UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXXIV
1997

STRONG OPERATOR CONVERGENCE AND SPECTRAL THEORY OF ORDINARY DIFFERENTIAL OPERATORS

BY JOACHIM WEIDMANN

Abstract. If A_n and A are self-adjoint operators such that A_n converges to A in the sense of strong resolvent convergence, then it is a classical result for the corresponding spectral resolutions E_n and E, that $E_n(\lambda) \xrightarrow{s} E(\lambda)$ for every λ which is not an eigenvalue of A. An extended version of this result, where A_n and A may operate on different Hilbert spaces, is used to show that isolated eigenvalues of singular Sturm–Liouville operators can be approximated by the eigenvalues of regular operators generated by the same differential expressions on smaller intervals. Counterexamples demonstrate that the choice of the boundary conditions for the regular operators is crucial. The abstract result is also used to develop a technique (related to the subordinacy method) for the proof of absolute continuity of the spectrum in certain intervals. — Similar results can be proved for discrete operators (Jacobi matrices) and for ordinary differential operators of any order and dimension.

1. Motivation. What can be said about the spectrum of a "complicated" self-adjoint operator by approximating it by "simple" operators, i. e. by operators with well known spectral properties? More concrete: what can be said about the spectrum of a singular differential operator approximated by regular operators.

To understand, what approximating should mean, we have first to talk about convergence. For *bounded* operators $A_n, A \in B(H)$ in a Hilbert space H, we have mainly the following notions of convergence

— *norm convergence*, $A_n \xrightarrow{n} A$, if $\|A_n - A\| \to 0$,
— *strong convergence*, $A_n \xrightarrow{s} A$, if $\|A_n f - Af\| \to 0 \;\; \forall f \in H$,
— *weak convergence*, $A_n \xrightarrow{w} A$, if $\langle A_n f - Af, g\rangle \to 0 \;\; \forall f, g \in H$.

For self-adjoint operators A_n and A these notions of convergence have quite different spectral implications:

— *norm convergence* implies $\sigma(A_n) \to \sigma(A)$; isolated eigenvalues λ of A of finite multiplicity are exactly the limits of eigenvalues of A_n (including multiplicity); the corresponding eigenprojections converge in norm.

— *strong convergence* implies $E_n(\lambda) \xrightarrow{s} E(\lambda)$ if λ is not an eigenvalue of A (see below); every $\lambda \in \sigma(A)$ is the limit of a sequence (λ_n) with $\lambda_n \in \sigma(A_n)$.
But: Not every limit of (λ_n) with $\lambda_n \in \sigma(A_n)$ lies in $\sigma(A)$ (neither if λ_n are [isolated] eigenvalues of A_n, nor if $\lambda_n \in \sigma(A_n)\backslash\sigma_{pp}(A_n)$ cf. Example 2).

— for *weak convergence* no reasonable spectral implication holds, as is shown by the following example:

EXAMPLE 1. Let $\{e_n\}$ be an orthonormal basis (ONB) of H, A_n the orthogonal projection onto the subspace spanned by $e_1 + e_n$,

$$A_n f \;:=\; \frac{1}{2}\langle e_1 + e_n, f\rangle(e_1 + e_n) \;=\; P_{e_1+e_n} f\,,$$

then

$$\sigma(A_n) \;=\; \{0,1\},\;\; A_n \xrightarrow{w} A := \frac{1}{2}P_{e_1},\;\; \text{but}\;\; \sigma(A) = \{0, \frac{1}{2}\}\,,$$

i.e. 1 is an eigenvalue of every A_n but not in the spectrum of A, while 1/2 is an eigenvalue of A, but not a limit of eigenvalues λ_n of A_n. #

In order to study differential operators, these notions are not useful in the form explained so far, since all these operators are unbounded. For self-adjoint operators reasonable notions appear to be the corresponding convergence notions for the resolvents,

$$(A_n - z)^{-1} \xrightarrow{n/s/w} (A - z)^{-1} \;\; \text{for}\;\; z \in \mathbf{C}\backslash\mathbf{R}\,.$$

