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ON THE WAŻEWSKI EQUATION

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1. Introduction. In 1947 Tadeusz Ważewski published a paper [1] whose aim was to give an estimate for the domain of existence of the implicit functions, and of the inverse maps as a special case. The paper appeared in the same volume of *Annales de la Société Polonaise de Mathématique* which contains his celebrated retract method. In this note we discuss only this special case of the inverse map. Ważewski's main idea was to apply the theory of ODE for that purpose. Namely, he suggested to study the differential equation

$$\dot{x} = Df(x)^{-1}w, \tag{W}$$

where f is a continuously differentiable map from an Euclidean space E into itself, $Df(x)$ stands for the derivative - jacobian matrix of f, and the parameter w is a vector in E. The equation (W) is well defined in the domain where $Df(x)$ is not singular. A solution $x(t)$ of (W) is the function whose image by f is linear:

$$f(x(t)) = f(x(0)) + wt. \tag{1}$$

We fix the initial condition $x(0) = a$ and denote by $x(t, a, w)$ the solution of (W) with this initial condition and by $T(a, w)$ the right-hand end of the maximal interval on which $x(t, a, w)$ exists. It follows from (1) that $x(t, a, w)$ is uniquely defined and thus depends continuously on a and w and therefore $T(a, w)$ is lower semi–continuous with respect to both variables. By (1) the map

$$g(y) = x(1, a, y - f(a)) \tag{2}$$

is the inverse map of f. That is, $f(g(y)) = f(x(1, a, y - f(a)) = f(a) + y - f(a) = y$. Clearly, g is defined on the set

$$S(a) = \{y : T(a, y - f(a)) > 1\}. \tag{3}$$

Notice that $S(a)$ is the maximal star shaped set with respect to $f(a)$ on which the inverse map exists. Another way the set $S(a)$ can be represented is:

$$S(a) = \{y : y = f(a) + wt, \ ||w|| = 1, \ 0 \le t < T(a, w)\}. \tag{3'}$$

The equation (W) is now known in the literature as the Ważewski equation and this name was first given to it by J. Sotomayor [2]. If we restrict ourselves to w from the unit sphere then any two solutions of (W) are either identical, or cross each other at only one point or are disjoint. If we assume that

$$\det Df(x) > 0 \text{ for each } x \tag{4}$$

then for any a and $w \neq 0$, $x(t, a, w) \to \infty$ if $t \to T(a, w)$. Denote by

$$\Omega(a) = \{p : p = x(t, a, w), \ 0 \le t < T(a, w), \ ||w|| = 1\}$$

the emission zone from a of the equation (W), in terminology of Ważewski. It is open and simply connected set. We notice that from (1) $f_{|\Omega(a)}$ is injective, $f(\Omega(a)) = S(a)$ and $f^{-1} = g$, where g is given by (2). In particular, if we are able to estimate from below $T(a, w)$ by r then the radius of a ball centered at $f(a)$ on which the inverse map f^{-1} exists is at least r. Such estimates has been obtained by Ważewski in [1].

This idea of Ważewski gives a short proof (see [1]) of the following clasical

HADAMARD THEOREM. *For a class C^1 map $f : E \to E$ assume (4) and*

$$||f(x)|| \to \infty \textit{ if } ||x|| \to \infty \tag{5}$$

then f is a diffeomorphism.

PROOF. From (5) it follows that $f(x(t, a, w)) \to \infty$ if $t \to T(a, w)$. From (1) we have $f(x(t, a, w)) \to f(a) + T(a, w)w$. Thus $T(a, w) = +\infty$ for each w and hence $S(a) = E$. Therefore f is one-to-one and onto E which completes the proof.

2. The Fessler–Gutierrez result. The aim of this note is to give another example where this simple idea of Ważewski could be applied. It concerns the recent result independently obtained by Robert Fessler [3] and Carlos Gutierrez [4].

THE FESSLER–GUTIERREZ THEOREM. *For E two dimensional if a class C^1 map $f : E \to E$ satisfies (4) and*

(6) *for $||x|| \geq M > 0$ the eigenvalues of $Df(x)$ are negative if they are real,*

where M is a fixed constant, then f is injective.

