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ON TOEPLITZ OPERATORS IN SEGAL–BARGMANN SPACES

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Abstract. This paper gathers most of known results concerning four topics related to (unbounded) Toeplitz operators T_φ in Segal–Bargmann spaces:

- the explicit description of the adjoint of T_φ,
- when do polynomials or exponents form a core for T_φ?
- closedness of T_φ,
- (essential) selfadjointness of T_φ.

A few new examples and facts on Toeplitz operators are also added.

The goal of this article is to give a survey of the theory of unbounded Toeplitz operators in the Segal–Bargmann space $\mathcal{B}_n$. The study of these operators has been initiated by Shapiro and Newman [20, 21] and soon afterwards by Berezin [3, 4]. Toeplitz operators in $\mathcal{B}_n$ are unitarily equivalent to some pseudodifferential ones in $L^2(\mathbf{R}^n)$ (cf. [1],[11]). Among them there are creation and annihilation operators, which are of great importance in Quantum Mechanics (cf. [1], [22], [23]). Bounded Toeplitz operators on $\mathcal{B}_n$ have been intensively investigated by Berger and Coburn (cf. [5, 6, 7]).

To begin with we introduce basic definitions and notations. An A is called a *linear operator* in a complex Hilbert space $\mathcal{H}$ if its domain $\mathcal{D}(A)$ is a linear subspace of $\mathcal{H}$ and $A : \mathcal{D}(A) \to \mathcal{H}$ is a linear mapping. As usual, A^* and $\overline{A}$ stand for the adjoint and the closure of A. We say that a linear subspace $\mathcal{D}$ of $\mathcal{D}(A)$ is a *core* for a closable operator A in $\mathcal{H}$ if $\overline{A} = (A|_{\mathcal{D}})^-$.

Consider $L^2(\mu)$, the Hilbert space of all complex Borel functions which are square integrable on $\mathbf{C}^n$ with respect to the measure μ given by the formula $\mathrm{d}\mu(z) = \pi^{-n}e^{-\|z\|^2}dV(z)$, where V is the Lebesgue measure in $\mathbf{C}^n$. The *Segal–Bargmann* space $\mathcal{B}_n$ (also called the Fock or the Fischer space, cf. [1], [10], [22], [23]) is the closed subspace of $L^2(\mu)$ composed of all analytic functions from $L^2(\mu)$. Denote by $\mathcal{E}_n$ the linear span of the set of functions

$e_a(z) := e^{\langle z,a\rangle}$, $a, z \in \mathbf{C}^n$, where $\langle \cdot, - \rangle$ is the canonical inner product in $\mathbf{C}^n$. It can be checked that $f(z) = (f, e_z)$ for $f \in \mathcal{B}_n$ and $z \in \mathbf{C}^n$, so $\mathcal{B}_n$ is a Reproducing Kernel Hilbert Space. We denote by $\mathcal{P}_n$ the space of all analytic polynomials in $\mathbf{C}^n$. Both $\mathcal{P}_n$ and $\mathcal{E}_n$ are dense subsets of $\mathcal{B}_n$. Let $\mathcal{A}_n$ be the algebra generated by $\mathcal{P}_n \cup \mathcal{E}_n$. The sequence $f_k(z) := z^k/\sqrt{k!}$, $k \in \mathbf{N}^n$, $z \in \mathbf{C}^n$, forms an orthonormal basis of $\mathcal{B}_n$, where according to the standard multiindex notation $z^k := z_1^{k_1} \cdot \ldots \cdot z_n^{k_n}$, $k! := k_1! \cdot \ldots \cdot k_n!$ and $|k| := k_1 + \ldots + k_n$ for $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$ and $k = (k_1, \ldots, k_n) \in \mathbf{N}^n$ ($\mathbf{N} = \{0, 1, 2, \ldots\}$). For a given $p = \sum_{|k| \leq N} a_k f_k \in \mathcal{P}_n$ we set $p^* := \sum_{|k| \leq N} \overline{a}_k f_k \in \mathcal{P}_n$ and $p(D) := \sum_{|k| \leq N} \frac{a_k}{\sqrt{k!}} \frac{\partial^{|k|}}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}$. The translation operator E_a, $a \in \mathbf{C}^n$, acts on functions $f : \mathbf{C}^n \to \mathbf{C}$ via $(E_a f)(z) = f(z + a)$ for $z \in \mathbf{C}^n$.

Let $\varphi : \mathbf{C}^n \to \mathbf{C}$ be a Borel function. The *Toeplitz operator* T_φ in $\mathcal{B}_n$ with the symbol φ is defined by $\mathcal{D}(T_\varphi) = \{f \in \mathcal{B}_n : \varphi \cdot f \in L^2(\mu)\}$ and $T_\varphi f = P(\varphi \cdot f)$ for $f \in \mathcal{D}(T_\varphi)$, where P stands for the orthogonal projection of $L^2(\mu)$ onto $\mathcal{B}_n$. If T_φ is densely defined, then T_φ is closable and $T_{\overline{\varphi}} \subseteq T_\varphi^*$.