Actually these notions are still not wide enough, since there are many interesting cases, where the operators A_n, A are defined on varying Hilbert spaces H_n, H. For this reason we come to the following definition: Assume that H_n, H are subspaces of one "large" Hilbert space K, $P_n := P_{H_n}$ and $P := P_H$ the corresponding orthogonal projections. We say that A_n *converges* to A in the sense of

— *norm resolvent convergence*, $A_n \xrightarrow{nrc} A$,
— *strong resolvent convergence*, $A_n \xrightarrow{src} A$,
— *weak resolvent convergence*, $A_n \xrightarrow{wrs} A$,

if $(A_n - z)^{-1} P_n \overset{n/s/w}{\longrightarrow} (A-z)^{-1} P$ with respect to norm / strong / weak convergence.

Norm resolvent convergence has the same nice properties as norm convergence. **But**, for most of the interesting questions it is **too** restrictive.

Weak resolvent convergence cannot be expected to be useful (after knowing the above example). In contrast to the other two cases it is not even true that, for bounded self-adjoint operators, $A_n \overset{w}{\longrightarrow} A$ is equivalent to $(A_n - z)^{-1} \overset{w}{\longrightarrow} (A-z)^{-1}$ for $z \in \mathbf{C} \setminus \mathbf{R}$ (as again the above example shows), and we even do not know, if $(A_n - z_0)^{-1} \overset{w}{\longrightarrow} (A - z_0)^{-1}$ implies $(A_n - z)^{-1} \overset{w}{\longrightarrow} (A-z)^{-1}$ for $z \neq z_0$. Anyhow it is noteworthy that sometimes under weak additional conditions weak (resolvent) convergence might imply strong (resolvent) convergence.

Therefore all our hope for reasonable applications lies on the notion of *strong resolvent convergence.* A useful criterion for strong resolvent convergence is:

THEOREM 1. *Assume the above situation for H_n, H, P_n, P and let A_n and A be self-adjoint operators in H_n and H, respectively. If there exists a core D_0 of A such that for every $f \in D_0$ there exists an $n(f)$ such that $f \in D(A_n)$ for $n \geq n(f)$ and $A_n f \to Af$ for $n \to \infty$, then $A_n \overset{src}{\longrightarrow} A$.*

PROOF. For $g = (A-z)f \in (A-z)D_0$ and sufficiently large n we have

$$\Big((A_n - z)^{-1} P_n - (A-z)^{-1} P\Big) g = (A_n - z)^{-1}(A - A_n)(A-z)^{-1}(A-z)f$$
$$= (A_n - z)^{-1}(A - A_n) f \longrightarrow 0,$$

since $(A - A_n)f \to 0$ for $f \in D_0$ and $\|(A_n - z)^{-1}\| \leq |\Im z|^{-1}$. Since $(A-z)D_0$ is dense in H and $(A_n - z)^{-1}$ is uniformly bounded, this implies the result. □

The following fundamental result is essentially due to F. RELLICH [4] (1937) (see also T. KATO [2]).

THEOREM 2. *Let A_n, A be self-adjoint operators, $A_n \overset{src}{\longrightarrow} A$ in the sense of the above definition. Then*

$$E_n(\lambda) P_n \overset{s}{\longrightarrow} E(\lambda) P \quad \textit{if} \quad \lambda \quad \textit{is not an eigenvalue of} \quad A.$$

PROOF. The main steps of the proof are as follows: By definition

$$(A_n - z)^{-1} P_n \overset{s}{\longrightarrow} (A-z)^{-1} P \quad \text{for} \quad z \in \mathbf{C} \setminus \mathbf{R}.$$

