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ON THE TANGENT BUNDLE OF A SCHEME

BY ERNST KUNZ

Abstract. This is a survey on properties of tangent bundles of schemes which are obtained by applying results of Avramov [2], Huneke [6], Huneke-Rossi [7], Simis-Vasconcelos [9], [10] and Vasconcelos [12], [13] about the symmetric algebra of a module to the module of Kaehler differentials and by rewriting them in the language of schemes. For unexplained notions and notation as well as for some results about schemes we refer the reader to [1]. As is now usual a scheme is what was formerly called a prescheme.

1. Tangent bundles. Let $f : X \to Y$ be a morphism of schemes. For $x \in X$ with $y := f(x)$ the $k(x)$-vector space

$$T_{X/Y}(x) := \mathrm{Der}_{\mathcal{O}_y}(\mathcal{O}_x, k(x))$$

is called the *(Zariski-)tangent space* of X/Y at x. Its elements, the $\mathcal{O}_y$-derivations $\delta : \mathcal{O}_x \to k(x)$, are called *tangent vectors* of X/Y at x.

Let $\Omega^1_{X/Y}$ be the $\mathcal{O}_X$-module of Kaehler differentials of X/Y ([1], IV.16.3.1). Then

$$T_{X/Y}(x) \cong \mathrm{Hom}_{k(x)}(\Omega^1_{\mathcal{O}_x/\mathcal{O}_y} \otimes_{\mathcal{O}_x} k(x), k(x)),$$

hence if $T_{X/Y}(x)$ is a finite dimensional $k(x)$-vector space its dimension is $\mu(\Omega^1_{\mathcal{O}_x/\mathcal{O}_y})$ where μ denotes the length of a minimal system of generators of a module.

The *tangent bundle* $T_{X/Y}$ of X/Y is by definition the fibre bundle

$$\mathbf{V}(\Omega^1_{X/Y}) = \mathrm{Spec}\,\mathbf{S}_{\mathcal{O}_X}(\Omega^1_{X/Y}),$$

see [1, IV, 16.5], where also the formal properties of the tangent bundle are described. This notion of tangent bundle is analogous to the corresponding concept of differential topology. For a different notion and its relation to the present one, see [11].

The *projection* $\mathrm{pr} : T_{X/Y} \to X$ of the tangent bundle onto its basis X is a morphism, which allows the following local description: Let $V = \operatorname{Spec} A$ be an affine open set in Y, $U = \operatorname{Spec} B$ an affine open set in X such that $f(U) \subset V$, then

$$\mathrm{pr}^{-1}(U) = \operatorname{Spec} \mathbf{S}_B(\Omega^1_{B/A}),$$

where $\mathbf{S}_B(\Omega^1_{B/A})$ denotes the symmetric algebra of the differential module $\Omega^1_{B/A}$ of the algebra B/A. Moreover, the restriction of pr to $\mathrm{pr}^{-1}(U)$ is the morphism induced on the spectra by the canonical injection $B \to \mathbf{S}_B(\Omega^1_{B/A})$. In particular,

$$\mathrm{pr}^{-1}(U) \cong T_{U/V}.$$

Since the symmetric algebra is compatible with base change the inverse image of $\operatorname{Spec} \mathcal{O}_x$ $(x \in X)$ by pr is $\operatorname{Spec} \mathbf{S}_{\mathcal{O}_x}(\Omega^1_{\mathcal{O}_x/\mathcal{O}_y})$, and the fibre of pr at x is

$$\mathrm{pr}^{-1}(x) = \operatorname{Spec} \mathbf{S}_{k(x)}(\Omega^1_{\mathcal{O}_x/\mathcal{O}_y} \otimes_{\mathcal{O}_x} k(x)).$$

By the universal property of the symmetric algebra the $k(x)$-rational points of $\mathrm{pr}^{-1}(x)$ are in natural one-to-one correspondence with the $k(x)$-linear maps $\Omega^1_{\mathcal{O}_x/\mathcal{O}_y} \otimes_{\mathcal{O}_x} k(x) \to k(x)$, hence with the tangent vectors of X/Y at x. For $\psi := f \circ \mathrm{pr} : T_{X/Y} \to Y$ and $y \in Y$ the fibre $\psi^{-1}(y)$ is the tangent bundle $T_{f^{-1}(y)/k(y)}$ of the $k(y)$-scheme $f^{-1}(y)$.

