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ON BERTINI-TYPE THEOREM FOR WEAKLY-NORMAL COMPLEX ANALYTIC SETS

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1. Introduction. The aim of the paper is to present an elementary proof of the following Bertini-type theorem for weakly-normal complex analytic sets

THEOREM 1. *If X is a weakly-normal locally analytic subset of $\mathbf{C}^n$ then there exists a fat subset M of the space of all affine hyperplanes in $\mathbf{C}^n$ such that for every $H \in M$ the intersection $X \cap H$ is again weakly-normal.*

In [M, Corollary II.6] an analogous Bertini-type theorem for normal and reduced locally analytic sets in $\mathbf{C}^n$ is proved.

Proofs for normal and reduced complex analytic sets are very similar, a Bertini-type theorem is deduced from two facts: a Sard-type theorem and the openness condition for the given property. In [M1, Thm. 18.] Manaresi proved that this kind of arguments may be applied to any local property of complex spaces (not being only reduced and normal).

The Sard-type theorem for reduced and normal complex analytic sets follows from the homological characterization of those properties, openness conditions were proved by Bănica in ([Ba]). Using a similar characterization Manaresi (in [M, Thm. 1.12]) proved a Sard-type theorem for weakly-normal complex analytic sets. The proof of the openness condition for weakly-normal complex analytic sets turned out to be much more difficult. Bingener and Flenner in [B-F] proved the openness condition for local properties satisfying

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certain conditions and verified these conditions for weakly-normal complex analytic sets. The proof is very complicated, it is based on the method of Stein compacts and uses a lot of algebraic geometry.

Our proof of the Bertini-type theorem for weakly-normal complex analytic sets is completely elementary and in fact is based on a careful examination of the proof of the Sard-type theorem (given by Manaresi), especially the class of "wrong fibers".

2. Homological characterization of reduced and normal complex spaces. Let X be a complex space. For $k = 0, 1, 2, \dots$, denote

$$S_k(\mathcal{O}_X) := \{p \in X : \text{ prof } \mathcal{O}_{X,p} \leq k\} \subset X.$$

In this situation $S_k(\mathcal{O}_X)$ is an analytic subset of X, for any k ([S-T]). These "singular subsets" S_k give a lot of information about the space X. In particular, we have the following characterization of reduced and normal spaces

LEMMA 1 [S-T]. *X is reduced iff*

$$\dim(\mathit{Sing}\ X \cap S_k(\mathcal{O}_X)) \leq k - 1,$$

for any $k \geq 0$.

LEMMA 2 [M]. *X is normal iff*

$$\dim(\mathit{Sing}\ X \cap S_k(\mathcal{O}_X)) \leq k - 2,$$

for any $k \geq 1$.

3. Weakly-normal complex spaces. A complex space X is said to be weakly-normal (or maximal) if every c-holomorphic function on X is holomorphic. Detailed information on weakly normal complex spaces may be found in [F] (see also [A-N]), we shall only give the following characterization here.

Let $(\tilde{X}, \pi)$, where $\pi: \tilde{X} \longrightarrow X$ is a finite map, be a normalization of X. Consider the reduction of the fiber product $R := (\tilde{X} \times_X \tilde{X})_{\text{red}}$, $\pi' : R \to X$.

Denote by $g_1, g_2: (\tilde{X} \times_X \tilde{X})_{\text{red}} \longrightarrow \tilde{X}$ the mappings induced by the projections $p_1, p_2: \tilde{X} \times_X \tilde{X} \longrightarrow \tilde{X}$. Let $\mathcal{O}_X$ and $\mathcal{O}_{\tilde{X}}$ denote the structure sheaves of X and $\tilde{X}$.

LEMMA 3, [M, (0.4.II)]. *The complex space $(X, \mathcal{O}_X)$ is weakly-normal iff the sequence*

$$0 \longrightarrow \mathcal{O}_X \underset{\pi^*}{\longrightarrow} \pi_* \mathcal{O}_{\tilde{X}} \underset{(g_1 - g_2)^*}{\longrightarrow} \pi'_* \mathcal{O}_R$$

is exact.

4. Bertini-type theorems for reduced and normal complex analytic sets. Now, let $f \in \mathcal{O}_X(X)$ be a holomorphic function on a complex space X. We shall denote $X_f := f^{-1}(0)$ and $\mathcal{O}_{X_f} := \mathcal{O}_X/(f \cdot \mathcal{O}_X)$. In this situation $(X_f, \mathcal{O}_{X_f})$ is a complex space. We shall denote it by X_f.

Following the ideas of [M] we can give a proof of a version of a Bertini-type theorem for reduced and normal complex analytic sets.

LEMMA 4. *Let X be a reduced (resp. normal) complex space. If $f \in \mathcal{O}_X(X)$ is such that*

(1) $Sing X_f \subset Sing X$

and

(2) X_f does not contain any irreducible component of sets $Sing\ X \cap S_k(\mathcal{O}_X)$ then the complex space X_f is reduced (resp. normal).

PROOF. We shall prove the lemma for a reduced complex space. By the assumption (2) the germ of f is not a zero divisor in any local ring $\mathcal{O}_{X,x}$, so for every $x \in X_f$ we have prof $(\mathcal{O}_{X,x}) =$ prof $(\mathcal{O}_{X_f,x}) + 1$ and consequently $S_k(\mathcal{O}_{X_f,x}) \subset S_{k+1}(\mathcal{O}_X)$ so by (1), we have $\text{Sing } X_f \cap S_k(\mathcal{O}_{X_f}) \subset \text{Sing } X \cap S_{k+1}(\mathcal{O}_X) \cap X_f$.