With the Stone–Weierstraß Theorem it follows that

$$\varphi(A_n)P_n \xrightarrow{s} \varphi(A)P \qquad \text{for} \quad \varphi \in C_\infty(\mathbf{R}),$$

where $C_\infty(\mathbf{R})$ is the space of complex valued continuous functions φ on $\mathbf{R}$ with $\lim_{|x|\to\infty} \varphi(x) = 0$. For intervals I with boundaries which are not eigenvalues of A this result can be extended to χ_I, i.e. $E_n(I)P_n = \chi_I(A_n)P_n \xrightarrow{s} \chi_I(A)P = E(I)P$. □

For $A_n \xrightarrow{src} A$ this result easily implies that for every $\lambda \in \sigma(A)$ there are $\lambda_n \in \sigma(A_n)$ with $\lambda_n \to \lambda$. But, on the other hand, from $\lambda_n \in \sigma(A_n)$ with $\lambda_n \to \lambda$ it does **not follow**, that $\lambda \in \sigma(A)$; this also does not hold if the λ_n are required to lie in $\sigma_c(A_n)$, $\sigma_{ac}(A_n)$ or $\sigma_e(A_n)$. This follows from the following example:

EXAMPLE 2. Define in $L^2(\mathbf{R})$ the operators $A_{j,n}$ by

$$A_{j,n} = -\frac{d^2}{dx^2} + V_{j,n} \qquad (n \in \mathbf{N},\ j = 1,2)$$

with

$$V_{1,n}(x) = \begin{cases} -1 & \text{if } n \le x \le n+1, \\ 0 & \text{otherwise,} \end{cases} \qquad V_{2,n}(x) = \begin{cases} -1 & \text{if } x \ge n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $A_{j,n} \xrightarrow{src} A = -\frac{d^2}{dx^2}$ for $n \to \infty$ $(j = 1,2)$, and $\sigma(A) = \sigma_{ac}(A) = [0,\infty)$, while $\sigma(A_{1,n}) = [0,\infty) \cup \{\mu\}$ with a simple eigenvalue $\mu \in [-1,0]$ and $\sigma(A_{2,n}) = \sigma_{ac}(A_{2,n}) = [-1,\infty)$. #

2. Regular approximations of singular operators. In what follows, we shall study the following interesting problem: Let

$$\tau f = \frac{1}{r}\Big\{ -(pf')' + qf \Big\} \quad \text{on} \quad (a,b)$$

with $r, p > 0$, $r, q, 1/p \in L^1_{\mathrm{loc}}(a,b)$ (the reader might think of r, p, q to be locally "nice" functions on (a,b)), T the maximal operator generated by τ in $L^2(a,b;r)$.

We say that τ is in the *limit point case* (lp) (*limit circle case* (lc)) at a, resp. b, if for every $z \in \mathbf{C}$ at most one solution (all solutions) of $(\tau - z)u = 0$ lies (ly) left, resp. right, in $L^2(a,b;r)$.

All self-adjoint realizations A_{v_a,v_b} (with separated boundary conditions) can be given as restrictions of T with domains

$$D(A_{v_a,v_b}) = \Big\{ f \in D(T) : [v_a, f]_a = 0 \quad \text{if } \tau \text{ is lc at } a, \qquad [v_b, f]_b = 0 \quad \text{if } \tau \text{ is lc at } b \Big\}$$

with no boundary condition at a/b, if τ is lp at a/b; the Lagrange bracket $[v, f]_x$ is defined by $[v, f]_x = \overline{v(x)}pf'(x) - \overline{pv'(x)}f(x)$ for $x \in (a, b)$, for $x = a/b$ this expression is defined as a limit (here and in what follows we use "a/b" as an abbreviation for "a resp. b"). v_a and v_b are non-trivial real solutions of $(\tau - \lambda_{a/b})v = 0$ with $\lambda_{a/b} \in \mathbf{R}$ (or, alternatively, $v_{a/b}$ are real functions from $D(T)$ such that $[v_{a/b}, f]_{a/b} = 0$ does not hold for every $f \in D(T)$).