Both authors prove this theorem by analyzing a non-injective map f satisfying (4). We will do the same using the Ważewski equation (W). This will serve us to explain the main idea of these two different proofs of Fessler and Gutierrez. By proving the theorem they solved the problem posed in 1960 by L. Markus and H. Yamabe [5] concerning global asymptotic stability of an autonomous ODE system on the plane. Let us mention at this point that there are two other papers [6] and [7] offering still different solutions of Markus–Yamabe problem.

3. Non-injective local diffeomorphism of the plane. From now on we assume that E is of dimension two, $f : E \to E$ is of class C^1, not injective and (4) holds. Without loss of generality we may assume that there are two different points a and b and a not crossing itself smooth curve $\varphi : [0, 1] \to E$, such that

(7) $\varphi(0) = a, \varphi(1) = b, f(a) = f(b) = 0$ and $f(\varphi(s))$, $0 < s \leq 1$, is injective.

PROPOSITION 1. *Conditions (4) and (7) imply that $\Omega(a) \cap \Omega(b) = \varnothing$.*

PROOF. Suppose the opposite. That is, there exist w_1, w_2, t_1, t_2, $||w_i|| = 1$, $t_i > 0$, $i = 1, 2$¡ such that

$$x(t_1, a, w_1) = x(t_2, b, w_2). \tag{8}$$

Then by (1) $f(a)+t_1w_1 = f(b)+t_2w_2$ and by (7) $t_1w_1 = t_2w_2$, hence $w_1 = \theta w_2$ and $t_1 = \theta t_2$, where $\theta = \pm 1$. The latter, the uniqueness property of (W), the identity $x(t, b, w_1) = x(\theta t, b, \theta w_2)$ and (8) implies that $x(t, a, w_1) = x(t, b, w_1)$ for $0 \leq t < t_1$ and in particular $a = b$. Hence a contradiction and the proof is complete.

From Proposition 1 it follows that both sets have nonempty boundary. We denote the trajectory of (W) passing through d by $I(d, w) = \{x(t, d, w) : -T(d, -w) < t < T(d, w)\}$ and by $I_+(d, w)$, $I_-(d, w)$ we denote the positive and negative half–trajectory, respectively.

DEFINITION 1. A trajectory $I(d, w)$ is *an extension* of $I_+(a, w)$ if there exist sequences $w_k \to w$, $a_k \to a$ and $0 < t_k < T(a_k, w_k)$ such that $x(t_k, a_k, w_k) \to d$.

PROPOSITION 2. *Let $d \in bd\Omega(a)$. Then*

(i) *$I(d, w_d)$ is an extension of $I_+(a, w_d)$, where $w_d = \frac{f(d)}{||f(d)||}$;*
(ii) *$I(d, w_d) \subset bd\Omega(a)$,*
(iii) *$||f(d)|| > T(a, w_d)$ and for any s, $||f(d)|| > s > T(a, w_d)$, $sw_d \in bdS(a)$.*

PROOF. Let $x_k \in \Omega(a)$ and $x_k \to d$ *if* $k \to \infty$, that is $x_k = x(t_k, a, w_k) \to d$. Then by (1) $f(x_k) = t_k w_k \to f(d)$ and if we assume that $||w_k|| = 1$ then the latter means that $w_k \to w_d$, $t_k \to ||f(d)||$. Thus (i) holds. From the continuity of solutions of (W) with respect to the initial condition and the parameter w it follows that $||f(d)|| < T(a, w_k)$ for k big enough and $x(||f(d)||, a, w_k) \to d$. The same holds also for $x(t + ||f(d)||, a, w_k) \to x(t, d, w_d)$ if $T(d, w_d) > t > -T(d, -w_d)$. If $x(t, d, w_d) \in \Omega(a)$ for some t then this would mean that $d = x(||f(d)||, a, w_d)$ and consequently that d is an interior point of $\Omega(a)$, thus (ii) is proved. Notice also that $f(x(t, a, w_k)) \to tw_d$ for $0 \le t < T(d, w_d) + ||F(d)||$. Therefore this implies that $x(t, d, w_d) \in bd\Omega(a)$ *if* $0 < t < T(d, w_d)$, that $||f(d)|| - T(d, -w_d) > T(a, w_d)$ and that for any fixed $s, sw_d \in bdS(a)$, if $s > T(a, w_d)$ since $sw_d \notin S(a)$. This completes the proof.