Recall definitions of related Toeplitz operators Π_φ and $\widetilde{T}_\varphi$. The domain of the operator Π_φ consists of all $f \in \mathcal{B}_n$ for which there exists $h \in \mathcal{B}_n$ such that $\int_{\mathbf{C}^n} (\varphi f - h)\overline{g}\, \mathrm{d}\mu = 0$ for all $g \in \mathcal{E}_n$; then we put $\Pi_\varphi f = h$. Taking $\mathcal{P}_n$ instead of $\mathcal{E}_n$ in the preceding definition we obtain the definition of $\widetilde{T}_\varphi$. It was shown in [13, Prop. 1.1 and Prop. 1.2] that $T_\varphi \subseteq \Pi_\varphi \subseteq \widetilde{T}_\varphi$ for any Borel φ and $T_\varphi = \Pi_\varphi = \widetilde{T}_\varphi$ for any entire φ. The equality $(T_\varphi|_{\mathcal{E}_n})^* = \Pi_{\overline{\varphi}}$ (resp. $(T_\varphi|_{\mathcal{P}_n})^* = \widetilde{T}_{\overline{\varphi}}$), which is true if $\mathcal{E}_n \subseteq \mathcal{D}(T_\varphi)$ (resp. $\mathcal{P}_n \subseteq \mathcal{D}(T_\varphi)$), gives us a chance to describe adjoints of Toeplitz operators (cf. [13, Prop. 1.4]). These two extensions of T_φ follow the general pattern. Let $\mathcal{V}$ be a dense linear subspace of $\mathcal{B}_n$. Define $T_{\mathcal{V},\varphi}$ to be the operator in $\mathcal{B}_n$ with the domain

$$\mathcal{D}(T_{\mathcal{V},\varphi}) := \{f \in \mathcal{B}_n : \exists h \in \mathcal{B}_n \, \forall v \in \mathcal{V} \int_{\mathbf{C}^n} (\varphi f - h)\overline{v}\, \mathrm{d}\mu = 0\};$$

since the function h is unique (by the density of $\mathcal{V}$), we set $T_{\mathcal{V},\varphi} f := h$ for $f \in \mathcal{D}(T_{\mathcal{V},\varphi})$. One can easily check that $T_\varphi \subseteq T_{\mathcal{V},\varphi}$. It turns out that $(T_\varphi|_{\mathcal{V}})^* = T_{\mathcal{V},\overline{\varphi}}$ whenever $\mathcal{V} \subseteq \mathcal{D}(T_\varphi)$. The family $\{T_{\mathcal{V},\varphi} : \mathcal{V}$ is a dense linear subspace of $\mathcal{B}_n\}$ is decreasing with respect to inclusion, but it may happen that the smallest member of this family is strictly greater than T_φ. This is the case for $\varphi(z) = (z\sqrt{1 - |z|})^{-1}\chi_\Delta(z)$, $z \in \mathbf{C}$, where χ_Δ is the characteristic function of the unit disc. Indeed, one can check that $1 \in \mathcal{D}(T_{\mathcal{B}_1,\varphi}) \setminus \mathcal{D}(T_\varphi)$ and $T_{\mathcal{B}_1,\varphi} 1 = 0$ (equivalently: $\varphi \notin L^2(\mu)$, $\varphi \cdot \overline{g} \in L^1(\mu)$ and $\int_{\mathbf{C}} \varphi \cdot \overline{g}\, \mathrm{d}\mu = 0$ for every $g \in \mathcal{B}_1$).

It is worth pointing out that the operators Π_φ and $\widetilde{T}_\varphi$ coincide provided $\mathcal{E}_n \subseteq \mathcal{D}(T_\varphi)$ (cf. [17, Th. 1.3(i)]). This is no longer true under weaker assumption $\mathcal{P}_n \subseteq \mathcal{D}(T_\varphi)$ (cf. [9, Th. 2.2]). If $\mathcal{E}_n \subseteq \mathcal{D}(T_\varphi)$, then also $\mathcal{A}_n \subseteq \mathcal{D}(T_\varphi)$

and $(T_\varphi|_{\mathcal{P}_n})^- = (T_\varphi|_{\mathcal{E}_n})^- = (T_\varphi|_{\mathcal{A}_n})^-$ (cf. [17, Th. 1.3(i)]). Another interesting feature concerning Π_φ is its integral description: $\mathcal{D}(\Pi_\varphi) = \{f \in \mathcal{B}_n :$ the function $z \mapsto \int_{\mathbf{C}^n} \varphi f \overline{e_z}\, d\mu$ belongs to $\mathcal{B}_n\}$ and $(\Pi_\varphi f)(z) = \int_{\mathbf{C}^n} \varphi f \overline{e_z}\, d\mu$ for $f \in \mathcal{D}(\Pi_\varphi)$ (see [13, Remark, p. 89]).