Let now $(a_n, b_n) \subset (a, b)$ with $a_n \searrow a$, $b_n \nearrow b$.

QUESTION: Is it possible to choose self-adjoint realizations A_n of τ in $L^2(a_n, b_n; r)$ such that $A_n \xrightarrow{src} A$ and (if possible) suitable spectral implications hold?

If we denote by T_n the maximal operator generated by τ in $L^2(a_n, b_n; r)$, then the following general result holds (cf. G. STOLZ and J. WEIDMANN [6]):

THEOREM 3. *Define* A_n *in* $L^2(a_n, b_n; r)$ *by*

$$D(A_n) = \Big\{ f \in D(T_n) : \textit{arbitrary self-adjoint boundary condition at } a_n/b_n, \ \textit{if } \tau \textit{ is lp at } a/b, \qquad [v_{a/b}, f]_{a_n/b_n} = 0 \ \textit{if } \tau \textit{ is lc at } a/b \Big\},$$

where $v_{a/b}$ *are the functions defining the above boundary conditions of* A *at* a/b. *Then* $A_n \xrightarrow{src} A$. *(If* τ *is lc at* a and b, *then* $(A_n - z)^{-1}P_n \to (A - z)^{-1}$ *with respect to Hilbert–Schmidt norm; the same holds if* $b_n = b$ *and* τ *is lc at* a, *or* $a_n = a$ *and* τ *is lc at* b. *This implies nice spectral convergence results, but it is, of course, not very interesting for applications.)*

The PROOF follows easily from the above mentioned criterion (Theorem 1) by noticing that

$$D_0 = \Big\{ f \in D(T) : f \equiv 0 \text{ near } a/b, \ \text{if } \tau \text{ is lp at } a/b, \qquad f \equiv c_{a/b}\, v_{a/b} \text{ near } a/b, \ \text{if } \tau \text{ is lc at } a/b \Big\}$$

is a core of A.

3. Applications to the absolutely continuous spectrum. In this form the result had been used long time ago (see e.g. J. WALTER [8], J. WEIDMANN [9], [10], [12]) in order to prove absolute continuity of the spectrum of certain classes of Sturm–Liouville operators in an interval I by showing that for $f \in L^2(a,b;r)$ with compact support in (a,b)

$$\|(E(\lambda'') - E(\lambda'))f\|^2 \le C_f|\lambda'' - \lambda'| \quad \text{for } \lambda', \lambda'' \in I.$$

The PROOFS of these results used

(i) the eigenfunction expansion of the finite dimensional projections $E_n(\lambda'') - E_n(\lambda')$,

(ii) estimates for the eigenfunctions of A_n corresponding to eigenvalues in $(\lambda', \lambda'']$ (uniformely in n and for λ in compact subintervals of I), and

(iii) estimates for the number of eigenvalues of A_n in $(\lambda', \lambda'']$, which follow from oscillation theory.

Recently the author proved a result, which extends the just mentioned ones, by a somewhat different, but similar method (cf. J. WEIDMANN [15], here only (i) and a more general version of (ii) are used, (iii) is not needed any more):

THEOREM 4. *Assume that* τ *is lp at* b, *and that for some* $c \in (a,b)$ *and* $I \subset \mathbf{R}$ *there exists a* $\vartheta > 0$ *and a function* $k : (c,b) \to \mathbf{R}$ *such that for all solutions* u *of* $(\tau - \lambda)u = 0$ *with*

$$|u(c)|^2 + |pu'(c)|^2 = 1 \quad \textit{and} \quad \lambda \in I$$

we have

$$\vartheta k(d) \le \int_c^d |u(x)|^2 r(x)\,dx \le k(d) \quad \textit{for} \quad d > c$$

(in short: for all $\lambda \in I$ *all solutions* are of the same size *near* b*). Then the spectrum of every self-adjoint realization of* τ *is purely absolutely continuous in* I*; actually the spectral measure is equivalent to the Lebesgue measure.*

PROOF. The assumption that all solutions are of the same size at b implies that either non or all solutions of $(\tau - \lambda)u = 0$ for $\lambda \in I$ lie in $L^2(a,b;r)$. Since τ is lp at b, the second alternative holds, which implies that every self-adjoint realization has purely continuous spectrum in I (cf. [14], Theorem 11.5).