REMARK. Notice that the argument of w_k is "on one side" of that of w_d. Notice also that for $||f(d)|| - T(d, -w_d) \ge t \ge T(a, w_d)$, $x(t, a, w_k)$ converges too but the limit may be infinity. If it is finite, then between $I(d, w_d)$ and $I_+(a, w_d)$ there is another trajectory which is also an extension of $I_+(a, w_d)$.

COROLLARY 1. *The boundary of $\Omega(a)$ is composed of full trajectories of (W), each unbounded at both ends. To each such boundary trajectory $I(d)$ there corresponds one connected component of $E \setminus \Omega(a)$, whose boundary is $I(d)$.*

Therefore there is a uniquely defined boundary trajectory of $\Omega(a)$ corresponding to b. Namely, the one which is the boundary of the connected component of $E \setminus \Omega(a)$ containing b. It is crossed by the curve φ at a point $d = \varphi(s_1)$, $0 < s_1 < 1$. We denote it shortly by $I(d)$. The corresponding boundary trajectory of $\Omega(b)$ we denote by $I(e)$, $e = \varphi(s_2)$, $s_1 < s_2 < 1$. The image of $I(d)$ by f is contained in the ray $r(w_d) = \{p : p = sw_d,\ s \ge 0\}$. Consider the set Λ of all trajectories $I(\varphi(s), w_d)$ such that $f(\varphi(s)) \in r(w_d)$; that is such that $f(I(\varphi(s), w_d)) \subset r(w_d)$. We notice that the set Λ is not empty since $I(d) \in \Lambda$ and finite since for any converging sequence $s_k \to s$ such that $f(\varphi(s_k)) \in r(w_d)$ and $f(\varphi(s)) \in r(w_d)$ all but finite $\varphi(s_k) \in I(\varphi(s), w_d)$; that is $I(\varphi(s_k), w_d) = I(\varphi(s), w_d)$ if k is big enough.
Next we will prove a lemma which will allow us to correct the curve φ connecting a with b to make the global picture simpler. Denote by

$$K = \{\gamma(s) = f(\varphi(s)) :\ 0 \le s \le 1\}.$$

By (7) K is a Jordan closed curve passing through zero.

LEMMA. *Suppose that there exist* $w \in E$, $0 \leq s_\alpha < s_\beta \leq 1$ *such that* $\beta = x(t_o, \alpha, w)$, *where* $0 < t_o < T(\alpha, w)$, $\alpha = \varphi(s_\alpha)$, $\beta = \varphi(s_\beta)$ *and assume that*

$$[f(\alpha), f(\beta)] \cap K \subset \{f(\varphi(s)) : s_\alpha \leq s \leq s_\beta\} \tag{9}$$

where $[f(\alpha), f(\beta)]$ *is the closed interval on the straight line passing through the end points. Put*

$$\varphi_*(s) = \varphi(s) \text{ if } 0 \leq s \leq s_\alpha \text{ or } s_\beta \leq s \leq 1$$

and

$$\varphi_*(s) = x(\frac{s - s_\alpha}{s_\beta - s_\alpha} t_o, \alpha, w) \text{ if } s_\alpha \leq s < s_\beta$$

then $f(\varphi_*(s))$ *is injective for* $0 < s \leq 1$.

PROOF. Assume that $0 < u < \nu \leq 1$. If both u and ν are either inside or outside $[s_\alpha, s_\beta]$ then clearly $f(\varphi_*(u)) \neq f(\varphi_*(\nu))$ because of (1) and (7). If $u \in [s_\alpha, s_\beta]$ then $f(\varphi_*(u)) \neq f(\varphi_*(\nu))$ because of (7) and (9).