In Table 1 we have gathered most of known results concerning three topics: the explicit form of T_φ^*, cores for T_φ and the closedness of T_φ. *Therein φ is assumed to be a Borel function on $\mathbf{C}^n$ such that $\mathcal{P}_n \subseteq \mathcal{D}(T_\varphi)$.* Further definitions and explanations related to Table 1 are given below. The entries of Table 1 marked with '$*$' contain a result of Borichev [8], which says that there is an entire function φ such that $\mathcal{P}_n \subseteq \mathcal{D}(T_\varphi)$ and $\mathcal{P}_n$ is not a core for T_φ (equivalently: $T_\varphi^* \neq \widetilde{T_{\overline{\varphi}}}$). Recently H. S. Shapiro has informed the first–named author about another example of that type (see Example 4).

Open question. Is it true that $T_\varphi^* = \Pi_{\overline{\varphi}}$ (or equivalently: $\mathcal{E}_n$ is a core for T_φ) for each entire function φ such that $\mathcal{E}_n \subseteq \mathcal{D}(T_\varphi)$?

Explanations and definitions to Table 1:

- Some of the facts placed in the table are not written explicitly in the literature. They can be deduced from appropriate references mentioned in the "SEE" column via discussion preceding the table. In particular, the following two facts are in frequent use:

(*) *If $\mathcal{P}_n \subseteq \mathcal{D}(T_\varphi)$ (resp. $\mathcal{E}_n \subseteq \mathcal{D}(T_\varphi)$), then $\mathcal{P}_n$ (resp. $\mathcal{E}_n$) is a core for T_φ if and only if $T_\varphi^* = \widetilde{T_{\overline{\varphi}}}$ (resp. $T_\varphi^* = \Pi_{\overline{\varphi}}$).*

(**) *If $\mathcal{E}_n \subseteq \mathcal{D}(T_\varphi)$, then $\mathcal{P}_n$ is a core for T_φ if and only if so is $\mathcal{E}_n$ (or equivalently $\mathcal{A}_n$).*

- 'T'='true'; 'F'='false (read: not always true)' — the number of the suitable counterexample is given in brackets.
- $\mathrm{Hol}(\mathbf{C}^n)$ is the set of all holomorphic functions on $\mathbf{C}^n$.
- If $\mathcal{X}$ and $\mathcal{Y}$ are sets of functions on $\mathbf{C}^n$, then we write $\mathcal{X}^* := \{\overline{\varphi} : \varphi \in \mathcal{X}\}$ and $\mathcal{X} \cdot \mathcal{Y} := \{\varphi \cdot \psi : \varphi \in \mathcal{X}, \psi \in \mathcal{Y}\}$.
- $\sum_{s=1}^k p_s(D)E_{a_s}$ is understood as a linear operator in $\mathcal{B}_n$ with the domain $\{f \in \mathcal{B}_n : \sum_{s=1}^k p_s(D)E_{a_s} f \in \mathcal{B}_n\}$ ($p_s \in \mathcal{P}_n$ and $a_s \in \mathbf{C}^n$).
- *φ satisfies condition* (JC1) if φ is entire, $\mathcal{E}_n \subseteq \mathcal{D}(T_\varphi)$ and for any $h \in \mathcal{D}(T_\varphi)$ there is $\varepsilon > 0$ such that $\sum_k \frac{1}{k!} \|(D^k\varphi)h\|^2(1+\varepsilon)^{|k|} < \infty$.
- *φ satisfies condition* (JC2) if φ is entire, $\mathcal{E}_n \subseteq \mathcal{D}(T_\varphi)$, $\mathcal{D}(T_\varphi) \subseteq \mathcal{D}(T_{D^s\varphi})$ and $\sum_k \frac{1}{k!} |(D^k\varphi)(w)|\, \|E_w D^k \varphi\| < \infty$ for $s \in \mathbf{N}^n$, $w \in \mathbf{C}^n$.
- *φ satisfies condition* (JC3) if there exists a constant $M > 0$ such that $|\varphi(z) - \varphi(w)| \leq M(1 + \|z - w\|)$ for all $z, w \in \mathbf{C}^n$. As is shown in Appendix, (JC3) is equivalent to the assumption of [14, Th. 3.3].
- φ is *t-radially symmetric* ($t \in \mathbf{R}_+^n$) if $\varphi(e^{i\vartheta t_1} z_1, \ldots, e^{i\vartheta t_n} z_n) = \varphi(z)$ for all $\vartheta \in \mathbf{R}$ and for almost every $z \in \mathbf{C}^n$ w.r.t. V ($\mathbf{R}_+ = \{t \in \mathbf{R} : t > 0\}$).