The essential part of the proof is, to prove the statement for the case where τ is regular at a: If A_α is defined by means of the boundary condition

$$u(a)\cos\alpha + pu'(a)\sin\alpha = 0, \tag{α}$$

and $u_\alpha(\lambda,\cdot)$ is the solution of $(\tau-\lambda)u=0$ satisfying the normalized initial conditions

$$u_\alpha(\lambda,a) = \sin\alpha, \qquad pu'_\alpha(\lambda,a) = -\cos\alpha$$

(i. e. u_α satisfies the above boundary condition at a) then A_α has a spectral representation

$$U_\alpha : L_2(a,b;r) \longrightarrow L_2(\mathbf{R},\mu_\alpha), \quad U_\alpha f(\lambda) = \operatorname*{l.i.m.}_{d\to b} \int_a^d u_\alpha(\lambda,x)f(x)r(x)\mathrm{d}x$$

with a uniquely determined Borel measure μ_α on $\mathbf{R}$.

We approximate A_α in the above sense by operators $A_{\alpha,d}$ on (a,d) with $d \nearrow b$. These have similar spectral representations with pure point measures μ_α concentrated on the eigenvalues of $A_{\alpha,d}$ (since the $A_{\alpha,d}$ have discrete spectra) with $\mu_{\alpha,d}(J) \to \mu_\alpha(J)$ for $d\to b$ and every interval $J\subset I$ (notice that — as mentioned above — A_α has no eigenvalues in I). From the eigenfunction expansion of the spectral resolutions $E_{\alpha,d}$ it follows that for every eigenvalue λ of $A_{\alpha,d}$ we have

$$\mu_{\alpha,d}(\{\lambda\}) = \|u_\alpha(\lambda,\cdot)\|_{a,d}^{-2},$$

and therefore for every $\alpha\in[0,\pi)$

$$\frac{N(J,d)-1}{k(d)} \le \mu_{\alpha,d}(J) \le \frac{N(J,d)+1}{\vartheta k(d)}$$

where $N(J,d) \sim \#\Big\{\text{eigenvalues of } A_{\alpha,d} \text{ in } J\Big\}$. This implies for $d\to b$ (where we use that from $J\subset\sigma(A_\alpha)$ it follows that $N(J,d)\to\infty$ for $d\to b$)

$$K(J) \le \mu_\alpha(J) \le \frac{K(J)}{\vartheta} \quad \text{for all } \alpha\in[0,\pi),\ J\subset I.$$

Hence all the measures $\mu_\alpha(\cdot)$ are uniformely equivalent on I.

Together with a result due to S. KOTANI [3], which tells that $\int_0^\pi \mu_\alpha(\cdot)\mathrm{d}\alpha$ is equivalent to the Lebesgue measure, this implies that every μ_α is equivalent to the Lebesgue measure.

For the general case (τ singular at a) an additional limit procedure is needed (cf. [15]). □

4. Approximation of the discrete spectrum. In order to prove convergence results for isolated eigenvalues we cannot — as above — work with arbitrary boundary conditions for A_n, as is demonstrated by the following

EXAMPLE 3. Let $\tau f = -f'' + qf$ on $\mathbf{R}$ with $q(x) \to 0$ for $|x| \to \infty$. There is a unique self-adjoint realization A with $\sigma_e(A) = [0, \infty)$ and possibly infinitely many negative eigenvalues accumulating at most at 0. Choose any real solution u of $(\tau - \mu)u = 0$ for some $\mu < 0$, and define A_n in $L^2(-n, n)$ by

$$D(A_n) = \left\{ f \in D(T_n) : [u, f]_{-n} = [u, f]_n = 0 \right\}.$$

Then for every n μ is an eigenvalue of A_n although in general μ is not an eigenvalue (and even not in the spectrum) of A. #