DEFINITION 2. We say that $I(d)$ is *transversal* to φ if (9) holds for α and β belonging to $I(d)$ and such that $\varphi(s) \in \Omega(a)$ for $s < s_\alpha$ and $\varphi(s) \notin \Omega(a) \cup I(d)$ for $s > s_\beta$.

PROPOSITION 3. *If* $I(d)$ *is transversal to* φ *then there is injective* $\varphi_* : [0, 1] \to E$ *satisfying the same condition (7) as* φ, *and crossing* $I(d)$ *only once.*

PROOF. Since α and β belong to the same trajectory of (W) thus for small enough $\varepsilon > 0$ the points $\alpha_* = \varphi(s_\alpha - \varepsilon)$ and $\beta_* = \varphi(s_\beta - \varepsilon)$ can be joined in a unique way by a solution curve of (W) and $[f(\alpha_*), f(\beta_*)] \cap K \subset \{f(\varphi(s)) : s_\alpha - \varepsilon \leq s \leq s_\beta + \varepsilon\}$. The latter follows from the continuity of solutions of (W) with respect to the initial condition and the parameter w. In other words, since $\beta \in \Omega(\alpha)$ then $\beta_* \in \Omega(\alpha_*)$ provided ε is small enough. The Lemma completes the proof.

It is obvious that if both $I(d)$ and $I(e)$ are transversal to φ then the global picture is simpler and easier to deal with. Thus the next proposition is of interest.

First we define for given φ a class of curves Ψ.

DEFINITION OF Ψ. A curve ψ belongs to $\Psi = \Psi(\varphi)$ if it is composed of a segment A of φ,

$$A = \{\varphi(s) : 0 \leq s_1 \leq s \leq s_2 \leq 1\}$$

and a segment B of the solution curve

$$B = \{x(t, \varphi(s_i), w) : 0 \leq t \leq t_0\}, \ i = 1 \text{ or } 2,$$

such that

$A \cap B = \{\varphi(s_i)\}$ and $f(\varphi(s_j)) = f(x(t_o, \varphi(s_i), w))$, where j is different from i.

Notice that $f(\psi)$ is a closed Jordan curve; that is, $f(\psi)$ is one-to-one except at the end points. In other words, Ψ is the class of regular injective maps from an interval into E whose images by f are close curves obtained from K by taking a segment of the latter and, if the segment is a proper subset of K, closing it by the interval of the straight line passing through the end points provided that this interval does not have other points in common with the segment of K in question and $f^{-1}(f(\psi(s_j))) \cap \Omega(\psi(s_i)) \setminus \psi(s_j) \neq \varnothing$, i equals either 1 or 2 and j different from i.

Notice that $\varphi \in \Psi$. For $\psi \in \Psi$ we denote by $a(\psi)$ and $b(\psi)$ the end points of it and by $I(d, \psi)$ the boundary trajectory of $\Omega(a(\psi))$ separating $b(\psi)$ from $a(\psi)$ and by $I(e, \psi)$ the boundary trajectory of $\Omega(b(\psi))$ separating $a(\psi)$ from $b(\psi)$. Each ψ can be oriented and the orientation we choose is that of φ. Our nearest aim is to prove the following.

PROPOSITION 4. *For given φ satisfying (7) there is $\psi \in \Psi(\varphi)$ such that both $I(d, \psi)$ and $I(e, \psi)$ are transversal with respect to ψ.*