- φ is *subradial* if there are $c, d, R > 0$ such that $|\varphi(e^{i\vartheta}z)| \leq c + d\,|\varphi(z)|$ for all $\vartheta \in \mathbf{R}$ and $\|z\| \geq R$.

TABLE 1.

IF	THEN			SEE
	$T_\varphi^* =$	the following are cores for T_φ	T_φ is closed	
$\varphi \in \mathrm{Hol}(\mathbb{C}^n)$,	?	$*$	T	[8], [13, Sec. 1]
$\varphi \in \mathrm{Hol}(\mathbb{C}^n)^*$,	$T_{\overline{\varphi}}$, $\Pi_{\overline{\varphi}}$, $\widetilde{T}_{\overline{\varphi}}$	$\mathcal{P}_n$	F (Ex. 2)	[17, Th. 1.4(i)]
$\varphi \in \mathcal{A}_n \cdot \mathcal{A}_n^*$,	$\Pi_{\overline{\varphi}}$, $\widetilde{T}_{\overline{\varphi}}$	$\mathcal{P}_n, \mathcal{E}_n, \mathcal{A}_n$	F (Ex. 2)	[17, Th. 1.4(ii)]
$\varphi \in \mathcal{A}_n$, $\varphi = \sum_{s=1}^k p_s e_{a_s}$,	$\sum_{s=1}^k p_s^*(D)E_{a_s}$	$\mathcal{P}_n, \mathcal{E}_n, \mathcal{A}_n$	T	[17, Th. 1.4]
$\varphi \in$ (JC1),	$\Pi_{\overline{\varphi}}$	$\mathcal{P}_n, \mathcal{E}_n, \mathcal{A}_n$	T	[13, Th. 2.2]
$\varphi \in$ (JC2),	$\Pi_{\overline{\varphi}}$	$\mathcal{P}_n, \mathcal{E}_n, \mathcal{A}_n$	T	[14, Th. 3.1]
$\varphi \in$ (JC3),	$T_{\overline{\varphi}}$	?	T	[14, Th. 3.3]
φ is t-radially symmetric,	$\overline{T}_{\overline{\varphi}}$, $\overline{\Pi}_{\overline{\varphi}}$, $\widetilde{T}_{\overline{\varphi}}$	$\mathcal{P}_n$	F (Ex. 1)	[17, Th. 4.1]
φ is subradial,	$\widetilde{T}_{\overline{\varphi}}$	$\mathcal{P}_n$	F (Ex. 1)	[17, Th. 1.3(iii)]
THE ONE VARIABLE CASE				
$\varphi \in \mathcal{P}_1$,	$T_{\overline{\varphi}}$, $\Pi_{\overline{\varphi}}$, $\widetilde{T}_{\overline{\varphi}}$	$\mathcal{P}_1, \mathcal{E}_1, \mathcal{A}_1$	T	[17, Prop. 3.1]
$\varphi \in \mathcal{P}_1^* \cdot \mathrm{Hol}(\mathbb{C})$,	$\overline{T}_{\overline{\varphi}}$	$*$	T	[17, Th. 3.2], [8]
$\varphi \in \mathcal{P}_1 \cdot \mathrm{Hol}(\mathbb{C})^*$,	$T_{\overline{\varphi}}$, $\Pi_{\overline{\varphi}}$, $\widetilde{T}_{\overline{\varphi}}$	$\mathcal{P}_1$	F (Ex. 4)	[17, Th. 3.2]
$\varphi \in \mathcal{P}_1 \cdot \mathcal{P}_1^*$,	$T_{\overline{\varphi}}$, $\Pi_{\overline{\varphi}}$, $\widetilde{T}_{\overline{\varphi}}$	$\mathcal{P}_1, \mathcal{E}_1, \mathcal{A}_1$	T	[17, Th. 3.2]
$\varphi \in \mathcal{P}_1 + \mathcal{P}_1^*$,	$\Pi_{\overline{\varphi}}$, $\widetilde{T}_{\overline{\varphi}}$	$\mathcal{P}_1, \mathcal{E}_1, \mathcal{A}_1$	F (Ex. 3)	[19]
$\varphi \in \mathcal{P}_1 + \mathcal{P}_1^*$, $\varphi = p + \overline{q}$, $\deg p < \deg q$,	$\overline{T}_{\overline{\varphi}}$, $\Pi_{\overline{\varphi}}$, $\widetilde{T}_{\overline{\varphi}}$	$\mathcal{P}_1, \mathcal{E}_1, \mathcal{A}_1$	T	[17, Th. 3.3] [19]
$\varphi \in \mathcal{P}_1 + \mathcal{P}_1^*$, $\varphi = \alpha p + \overline{p}$, $\alpha \in \mathbb{C}$, $\lvert\alpha\rvert > 1$,	$\Pi_{\overline{\varphi}}$, $\widetilde{T}_{\overline{\varphi}}$	$\mathcal{P}_1, \mathcal{E}_1, \mathcal{A}_1$	T	[9, Th. 3.4]
$\varphi \in \mathcal{P}_1 + \mathcal{P}_1^*$, $\varphi = p + \overline{p}$,	$\Pi_{\overline{\varphi}}$, $\widetilde{T}_{\overline{\varphi}}$	$\mathcal{P}_1, \mathcal{E}_1, \mathcal{A}_1$	F (Ex. 3)	[19]