By assumption (2) and Lemma 1 this gives $\dim(\text{Sing } X_f \cap S_k(\mathcal{O}_{X_f})) \leq \dim(\text{Sing} X \cap S_{k+1}(\mathcal{O}_X)) - 1 \leq k + 1 - 1 - 1 = k - 1$. By Lemma 1, this completes the proof.

The proof for a normal space is similar (we use Lemma 2 instead of Lemma 1).

From this lemma, there easily follows Bertini–type Theorem for normal and reduced complex analytic sets.

THEOREM 2, [M, Cor. II.6]. *If X is a normal (resp. reduced) locally analytic subset of $\mathbf{C}^n$ then there exists a fat subset M of the space of all affine hyperplanes in $\mathbf{C}^n$ such that for every $H \in M$ the intersection $X \cap H$ is again normal (resp. reduced).*

5. Bertini-type theorem for weakly-normal analytic sets. We shall preserve the notation introduced in previous sections. Moreover, for simplicity we shall denote $\tilde{X}_f := \tilde{X}_{f \circ \pi}$, $R_f := R_{f \circ \pi'}$.

LEMMA 5. *Let $(X, \mathcal{O}_X)$ be a complex space, and let*

$$0 \longrightarrow \mathcal{F} \underset{\alpha}{\longrightarrow} \mathcal{G} \underset{\beta}{\longrightarrow} \mathcal{H}$$

be an exact sequence of coherent analytic sheaves. Then there exists a sequence of analytic subsets $\{X_i\}_{i \in I}$ of X such that for any holomorphic function $f \in \mathcal{O}_X(X)$ satisfying condition:

X_f does not contain any of the sets X_i

the sequence

$$0 \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_f} \underset{\alpha}{\longrightarrow} \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_f} \underset{\beta}{\longrightarrow} \mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_f}$$

is exact.

PROOF. Take as X_i all analytic varieties associated with the sheaves $\mathcal{G}/\alpha(\mathcal{F})$, $\mathcal{H}/\mathcal{I}m\beta$ ([S]) and irreducible components of space X. Then apply the proof of [M (1.8.)].

LEMMA 6. *Let X be a weakly-normal complex space. There exists a sequence of analytic subsets $\{X_i\}$ of X such that if $f \in \mathcal{O}_X(X)$ is such that*

(1) *f is not zero on any of sets X_i,*
(2) *Sing $R_f* \subset$Sing R,*
(3) *Sing $\tilde{X}_f \subset$Sing $\tilde{X}$,*

then the complex space $(X_f, \mathcal{O}_{X_f})$ is weakly-normal.

PROOF. Let us take the exact sequence

$$0 \longrightarrow \mathcal{O}_X \underset{\pi^*}{\longrightarrow} \pi_* \mathcal{O}_{\tilde{X}} \underset{(g_1-g_2)^*}{\longrightarrow} \pi'_* \mathcal{O}_R.$$

There exists a sequence of analytic subsets $\{X_i\}$ of X such that for any holomorphic function $f \in \mathcal{O}_X(X)$ which is not zero on any of sets X_i we have

(a) the sequence

$$0 \longrightarrow \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X_f} \underset{\pi^*}{\longrightarrow} \pi_* \mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_f} \underset{(g_1-g_2)^*}{\longrightarrow} \pi'_* \mathcal{O}_R \otimes_{\mathcal{O}_X} \mathcal{O}_{X_f}$$

is exact,

(b) the space $(\tilde{X}_f, \mathcal{O}_{\tilde{X}_f})$ is normal,
(c) the space $(R_f, \mathcal{O}_{R_f})$ is reduced,
(d) the set X_f contains no irreducible component of X and Sing$X$.

Now, we have

$$\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X_f} = \mathcal{O}_{X_f},$$

$$\pi_* \mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_f} = \pi_* \mathcal{O}_{\tilde{X}_f},$$

$$\pi'_* \mathcal{O}_R \otimes_{\mathcal{O}_X} \mathcal{O}_{X_f} = \pi'_* \mathcal{O}_{R_f}.$$

In this situation, by (b) and (d)

$$\pi | \tilde{X}_f : \tilde{X}_f \to X_f$$

is a normalization.

Using the universal property of the fiber product and (c) we get

$$(\tilde{X}_f \times_{X_f} \tilde{X}_f)_{\text{red}} = R_f.$$

Consequently, we can write the exact sequence (a) in the following form

$$0 \longrightarrow \mathcal{O}_{X_f} \longrightarrow \mathcal{O}_{\tilde{X}_f} \longrightarrow (\mathcal{O}_{\tilde{X}_f \times_{X_f} \tilde{X}_f})_{\text{red}}$$

which completes the proof.

PROOF OF THEOREM 1. Let X be a weakly–normal analytic subset of $\mathbf{C}^n$. From the Sard theorem it follows that there exists a fat family M_1 of hyperplanes in $\mathbf{C}^n$ such that for any $H \in M_1$ we have

$$\text{Sing}\,(X \cap H) \subset \text{Sing}\,X \cap H.$$

Now, let $\{X_i\}$ be a sequence of analytic subsets of X satysfying the assertions of Lemma 6. The family M_2 of those affine hyperplanes in $\mathbf{C}^n$ which do not contain any X_i is fat.

Then the intersection $M := M_1 \cap M_2$ is also fat. Take any $H \in M$ and let f be an equation of H. Since the mappings π and π' are biholomorphisms outside the singular locus Sing X of X, we have Sing $\tilde{X}_f \subset$ Sing $\tilde{X}$ and Sing $R_f \subset$ Sing R. Therefore, by Lemma 6, the set $X \cap H = X_f$ is weakly–normal. This proves the Theorem.

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