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ON THE WAŻEWSKI EQUATION

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1. Introduction. In 1947 Tadeusz Ważewski published a paper [1] whose aim was to give an estimate for the domain of existence of the implicit functions, and of the inverse maps as a special case. The paper appeared in the same volume of *Annales de la Société Polonaise de Mathématique* which contains his celebrated retract method. In this note we discuss only this special case of the inverse map. Ważewski's main idea was to apply the theory of ODE for that purpose. Namely, he suggested to study the differential equation

$$\dot{x} = Df(x)^{-1}w, \tag{W}$$

where f is a continuously differentiable map from an Euclidean space E into itself, $Df(x)$ stands for the derivative - jacobian matrix of f, and the parameter w is a vector in E. The equation (W) is well defined in the domain where $Df(x)$ is not singular. A solution $x(t)$ of (W) is the function whose image by f is linear:

$$f(x(t)) = f(x(0)) + wt. \tag{1}$$

We fix the initial condition $x(0) = a$ and denote by $x(t, a, w)$ the solution of (W) with this initial condition and by $T(a, w)$ the right-hand end of the maximal interval on which $x(t, a, w)$ exists. It follows from (1) that $x(t, a, w)$ is uniquely defined and thus depends continuously on a and w and therefore $T(a, w)$ is lower semi–continuous with respect to both variables. By (1) the map

$$g(y) = x(1, a, y - f(a)) \tag{2}$$

is the inverse map of f. That is, $f(g(y)) = f(x(1,a,y-f(a)) = f(a)+y-f(a) = y$. Clearly, g is defined on the set

$$S(a) = \{y : T(a, y - f(a)) > 1\}. \tag{3}$$

Notice that $S(a)$ is the maximal star shaped set with respect to $f(a)$ on which the inverse map exists. Another way the set $S(a)$ can be represented is:

$$S(a) = \{y : y = f(a) + wt,\ ||w|| = 1,\ 0 \le t < T(a,w)\}. \tag{3'}$$

The equation (W) is now known in the literature as the Ważewski equation and this name was first given to it by J. Sotomayor [2]. If we restrict ourselves to w from the unit sphere then any two solutions of (W) are either identical, or cross each other at only one point or are disjoint. If we assume that

$$\det Df(x) > 0 \text{ for each } x \tag{4}$$

then for any a and $w \neq 0$, $x(t,a,w) \to \infty$ if $t \to T(a,w)$. Denote by

$$\Omega(a) = \{p : p = x(t,a,w),\ 0 \le t < T(a,w),\ ||w|| = 1\}$$

the emission zone from a of the equation (W), in terminology of Ważewski. It is open and simply connected set. We notice that from (1) $f_{|\Omega(a)}$ is injective, $f(\Omega(a)) = S(a)$ and $f^{-1} = g$, where g is given by (2). In particular, if we are able to estimate from below $T(a,w)$ by r then the radius of a ball centered at $f(a)$ on which the inverse map f^{-1} exists is at least r. Such estimates has been obtained by Ważewski in [1].

This idea of Ważewski gives a short proof (see [1]) of the following clasical

HADAMARD THEOREM. *For a class C^1 map $f : E \to E$ assume (4) and*

$$||f(x)|| \to \infty \text{ if } ||x|| \to \infty \tag{5}$$

then f is a diffeomorphism.

PROOF. From (5) it follows that $f(x(t,a,w)) \to \infty$ if $t \to T(a,w)$. From (1) we have $f(x(t,a,w)) \to f(a) + T(a,w)w$. Thus $T(a,w) = +\infty$ for each w and hence $S(a) = E$. Therefore f is one-to-one and onto E which completes the proof.

2. The Fessler–Gutierrez result. The aim of this note is to give another example where this simple idea of Ważewski could be applied. It concerns the recent result independently obtained by Robert Fessler [3] and Carlos Gutierrez [4].

THE FESSLER–GUTIERREZ THEOREM. *For E two dimensional if a class C^1 map $f : E \to E$ satisfies (4) and*

(6) *for $||x|| \geq M > 0$ the eigenvalues of $Df(x)$ are negative if they are real,*

where M is a fixed constant, then f is injective.

Both authors prove this theorem by analyzing a non-injective map f satisfying (4). We will do the same using the Ważewski equation (W). This will serve us to explain the main idea of these two different proofs of Fessler and Gutierrez. By proving the theorem they solved the problem posed in 1960 by L. Markus and H. Yamabe [5] concerning global asymptotic stability of an autonomous ODE system on the plane. Let us mention at this point that there are two other papers [6] and [7] offering still different solutions of Markus–Yamabe problem.

3. Non-injective local diffeomorphism of the plane. From now on we assume that E is of dimension two, $f : E \to E$ is of class C^1, not injective and (4) holds. Without loss of generality we may assume that there are two different points a and b and a not crossing itself smooth curve $\varphi : [0,1] \to E$, such that

(7) $\varphi(0) = a, \varphi(1) = b, f(a) = f(b) = 0$ and $f(\varphi(s))$, $0 < s \leq 1$, is injective.

PROPOSITION 1. *Conditions (4) and (7) imply that $\Omega(a) \cap \Omega(b) = \varnothing$.*

PROOF. Suppose the opposite. That is, there exist w_1, w_2, t_1, t_2, $||w_i|| = 1$, $t_i > 0$, $i = 1,2$¡ such that

$$x(t_1, a, w_1) = x(t_2, b, w_2). \tag{8}$$

Then by (1) $f(a)+t_1w_1 = f(b)+t_2w_2$ and by (7) $t_1w_1 = t_2w_2$, hence $w_1 = \theta w_2$ and $t_1 = \theta t_2$, where $\theta = \pm 1$. The latter, the uniqueness property of (W), the identity $x(t, b, w_1) = x(\theta t, b, \theta w_2)$ and (8) implies that $x(t, a, w_1) = x(t, b, w_1)$ for $0 \leq t < t_1$ and in particular $a = b$. Hence a contradiction and the proof is complete.

From Proposition 1 it follows that both sets have nonempty boundary. We denote the trajectory of (W) passing through d by $I(d, w) = \{x(t, d, w) : -T(d, -w) < t < T(d, w)\}$ and by $I_+(d, w)$, $I_-(d, w)$ we denote the positive and negative half–trajectory, respectively.

DEFINITION 1. A trajectory $I(d, w)$ is *an extension* of $I_+(a, w)$ if there exist sequences $w_k \to w$, $a_k \to a$ and $0 < t_k < T(a_k, w_k)$ such that $x(t_k, a_k, w_k) \to d$.