Examples to Table 1:

1) $\varphi(z) = e^{a|z|^2}$, $z \in \mathbf{C}$, $a \in (0, 1/2)$ (cf. [17, Ex. 6.1]).
2) $\varphi(z) = \overline{z_1 z_2}$, $z \in \mathbf{C}^n$, $n \geq 2$ (cf. [17, Ex. 6.3]).
3) $\varphi(z) = \operatorname{Re}(z^s)$, $z \in \mathbf{C}$, $s \in \mathbf{N}$, $s \geq 2$ (cf. [17, Ex. 6.2]).
4) $\overline{\varphi(z)} = \dfrac{e^{z^2/2} - 1}{\prod_{k=1}^{\infty}(1 - \frac{z}{z_k})}$, $z \in \mathbf{C}$, where $\{z_k\}_{k=1}^{\infty}$ is a sequence of nonzero complex numbers chosen among the zeros of $e^{z^2/2} - 1$ so that $|z_{k+1}| > 2|z_k|$ for every $k \geq 1$.

Example 4 needs an explanation. The entire function $\overline{\varphi}$ has been constructed by H. S. Shapiro in order to show that $\mathcal{P}_1$ is not a core for $T_{\overline{\varphi}}$ though $\mathcal{P}_1 \subseteq \mathcal{D}(T_{\overline{\varphi}})$ (in fact we can take any co–analytic function on $\mathbf{C}$ sharing this two properties, see [8] for other examples). Hence the fact that T_φ is not closed for such φ follows from

PROPOSITION 1. *If $\varphi \in \operatorname{Hol}(\mathbf{C}^n)^*$, $\mathcal{P}_n \subseteq \mathcal{D}(T_\varphi)$ and T_φ is closed, then $\mathcal{P}_n$ is a core for T_φ and $T_{\overline{\varphi}}$, $T_{\overline{\varphi}}^* = T_\varphi = \Pi_\varphi = \widetilde{T}_\varphi$ and $T_\varphi^* = T_{\overline{\varphi}} = \Pi_{\overline{\varphi}} = \widetilde{T}_{\overline{\varphi}}$.*

PROOF. Since T_φ and $T_{\overline{\varphi}}$ are closed and $\mathcal{P}_n$ is a core for T_φ (cf. [17, Th. 1.4(i)]), Proposition 2.2 in [17] implies that $\mathcal{P}_n$ is a core for $T_{\overline{\varphi}}$, i.e. $T_{\overline{\varphi}} = (T_{\overline{\varphi}}|_{\mathcal{P}_n})^-$. Taking adjoints and applying [13, Prop. 1.4] we get $T_{\overline{\varphi}}^* = \widetilde{T}_\varphi$. The equality $T_{\overline{\varphi}}^* = T_\varphi$ follows from [17, Th. 1.4(i)]. Hence $T_\varphi = \Pi_\varphi = \widetilde{T}_\varphi$, because always $T_\varphi \subseteq \Pi_\varphi \subseteq \widetilde{T}_\varphi$. The equalities $T_\varphi^* = T_{\overline{\varphi}} = \Pi_{\overline{\varphi}} = \widetilde{T}_{\overline{\varphi}}$ can be deduced from [17, Th. 1.4(i)]. □

As is mentioned in the Table 1, the operator $T_{p+\overline{q}}$ is closed provided $p, q \in \mathcal{P}_1$ and $\deg p < \deg q$. The following result shows that $T_{p+\overline{q}}$ may also be closed in case $\deg p > \deg q$. Note that $T_{p+\overline{q}} = T_p + T_{\overline{q}}$ whenever $p, q \in \mathcal{P}_1$ and $\deg p \neq \deg q$ (cf. [9, Prop. 3.5]).

