ is biholomorphic. From the Theorem of Chow, $F_\alpha = T^\alpha_{(k)}$ for $T^\alpha \in L_0(X, X)$, so $\mathbf{C}T^\alpha \in \mathbf{P}GL(n)$. The mapping $\alpha \mapsto \mathbf{C}T^\alpha$ is a group homomorphism because of (1) and (2). It is holomorphic because if we take $n+1$ points in general position in the projective space $\mathbf{P}^{n-1}(X) : \mathbf{C}a^i = \mathbf{C}(a^i_0, \ldots a^i_{n-1})$ for $i = 0, \ldots n$, then we have

$$\mathbf{C}a^i = E^i_1 \cap \cdots \cap E^i_r$$

(i.e. the line $\mathbf{C}a^i$ is an intersection of a finite number of k-dimensional subspaces of X). From Proposition 1 (Section 1) the mapping $\alpha \mapsto F_\alpha(E^i_1) \cap \cdots \cap F_\alpha(E^i_r) = T^\alpha(E^i_1 \cap \cdots \cap E^i_r)$ is holomorphic and by Cramer's rules the system:

$$\mathbf{C}T^\alpha(a^i) = T^\alpha(E^i_1 \cap \cdots \cap E^i_r) \quad ; \quad i = 0, \ldots, n$$

has a unique holomorphic solution $\alpha \mapsto \mathbf{C}T^\alpha$.

On the other hand, if $\epsilon : M \to \mathbf{P}GL(n)$ is a holomorphic homomorphism, then the mapping $F : (\alpha, E) \mapsto \epsilon(\alpha)(E)$ is holomorphic directly by the Hartogs' Theorem on separate analyticity [7].

Case II. Just as in Case I, the mapping $\alpha \mapsto F_\alpha$ is a group homomorphism because of (1) and (2).

LEMMA. *Let* $\epsilon : M \to \mathcal{G}$ *be a group homomorphism such that* $F\colon (\alpha, E) \mapsto \epsilon(\alpha)(E)$ *is a continuous mapping. Then if we consider the topology* $\tau = \{\emptyset, G_1, G_2, \mathcal{G}\}$ *on* $\mathcal{G}$, *the mapping* ϵ *is continuous.*

PROOF OF THE LEMMA. It is enough to prove that there exists a neighbourhood U_1 of $1 \in M$ such that

$$\alpha \in U_1 \Rightarrow \epsilon(\alpha) \in G_1.$$

Let us suppose that there exists a sequence $\{\alpha_n\}$ of points of M tending to $1 \in M$ such that $\epsilon(\alpha_n) \in G_2$. Then $F(\alpha_n, E)$ tends to E for any fixed

$E \in \mathbf{G}_k(X)$ and $\epsilon(\alpha_n) = \tau_k^{-1} \circ L^n$ where $L^n : X \to X^*$ is a sequence of linear isomorphisms. Taking a subsequence if necessary, we may suppose that $\dfrac{L^n(e_1)}{|L^n(e_1)|}$ tends to a unit covector $v = c_1 e_1^* + \cdots + c_n e_n^*$. Thus we may assume that there exists an r such that $c_r \neq 0$.

Now if we take $E_\gamma = \sum_{i=1}^k \mathbf{C} e_{\gamma_i}$, such that $e_1, e_r \in E_\gamma$ and if we consider the inverse chart $\phi_{E_\gamma, E_{\gamma'}}$ (cf. Section 1, (*)), we have $F(\alpha_n, E_\gamma) = \widehat{f}_n$, $\dfrac{L^n(e_1)}{|L^n(e_1)|}(p_r(f_n)) = 0$. But on the other hand it tends to $v(e_r) = c_r \neq 0$. This contradiction completes the proof of the Lemma.

From the Lemma and the Theorem of Chow, F_α is either of the form $T^\alpha_{(k)}$ or $\tau_k^{-1} \circ T'^\alpha_{(k)}$ for $\alpha \in U$, where U is an open subset of M. Hence in the local charts (cf. Section 2, (**)), we can apply Cramer's rules as in Case I and similarly by Hartogs' Theorem we complete the proof of the Theorem.

4. Corollaries

Now we shall state a special case of the Theorem (for $M = \mathbf{P}GL(n)$) which is related to Kleiman's Transversality Theorem [4], [10] Chapitre III).

REMARK. Every holomorphic mapping $F : \mathbf{P}GL(n) \times \mathbf{G}_k(X) \to \mathbf{G}_k(X)$ with the properties (1), (2) (cf. Section 3) is induced by an analytic endomorphism $\epsilon : \mathbf{P}GL(n) \to \mathbf{P}GL(n)$. Moreover the holomorphic endomorphisms are only of the three following forms:

(i) $\alpha \mapsto \beta \circ \alpha \circ \beta^{-1}$

(ii) $\alpha \mapsto \beta \circ (\alpha^t)^{-1} \circ \beta^{-1}$

(iii) $\alpha \mapsto 1$.

The automorphisms of the classical group $\mathbf{P}GL(n)$ are described by J. Dieudonné [3] Chapitre IV. We will prove that if a holomorphic endomorphism $\epsilon : \mathbf{P}GL(n) \to \mathbf{P}GL(n)$ is not of the form (iii) then it must be an automorphism.

First let us recall from group theory that $\mathbf{P}GL(n)$ is a simple group (see e.g. O'Meara [8]). $\operatorname{Ker} \epsilon \subset \mathbf{P}GL(n)$ is a normal subgroup. Hence if ϵ is not of the form (iii) it is injective, but $\mathbf{P}GL(n)$ is connected and from Proposition 2 Section 1 , $\epsilon(\mathbf{P}GL(n))$ is an open subgroup. Moreover $\mathbf{P}GL(n) \setminus \epsilon(\mathbf{P}GL(n)) = \bigcup_{\alpha \notin \epsilon(\mathbf{P}GL(n))} \alpha.\epsilon(\mathbf{P}GL(n))$ and it is open as well. Hence ϵ is an automorphism.

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