On the other hand, if we choose *suitable boundary* conditions for A_n, then the isolated eigenvalues and the corresponding eigenfunctions converge:

EXAMPLE 4. For the above τ and A define A_n by means of the Dirichlet boundary condition at $-n$ and n,

$$D(A_n) = \left\{ f \in D(T_n) : f(-n) = f(n) = 0 \right\}.$$

Then $A_n \geq A_{n+1} \geq A$ (in the sense of quadratic forms) and $A_n \stackrel{src}{\longrightarrow} A$ (cf. [6]). This implies

$$\begin{aligned} E_n(\lambda)P_n &\stackrel{s}{\longrightarrow} E(\lambda) \text{ if } \lambda \text{ is not eigenvalue of } A, \text{ and} \\ \dim E_n(\lambda) &= \dim E_n(\lambda)P_n \leq \dim E(\lambda) \quad \text{for all } n \text{ and } \lambda. \end{aligned}$$

Using the simple lemma (cf. T. KATO [2], Lemma VIII.1.24) "Q_n, Q orthogonal projections, $Q_n \stackrel{s}{\longrightarrow} Q$, $\dim Q_n \leq \dim Q < \infty \Longrightarrow \dim Q_n = \dim Q$ for large n and $\|Q_n - Q\| \to 0$" this implies $\|E_n(\lambda)P_n - E(\lambda)\| \to 0$ for $n \to \infty$ if $\lambda < 0$ is not an eigenvalue of A; therefore the negative eigenvalues of A are exactly the limits of the negative eigenvalues of A_n, and the corresponding eigenprojections converge in norm. #

The following example shows that the choice of boundary conditions in the above example is not suitable in all cases.

EXAMPLE 5. Let A be the unique self-adjoint operator generated by

$$\tau f = -f'' + q_p f + qf \qquad \text{on } \mathbf{R}$$

with q_p periodic and $q(x) \longrightarrow 0$ for $|x| \longrightarrow \infty$. Then the spectrum of A consists of bands with (probably) isolated eigenvalues in the gaps between the bands. If the A_n are defined as in the preceeding example, then all eigenvalues $\lambda_k^{(n)}$ are travelling downwards to fill up the bottom band of the spectrum of A; convergence is impossible (cf. the figure in G. STOLZ and J. WEIDMANN [6]). #

QUESTIONS: In Example 4 monotonicity was an important argument; on the other hand this monotonicity makes convergence impossible in Example 5. Which other boundary conditions are possible in Example 4 in order to approximate the discrete eigenvalues below 0? Can one, in general, approximate the eigenvalues below the essential spectrum? How is it possible to approximate eigenvalues in gaps of the essential spectrum (as in the problem of Example 5)? — The answers to these questions might allow to apply the corresponding results to Dirac systems; in fact they do:

THEOREM 5. (G. STOLZ and J. WEIDMANN [6]) *Let τ be lp at a and b, A the unique self-adjoint realization, $I = [\alpha, \beta]$ a compact interval contained in a gap of the essential spectrum of A, $\lambda_{a/b} \in I$ and $u_{a/b}$ solutions of $(\tau - \lambda_{a/b})u = 0$ with $u_{a/b}$ left/right in $L^2(a, b; r)$, $[a_n, b_n] \subset (a, b)$,*

$$D(A_n) = \Big\{ f \in D(T_n) : [u_a, f]_{a_n} = [u_b, f]_{b_n} = 0 \Big\}.$$

Then the eigenvalues of A in I are exactly the limits of eigenvalues of A_n in I. The eigenfunctions converge in norm.

PROOF. Without restriction we may assume that the boundary points of I are not eigenvalues of A, hence $E_n(I)P_n \xrightarrow{s} E(I)$. Below we prove $\dim E_n(I) \leq \dim E(I)$. This implies $\|E_n(t)P_n - E(t)\| \to 0$ and $\dim E_n(I) = \dim E(I) < \infty$ for large n, which implies the statement easily.