PROPOSITION 2. *If $p, q \in \mathcal{P}_1$ and $\deg q \leq 1$, then $T_{p+\overline{q}}$ is closed.*

PROOF. We can assume without loss of generality that $q(z) = az$ for $z \in \mathbf{C}$. We show that

$$\|(p + \overline{q})f\|^2 \leq \max\{1, |a|^2\}\,(\|f\|^2 + \|T_{p+\overline{q}}f\|^2), \quad f \in \mathcal{P}_1. \tag{1}$$

Indeed, applying [21, Th. 3] to the product qf, we get

$$\begin{aligned}
\|(p + \overline{q})f\|^2 &= \|pf\|^2 + 2\operatorname{Re}(pf, \overline{q}f) + \|qf\|^2 \\
&= \|pf\|^2 + 2\operatorname{Re}(pf, q^*(D)f) + \|q^*(D)f\|^2 + \|q^{*\prime}(D)f\|^2 \\
&= \|pf + q^*(D)f\|^2 + |a|^2\|f\|^2 \\
&\leq \max\{1, |a|^2\}\,(\|f\|^2 + \|pf + q^*(D)f\|^2), \quad f \in \mathcal{P}_1,
\end{aligned}$$

which proves (1). Since $\mathcal{P}_1$ is a core for $T_{p+\overline{q}}$, Proposition 2.3 in [17] completes the proof. □

We now turn to symmetric Toeplitz operators. It is well–known that a densely defined Toeplitz operator T_φ is always symmetric provided φ is real–valued. Moreover, if $\varphi(z) = \varphi(\overline{z})$ a.e. $[V]$, then the deficiency indices of T_φ are equal (cf. [13, Prop. 3.1]). In Table 2 we present conditions implying the (essential) selfadjointness of T_φ. The eighth example in Table 2 is only a sample of which can be deduced from [17, Lemma 5.3].

Recall that a nonzero polynomial $p(z) = \sum_k c_k z^k$ is said to be *t-homogeneous* of t-degree d ($t \in \mathbf{R}^n_+$) if $\sum_{j=1}^n t_j k_j = d$ for every $k \in \mathbf{N}^n$ such that $c_k \neq 0$ (cf. [10]). If $t = (1, \ldots, 1)$, then t-homogeneous polynomials are homogeneous in the usual sense. It is easily seen that every homogeneous polynomial is subradial and that cz^0 is the only polynomial which is simultaneously t–homogeneous and t–radially symmetric.

TABLE 2.

IF	THEN T_φ IS	SEE
$\varphi \in$ (JC3) is real–valued,	selfadjoint	[14, Th. 3.3]
φ is t-radially symmetric, real–valued and $\mathcal{P}_n \subseteq \mathcal{D}(T_\varphi)$,	essentially selfadjoint	[17, Th. 4.1(ii)]
$\varphi = \operatorname{Re} \sum_{j=1}^k p_j \overline{q_j}$, where $p_j, q_j \in \mathcal{P}_n$ are t-homogeneous with t-deg $p_j = t$-deg q_j,	—||—	—||—
$\varphi = \sum_{j=1}^k \lvert p_j \rvert^{s_j}$, where $p_j \in \mathcal{P}_n$ are t–homogeneous and $s_j \in \mathbb{R}_+$,	—||—	—||—
$\varphi = \lvert p \rvert^2$, where $p \in \mathcal{P}_n$ is t-homogeneous,	—||—	—||—
$\varphi = \lvert p \rvert^2$, where $p \in \mathcal{P}_n$ and $n \geq 2$,	?	
$\varphi = \lvert p \rvert^2$, where $p \in \mathcal{P}_1$,	selfadjoint	[17, Th. 3.2(iii)]
$\varphi = \lvert p_1(\xi_1) \cdot \ldots \cdot p_s(\xi_s) \rvert^2$ or $\varphi = \lvert p_1(\xi_1) \rvert^2 + \ldots + \lvert p_s(\xi_s) \rvert^2$, where $p_j \in \mathcal{P}_1$, $\xi_j \in \mathcal{P}_n$ with $\deg \xi_j = 1$ and the vectors $\{(\frac{\partial}{\partial z_1}\xi_j, \ldots, \frac{\partial}{\partial z_n}\xi_j)\}_{j=1}^s$ are pairwise orthogonal in $\mathbb{C}^n$,	—||—	[17, Prop. 5.4]
$\varphi = \operatorname{Re} p$, where $p \in \mathcal{P}_1$ and $\deg p \leq 2$,	essentially selfadjoint	[13, Th. 3.4], [18, Cor. 5.4]
$\varphi(z, w) = \operatorname{Re}(z \cdot w)$ for $z, w \in \mathbb{C}$,	—||—	[17, Ex. 6.6]

The last but one example in Table 2 is optimum taking account the degree of the polynomial p, because $T_{\mathrm{Re}\,p}$ is not essentially selfadjoint if $p(z) = z^3$ for $z \in \mathbf{C}$ (cf. [13, Ex. 3.6]). In this particular case $\overline{T}_{\mathrm{Re}\,p} \subsetneq \widetilde{T}_{\mathrm{Re}\,p}$.