PROOF. Suppose now that either $I(d, \psi)$ or $I(e, \psi)$, say $I(d, \psi)$, is not transversal to ψ. Then there exist three points α, β and δ belonging to ψ such that α is the first point ψ meets $I(d, \psi)$, β is the last point and δ is the first after β such that $f(\delta)$ belongs to the interval $[f(\alpha), f(\beta)]$; that is $f(\delta) = \lambda f(\alpha) + (1 - \lambda) f(\beta)$, $0 < \lambda < 1$. The latter is a consequence of the assumption that the inclusion (9) does not hold. We obtained in that way a curve $\psi_1 \in \Psi$ whose component A is the segment of φ between β and δ and B is the segment of $I(d, \psi)$ between β and γ where γ is the unique point of $I(d, \psi)$ such that $f(\gamma) = f(\delta)$. If $I(e, \psi)$ is not transveral to ψ then β and δ are the first and the last point in common of ψ and $I(e, \psi)$ and α is such that $f(\alpha) = \lambda f(\delta) + (1 - \lambda) f(\beta)$, $0 < \lambda < 1$ and no other point of ψ between α and β has this property. The curve ψ_1 is now composed of the segments $[\alpha, \beta]$ of φ and $[\beta, \gamma]$ of $I(e, \psi)$, where $\gamma \in I(e, \psi)$ and $f(\gamma) = f(\alpha)$. We claim that all three points α, β and δ belong to the segment A of ψ; that is, belong to φ. Without loss of generality we may assume that the segment B of ψ is first. Since B is a piece of a trajectory of (W) passing through $a(\psi)$ thus $f(B) \subset \Omega(a(\psi))$ hence $\alpha \notin B$ and the same holds for β and δ provided $I(d, \psi)$

was used to define these three points. If $I(e,\psi)$ is used and α were in B then there would exist a γ in B different from $a(\psi)$ such that either $f(\gamma) = f(\beta)$ or $f(\gamma) = f(\delta)$ which contradicts the asumption that $f(\psi)$ is a Jordan curve. The latter holds because both $f(B)$ and $[f(\delta), f(\beta)]$ lie in a straight line passing through $f(a(\psi))$ and therefore non-empty intersection implies that one of the end point of $[f(\delta), f(\beta)]$ belongs to $f(B)$. We can repeat this construction and obtain a sequence ψ_k contained in Ψ provided one of the separating trajectories $I(d, \psi_{k-1})$ or $I(e, \psi_{k-1})$ is not transversal to ψ_{k-1}. Thus to prove Proposition 4 it is enough to show that this sequence is finite. Suppose it is not finite then we have three sequences $\{\alpha_k\}$, $\{\beta_k\}$ and $\{\delta_k\}$ contained in φ and satisfying the following relations

$$\alpha_k < \beta_k < \delta_k,$$

and either

$$(10) \quad [\alpha_k, \delta_k] \subset [\alpha_{k-1}, \beta_{k-1}],\ f(\alpha_{k-1}) = \lambda f(\beta_{k-1}) + (1-\lambda) f(\delta_{k-1}),\ 0 < \lambda < 1$$

or

(11)

$$[\alpha_k, \delta_k] \subset [\beta_{k-1}, \delta_{k-1}],\ f(\delta_{k-1}) = \mu f(\alpha_{k-1}) + (1-\mu) f(\beta_{k-1}),\ 0 < \mu < 1.$$

In both cases we conclude that α_k and δ_k are monotone hence convergent and we denote the limits by $\alpha_\#$ and $\delta_\#$, respectively. If (10) holds for infinitely many k then from the inequality $\delta_k < \beta_{k-1} < \delta_{k-1}$ there is a subsequence for which also β_{k-1} converges to $\delta_\#$. Then from the second part of (10) we obtain that $f(\alpha_\#) = f(\delta_\#)$. The latter, injectivity of $f(\varphi(s))$ if $0 < s < 1$ and $\alpha_\# \neq a$, $\delta_\# \neq b$ we conclude that $\alpha_\# = \delta_\#$. Hence $\beta_k \to \alpha_\# = \delta_\#$, too. This implies that for k big enough α_k, β_k, δ_k and the segment $[\beta_k, \delta_k]$ of the trajectory of (W) uniquely determined by those two points are all contained in $\Omega(\alpha_\#)$ on which f is injective. Therefore the second part of (10) implies that α_k belongs to the trajectory of (W) passing through β_k and δ_k which contradicts the definition of α_k. If (11) holds for infinitely many k then the proof is quite analogous and we omit it.