Let $\lambda_1, \dots, \lambda_k$ $(k = k(n))$ be the eigenvalues of A in I, $\varphi_1, \dots, \varphi_k$ the corresponding orthonormal eigenfunctions (this is an ONB of $R(E_n(I))$). The functions φ_j on (a_n, b_n) can be extended to (a, b) by

$$\psi_j(x) = \begin{cases} c_{a,j} u_a(x) & \text{for } a < x \leq a_n, \\ \varphi_j(x) & \text{for } a_n \leq x \leq b_n, \\ c_{b,j} u_b(x) & \text{for } b_n \leq x < b \end{cases}$$

in such a way that $\psi_j \in D(A)$. Then the span M of $\{\psi_1, \dots, \psi_k\}$ is k-dimensional. Every $\psi \in M$ is of the form

$$\psi(x) = \begin{cases} c_a u_a(x) & \text{for } a < x \leq a_n, \\ \sum_{j=1}^k c_j \varphi_j(x) & \text{for } a_n \leq x \leq b_n, \\ c_b u_b(x) & \text{for } b_n \leq x < b. \end{cases}$$

and therefore, by an easy calculation,

$$\Big\| \Big(A - \frac{\alpha + \beta}{2}\Big)\psi \Big\| \leq \frac{\beta - \alpha}{2} \|\psi\| \quad \text{for} \quad \psi \in M.$$

This implies the desired inequality $\dim E(I) \geq k = \dim E_n(I)$. □

REMARKS. (i) The case of lc at both end points is not interesting, since in this case we have (for every suitable approximation in the sense of strong resolvent convergence) automatically norm resolvent convergence (cf. G. STOLZ and J. WEIDMANN [6]).

(ii) The case of (singular) lc at one end point and lp at the other end point is proved in two steps (cf. again [6]), the first step uses norm resolvent convergence and the second one is similar to the above proof.

(iii) It is of course in general a problem to find (numerically) the L^2–solution of $(\tau - \lambda_{a/b})u = 0$. If $q = q_0 + q_1$ with $q_1(x)/r(x) \to 0$ for $x \to a/b$, then the solution $u_{a/b}$ in Theorem 5 can be replaced by corresponding solutions of $(\tau_0 - \lambda)u = 0$, where $\tau_0 = \tau - q_1/r$ (cf. [6], Corollary 3).

(iv) The situation is much easier in the case of one regular end point (say a) and one limit point end point (say b). In this case for the approximating operators A_n we may use intervals (a, b_n) with $b_n \nearrow b$. The natural choice of the boundary condition at b_n (corresponding to Theorem 5) would be $[u, f]_{b_n} = 0$ with an L^2–solution u of $(\tau - \lambda)u = 0$, λ in the gap. But, as mentioned in (iii), the difficulty is to find (numerically) this L^2–solution. If we simply choose u to be the solution of $(\tau - \lambda)u = 0$ satisfying the boundary condition at a, then there are two possibilities:

(α) If u is in L^2 (by accident), then by Theorem 5 the eigenvalues of A_n converge to the true eigenvalues of A (in the gap).

(β) If u is not in L^2, then λ is an eigenvalue of every A_n although λ is not an eigenvalue of A. By a simple dimensional argument this is the only additional limit occuring. Therefore after applying this method for two different λ from the gap, we are able to tell which are the true eigenvalues of A.

(v) In the special case of $\tau f = -f'' + qf$ on $[0, \infty)$ with $q(x) \to 0$ for $x \to \infty$ the above question "Which other boundary conditions are possible in Example 4 in order to approximate the discrete eigenvalues below 0?" can now be answered by using (iii): For the boundary condition at b_n every L^2–solution of $(\tau_0 - \lambda)u = -u'' - \lambda u = 0$ with $\lambda < 0$, $u(x) = \exp(-\sqrt{-\lambda}x)$, may be used, i. e. the boundary condition $\mu f(b_n) + f'(b_n) = 0$ for every $\mu = \sqrt{-\lambda} > 0$. The Dirichlet boundary condition of Example 4 is not included, but it is the limit case for $\lambda \to -\infty$.