The theory of unbounded Toeplitz operators in $\mathcal{B}_n$ is still being developed in many aspects. In [13, 14] Janas has found some relations concerning spectra of T_φ, sufficient conditions for the compactness of resolvent of T_φ and sufficient conditions for $\overline{T}_\varphi$ to be a generator of a continuous semigroup of contractions. We also refer the reader to [24] and [14] for some applications of the theory of Toeplitz operators to Fischer pairs of polynomials.

Concluding the article, let us mention two (essentially different) attempts of developing the theory of bounded Toeplitz operators on Segal–Bargmann type spaces of entire functions in infinitely many variables (cf. [15, 16]). Both of them depend on the choice of a model space for the Canonical Commutation Relations (CCR). Particular model spaces for CCR have been constructed in [2], [23] and [15].

Appendix. The following lemma has been isolated so as to make it more convenient to further use. It offers two equivalent forms of condition (JC3). From now on, $\varphi : \mathbf{C}^n \to \mathbf{C}$ is an arbitrary function.

LEMMA. *The following conditions are equivalent*

(i) $\exists\, M > 0\ \forall z, w \in \mathbf{C}^n : |\varphi(z) - \varphi(w)| \leq M\, e^{\|z-w\|^2/4}$,
(ii) $\exists\, 0 < r < \infty : \sup\{|\varphi(z) - \varphi(w)| : z, w \in \mathbf{C}^n,\ \|z - w\| \leq r\} < \infty$,
(iii) $\exists\, M > 0\ \forall z, w \in \mathbf{C}^n : |\varphi(z) - \varphi(w)| \leq M\,(1 + \|z - w\|)$.

If (i) *holds, then there is* $C > 0$ *such that* $|\varphi(z)| \leq C(1 + \|z\|)$ *for* $z \in \mathbf{C}^n$.

PROOF. (ii)⇒(iii). Set $K := \sup\{|\varphi(z) - \varphi(w)| : z, w \in \mathbf{C}^n,\ \|z - w\| \leq r\}$. Pick $z, w \in \mathbf{C}^n$ such that $\|z - w\| > r$. Define m to be the smallest integer greater than or equal to $\|z - w\|\, r^{-1}$. Set $z_j = z + jr\|z - w\|^{-1}(w - z)$ for $j = 0, \ldots, m-1$ and $z_m = w$. One can show that $\|z_{j+1} - z_j\| \leq r$ for $j = 0, \ldots, m-1$. Since $K < \infty$, we have

$$\begin{aligned} |\varphi(z) - \varphi(w)| &\leq \sum_{j=0}^{m-1} |\varphi(z_{j+1}) - \varphi(z_j)| \\ &\leq Km \leq K(1 + \|z - w\| r^{-1}) \\ &\leq K \max\{1, r^{-1}\}(1 + \|z - w\|), \quad \|z - w\| > r. \end{aligned}$$

Consequently (iii) holds with $M := K \max\{1, r^{-1}\}$.

(iii)⇒(i). This is clear because

$$1 + x \leq \frac{3}{2} \max\{1, r^{-1}\} e^{rx^2}, \quad x \geq 0,\ r > 0. \tag{2}$$

Since (i)⇒(ii) is obvious, all conditions (i), (ii) and (iii) are equivalent. If (iii) holds, then $|\varphi(z)| \leq |\varphi(0)| + |\varphi(z) - \varphi(0)| \leq |\varphi(0)| + M(1 + \|z\|) \leq (|\varphi(0)| + M)(1 + \|z\|)$ for $z \in \mathbf{C}^n$. This completes the proof. □

Comments:

A) Condition (i) has been introduced by Janas in [12, Prop. 6]. Equivalence (i)⇔(iii) has been noticed by J. Peetre (cf. [13, Remark, p. 95]).

B) If φ satisfies condition (i), then, by Lemma and (2), there is $C > 0$ such that $|\varphi(z)| \leq \frac{3}{2} C \max\{1, r^{-1}\} e^{r\|z\|^2}$ for all $z \in \mathbf{C}^n$ and $r > 0$. Hence, following the notation of [14], (++) implies (+) (conditions (+) and (++) constitute the assumption of [14, Th. 3.3]). In other words, the assumption of [14, Th. 3.3] is equivalent to (++) (= (i) in our terminology).