4. Conclusions. We assume now that both $I(d)$ and $I(e)$ are crossed by φ only once and the crossing points are d and e, respectively. Because of Propositions 3 and 4 this assumption does not restrict the generality of our consideration. We describe the picture of φ in this case in some more details and point out the connections with the proofs of Fessler and Gutierrez of their result we mentioned. We have three subcases to consider: 1. $e = d$ hence also

$I(d) = I(e)$ and $w_e = w_d$, 2. $w_e \neq w_d$, 3. $w_e = w_d$ but $I(d)$ different from $I(e)$,

Case 1, $e = d$. Denote by $d_\# = \varphi(s_\#)$, where $0 \leq s_\# < s_1$, $f(d_\#) \in r(w_d)$ while $f(\varphi(s)) \notin r(w_d)$ if $s_\# < s < s_1$. In an analogous way we define $e^\# = \varphi(s^\#)$, $s_2 < s^\# \leq 1$. We add the trajectory $I_-(d, w_d)$ to the curve L composed of the curve $\varphi(s)$, $s_\# < s < s^\#$ and two half trajectories $I_+(d_\#, w_d)$ and $I_+(e^\#, w_d)$. The foliation given by trajectories of (W) with fixed $w = w_d$ on the set D bounded by L and containing $I_-(d, w_d)$ gives us two *half-Reeb components* of (W) both on the same side of φ. Let us recall that $A \subset R^2$ is a half-Reeb component of (W) if the foliation on A determined by trajectories of (W) (with fixed w) is topologically equivalent to the foliation on $B = \{(x, y) : -1 \leq x \leq 1,\ y \geq 0\}$ given by the one parameter family of functions: $y = c + \frac{1}{x^2 - 1}$. This follows from Proposition 2. Indeed $I_-(d, w_d)$ is an extension of both $I_+(d_\#, w_d)$ and $I_+(e^\#, w_e)$ which means that a trajectory starting in D near $I_+(d_\#, w_d)$ or near $I_+(e^\#, w_e)$ stays near $I_-(d, w_d)$, also. In particular, each such trajectory tends to infinity at both ends on the other side of φ (outside L). The main tool of Gutierrez proof of the theorem is a proposition (see [4, Proposition 2.1, p.631]) that (6) rules out the existence of such two half-Reeb components of the phase portrait of (W) for any fixed w.

Case 2, $w_d \neq w_e$. Consider the curve K. Both $f(d_\#)$ and $f(d)$ belong to K and either $f(I_+(d_\#, w_d))$ or $f(I_+(d, w_d))$ is contained in the interior of K and only one has this property. Hence the other is in the exterior of K. Denote by d_o that point, of the two d and $d_\#$. Such that $f(I_+(d_o, w_d))$ is contained in the interior of K. Put $d_1 = d$ if $d_o = d_\#$. If $d_o = d_1$ then we put $d_1 \in \varphi$ such that $f(d_1) = \tau_m w_d$, where $\tau_m = \max\{\tau | \tau w_d \in K\}$. We notice that the segment of K between $f(d_o)$ and $f(d_1)$ not containing $f(a)$ lays on "one side" of the ray $r(w_d)$; by that we mean that the tangent vectors to K at $f(d_o)$ and $f(d_1)$, which without loss of generality can be assumed to be perpendicular to the ray, have opposite orientation. Therefore the rotation of the tangent vector to K between $f(d_o)$ and $f(d_1)$ is equal π. Similarly we denote by e_o one of the two points e and $e^\#$ namely, the one for which $f(I_+(e_o, w_e))$ is contained in the interior of K and by $e_1 \in \varphi$ such point that $e_1 = e$ if $e_o = e^\#$; if $e_o = e$ then $e_1 \in \varphi$ and $f(e_1) \in K \cap r(w_e)$ and it is the last with these properties above $f(e_o)$. We denote by L the curve composed of φ between d_o and e_o and two half trajectories $I_+(d_o, w_d)$ and $I_+(e_o, w_e)$. The curve L is an injective proper piecewise smooth image of R into E. By a local change of the curve L we can make the crossings of $f(L)$ with the rays $r(w_d) = \{p : p = s w_d,\ s \geq 0\}$ and $r(w_e) = \{p : p = s w_e,\ s \geq 0\}$ perpendicular. Let us fix the orientation of φ such that the orientation of K is positive hence the rotation of the tangent vector to K is $+2\pi$. The orientation of $f(L)$ is determined by that of K.