5. Concluding remarks. The same proofs with only obvious changes hold for Dirac systems and for discrete operators (Jacobi matrices). A full generalization of the results concerning the approximation of isolated eigenvalues to arbitrary singular ordinary differential operators of any order and any dimension is given in G. STOLZ and J. WEIDMANN [7].

In a similar way it is possible to study partial differential operators (especially $-\Delta + q$) on varying domains Ω in $\mathbf{R}^m$ in order to prove that the spectrum, the eigenvalues and the eigenfunctions depend continuously on Ω (cf. P. STOLLMANN [5] and J. WEIDMANN [15]).

References

1. Bailey, P., Everitt, N., Weidmann, J., Zettl, A., *Regular Approximations of Singular Sturm–Liouville Problems*, Res. Math. **23** (1993), 3–22.
2. Kato, T., *Perturbation Theory for Linear Operators*, Grundlehren, Vol. 132, Springer-Verlag, Berlin/Heidelberg/New York, 1976.
3. Kotani, S., *Lyapunov exponents and spectra for one-dimensional random Schrödinger operators*, Contemp. Math. **50** (1986), 277–286.
4. Rellich, F., *Störungstheorie der Spektralzerlegung I–V*, Math. Ann. **113, 113, 116, 117, 118** (1937, 1937, 1939, 1940, 1942), 600–619, 677–685, 555–570, 356–382, 462–484.
5. Stollmann, P., *Störungstheorie von Dirichletformen mit Anwendungen auf Schrödingeroperatoren*, Habilitationsschrift, Frankfurt, 1994.
6. Stolz, G., Weidmann, J., *Approximation of Isolated Eigenvalues of Ordinary Differential Operators*, J. Reine Angew. Math. **445** (1993), 31–44.
7. Stolz, G., Weidmann, J., *Approximation of Isolated Eigenvalues of General Singular Ordinary Differential Operators*, To appear in Res. Math..
8. Walter, J., *Absolute Continuity of the Essential Spectrum of* $-d^2/dt^2 + q(x)$ *without Monotony of* q, Math. Z. **129** (1972), 83–94.
9. Weidmann, J., *Zur Spektraltheorie von Sturm–Liouville–Operatoren.*, Math. Z. **98** (1967), 268–302.
10. Weidmann, J., *Oszillationsmethoden für Systeme gewöhnlicher Differentialgleichungen*, Math. Z. **119** (1971), 349–373.
11. Weidmann, J., *Continuity of the eigenvalues of self-adjoint operators with respect to the strong operator topology*, Integral Equations Operator Theory **3** (1980), 138-142.
12. Weidmann, J., *Absolut stetiges Spectrum bei Sturm–Liouville–Operatoren und Dirac–Systemen*, Math. Z. **180** (1982), 423-427.
13. Weidmann, J., *Stetige Abhängigkeit der Eigenwerte und Eigenfunktionen elliptischer Differentialoperatoren vom Gebiet.*, Math. Scand. **54** (1984), 51-69.
14. Weidmann, J., *Spectral Theory of Ordinary Differential Operators*, Lecture Notes in Mathematics, Vol. 1258, Springer-Verlag, Berlin/Heidelberg/New York/London/Paris/Tokyo, 1987.
15. Weidmann, J., *Uniform Nonsubordinacy and the Absolutely Continuous Spectrum of Ordinacy Differential Operators*, Analysis **16** (1996), 89–99.

Received July 2, 1996

Fachbereich Mathematik
Johann Wolfgang Goethe–Universität
D–60054 Frankfurt am Main
e-mail: weidmann@math.uni-frankfurt.de