C) Set $\rho_\varphi(r) = \sup\{|\varphi(z) - \varphi(w)| : z, w \in \mathbf{C}^n,\ \|z - w\| \leq r\}$ for $r \geq 0$. The function $\rho_\varphi : \overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+ \cup \{\infty\}$ is monotonically increasing and $|\varphi(z) - \varphi(w)| \leq \rho_\varphi(\|z - w\|)$ for $z, w \in \mathbf{C}^n$. It is clear, due to Lemma, that φ satisfies (i) if and only if $\rho_\varphi(r) < \infty$ for at least one $r > 0$ (equivalently: $\rho_\varphi(r) < \infty$ for every $r \geq 0$). Another equivalent formulation of condition (i) is: $\forall z, w \in \mathbf{C}^n : |\varphi(z) - \varphi(w)| \leq \rho(\|z - w\|)$, where $\rho : \overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$ is any function bounded in some neighbourhood of 0. One can construct a positive bounded function φ of the class $\mathcal{C}^\infty$ on $\mathbf{R}^{2n}$ for which $\rho_\varphi \equiv 1$ on $\mathbf{R}^{2n} \setminus \{0\}$. On the other hand, condition (i) does not imply the continuity of φ. Note that φ is uniformly continuous if and only if ρ_φ is continuous at 0; if this is the case, then $\rho_\varphi(r) < \infty$ for every $r \geq 0$ and ρ_φ is continuous. In other words, each uniformly continuous function $\varphi : \mathbf{C}^n \to \mathbf{C}$ satisfies condition (JC3).

References

[1] Bargmann V., *On a Hilbert space of analytic functions and an associated integral transform*, Comm. Pure Appl. Math. **14** (1961), 187–214.

[2] Bargmann V., *Remarks on a Hilbert space of analytic functions*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 199–204.

[3] Berezin F. A., *Vick and anti-Vick symbols of operators*, Mat. Sb. **86** (1971), 578–610.

[4] Berezin F. A., *Covariant and contravariant symbols of operators*, Math. USSR Izv. **6** (1972), 1117–1151.

[5] Berger C. A., Coburn L. A., *Toeplitz operators and quantum mechanics*, J. Funct. Anal. **68** (1986), 273–299.

[6] Berger C. A., Coburn L. A., *Toeplitz operators on the Segal–Bargmann space*, Trans. Amer. Math. Soc. **301** (1987), 813–829.

[7] Berger C. A., Coburn L. A., *Heat flow and Berezin–Toeplitz estimates*, Amer. J. Math. **116** (1994), 563–590.

[8] Borichev A., *The polynomial approximation property in Fock–type spaces*, to appear in Math. Scand.

[9] Cichoń D., *Notes on unbounded Toeplitz operators in Segal–Bargmann spaces*, Ann. Polon. Math. **3** (1996), 227–235.

[10] Fischer E., *Über die Differentiationsprozesse der Algebra*, J. Math. **148** (1917), 1–78.

[11] Guillemin V., *Toeplitz operators in n-dimensions*, Integral Equations Operator Theory **7** (1984), 145–205.

[12] Janas J., *Toeplitz and Hankel operators on Bargmann spaces*, Glasgow Math. J. **30** (1988), 315–323.

[13] Janas J., *Unbounded Toeplitz operators in the Bargmann–Segal space*, Studia Math. **99** (1991), 87–99.

[14] Janas J., *Unbounded Toeplitz operators in the Segal–Bargmann space, III*, submitted.

[15] Janas J., Rudol K., *Toeplitz operators on the Segal–Bargmann space of infinitely many variables*, Oper. Theory Adv. Appl. **43** (1990), 217–228.

[16] Janas J., Rudol K., *Toeplitz operators in infinitely many variables*, Topics in Operator Theory, Oper. Algebras and Applications (A. Gheondea et al., eds.), XV-th Internat. Conf. in Operator Theory, Timişoara 1994, IMAR, Bucharest, 1995, pp. 147-160.

[17] Janas J., Stochel J., *Unbounded Toeplitz operators in the Segal–Bargmann space, II*, J. Funct. Anal. **126** (1994), 418–447.

[18] Janas J., Stochel J., *Selfadjoint operator matrices with finite rows*, Ann. Polon. Math. **66** (1997), 155–172.

[19] Janas J., Stochel J., *Unbounded Toeplitz operators in the Segal–Bargmann space, IV*, in preparation.

[20] Newman D. J., Shapiro H. S., *Certain Hilbert spaces of entire functions*, Bull. Amer. Math. Soc. **72** (1966), 971–977.

[21] Newman D. J., Shapiro H. S., *Fischer spaces of entire functions*, Entire functions and related parts of analysis (J. Koorevaar, ed.), AMS Proc. Symp. Pure Math., vol XI, Providence, 1968, pp. 360–369.

[22] Segal I. E., *Lectures at the Summer Seminar on Applied Math.*, Boulder, Colorado, 1960.

[23] Segal I. E., *The complex wave representation of the free boson field*, Adv. in Math., Suppl. Studies **3** (1978), 321–343.

[24] Shapiro H. S., *An algebraic theorem of E. Fischer and the holomorphic Goursat problem*, Bull. London Math. Soc. **21** (1989), 513–537.

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