Both rays are crossed in the positive direction. Therefore the rotation of the tangent vector to $f(L)$ from $f(d_1)$ to $f(e_1)$ is either α or $2\pi - \alpha$ where α is the angle between w_d and w_e; from $f(d_o)$ to $f(d_1)$ equals π between any point of the intersection $f(L) \cap r(w_d)$ and $f(d_1)$ is equal to $\frac{3\pi}{2}$ and the same is the rotation between $f(e_1)$ and any point of the intersection $f(L) \cap r(w_e)$. Hence we have:

PROPOSITION 5. *The rotation of the tangent vector to $f(L)$, where L is the curve described above, from any point of $f(L) \cap r(w_d)$ to any point of $f(L) \cap r(w_e)$ is constant and greater than 3π.*

The main idea of Fessler's proof is to show that a curve L, having the properties described in Proposition 5, exists if f satisfying (4) is not injective (see [3], Theorems 2 and 3, p.69). He proves that on such curve there is a point $p \in L$ such that the tangent vector ν to $f(L)$ at the point $f(p)$ is equal μu where u is the tangent vector to L at p and $\mu > 0$. This leads him to the contradiction with (6).

Case 3, $e \neq d$, $w_e = w_d$. We assume that $f(d) < f(e)$ with respect to the order in $r(w_d)$ induced by w_d. Consider the part P of K between $f(d)$ and $f(e)$. As before we assume that the tangent vectors to K at $f(d)$ and $f(e)$ are perpendicular to $r(w_d)$. First we consider the case where the direction of crossing $r(w_d)$ at $f(e)$ is opposite to that at $f(d)$. We assume also that $P \cap r(w_d) = \{f(d), f(e)\}$. If this assumption is not satisfied then replacing $f(d)$ by the maximum of $(P \cap r(w_d)) \setminus f(e)$ we make it hold true. Denote by D the set bounded by P and the interval $[f(d), f(e)]$ of $r(w_d)$ and by Σ the union of intervals $[f(d), p]$ contained in D where $p \in P$ and $p \neq f(d)$. If $\Sigma \subset f(\Omega(d))$ then the picture is like in the case 1. The trajectory $I(e, w_d)$ is an extension of both $I_+(e^\#, w_d)$ and $I(d, w_d)$. Thus the phase portrait of (W) for $w = w_d$ gives two *half-Reeb components* of (W) on one side of φ. If the above inclusion does not hold then there is a w_o such that $f(I_+(d, w_o)) \setminus f(d) \subset \text{int}D$ and the curve L composed of the segment of φ between d and $e^\#$ and two trajectories $I_+(d, w_o)$ and $I_+(e^\#, w_d)$ have the same properties as the one in the case 2. That is the rotation of the tangent vectors to L between any two points close to infinity and at opposite ends of it is constant and greater than 3π. Thus Fessler's arguments applies. Other possible configurations of K in this case can be dealt with in an analogous way and we omit the details.

REMARK. The sketch of the proof of the Fessler–Gutierrez result we presented is not complete. We left untouched the problem of moving the curve L outside a fixed compact set so that (6) can be applied. It seems that this problem is more complicated in the Fessler case, however, the tool Gutierrez works with, that is that existence of two half-Reeb components of (W) contradicts (6), needs more effort to be established. On the other hand, if there were a common Fessler-Gutierrez proof then it should be simpler since one could avoid extra effort to proving the existence of Gutierrez two half-Reeb components of (W) picture while the Fessler curve L is easy to establish and vice versa.

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