A *vector field* on X/Y is by definition a section $v : X \to T_{X/Y}$ in the tangent bundle i.e. a morphism such that $\mathrm{pr} \circ v = \mathrm{id}_X$. For $V = \operatorname{Spec} A$, $U = \operatorname{Spec} B$ as above the restriction $v|_U : U \to T_{U/V}$ is the map induced by a B-algebra homomorphism $\mathbf{S}_B(\Omega^1_{B/A}) \to B$. By the universal property of the symmetric algebra these homomorphisms are in one-to-one correspondence with the elements of

$$\operatorname{Hom}_B(\Omega^1_{B/A}, B) = \operatorname{Der}_A(B, B).$$

Thus for a morphism $f : X \to Y$ of affine schemes the vector fields are in one-to-one correspondence with the $\Gamma(Y, \mathcal{O}_Y)$-derivations of $\Gamma(X, \mathcal{O}_X)$.

Similarly for $x \in X$, $y = f(x)$ a vector field v induces a vector field v_x on the preimage $\operatorname{Spec} \mathbf{S}_{\mathcal{O}_x}(\Omega^1_{\mathcal{O}_x/\mathcal{O}_y})$ of $\operatorname{Spec} \mathcal{O}_x$ in $T_{X/Y}$ which corresponds to an $\mathcal{O}_y$-derivation $\delta_x : \mathcal{O}_x \to \mathcal{O}_x$. Let $v(x)$ be the composition of δ_x with the canonical residue map $\mathcal{O}_x \to k(x)$. Then $v(x) \in T_{X/Y}(x)$ is called the tangent vector given by the vector field v at x.

LEMMA 1.1. *Let v, v' be vector fields of X/Y. Assume X is a reduced scheme. Then $v = v'$ if and only if $v(x) = v'(x)$ for all $x \in X$.*

PROOF. It suffices to prove the lemma for affine schemes $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Let δ, δ' be the A-derivations of B corresponding to v and v'. It suffices to show that $\delta = 0$ if $v(x) = 0$ for all $x \in X$.

Let $b \in B$ be given with image b_x in $\mathcal{O}_x$. Let $\delta_x \in \operatorname{Der}_{\mathcal{O}_y}(\mathcal{O}_x, \mathcal{O}_x)$ be the natural extension of δ by the quotient rule for derivations. Then $v(x)(b_x)$ is the residue class of $\delta_x(b_x)$ in $k(x)$. If $v(x) = 0$ this implies $\delta_x(b_x) \in \mathfrak{m}_x$ and $\delta(b) \in \mathfrak{p}$ where $\mathfrak{p}$ is the prime ideal of B corresponding to x. Now, if $v(x) = 0$ for all $x \in X$, we have $\delta(b) \in \bigcap_{\mathfrak{p} \in \operatorname{Spec} B} \mathfrak{p}$, hence $\delta(b) = 0$ since B is reduced. It follows that $\delta = 0$.

PROPOSITION 1.2. *Assume X is an integral scheme. Then $T_{X/Y}$ is integral if and only if for all $i \in \mathbf{N}$ the $\mathcal{O}_X$-module $\mathbf{S}_i(\Omega^1_{X/Y})$ is torsion free.*

PROOF. Let ξ be the generic point of X, let $\eta := f(\xi)$ and $L := \mathcal{O}_\xi$ the field of rational functions on X. Then $\mathbf{S}_L(\Omega^1_{L/\mathcal{O}_\eta})$ is a polynomial algebra over L. The condition of torsion freeness is equivalent with the canonical map $\mathbf{S}_i(\Omega^1_{\mathcal{O}_x/\mathcal{O}_y}) \to \mathbf{S}_i(\Omega^1_{L/\mathcal{O}_\eta})$ being injective for all $x \in X$, $y = f(x)$, hence with $\mathbf{S}(\Omega^1_{\mathcal{O}_x/\mathcal{O}_y})$ being an integral domain. But this is equivalent with $T_{X/Y}$ being an integral scheme.

PROPOSITION 1.3. *Assume Y is locally noetherian and X/Y is locally of finite type. Then $T_{X/Y}$ is smooth over Y if and only if X/Y is. If X/Y is smooth and equidimensional of dimension d then $T_{X/Y}/Y$ is equidimensional of dimension $2d$.*

We shall use the following lemma (see [13], Prop. 1.1).

LEMMA 1.4. *Let $(R, \mathfrak{m})$ be a noetherian local ring, M a finitely generated R-module, $S := \mathbf{S}_R(M)$ and $\mathfrak{M} := \mathfrak{m} \oplus S_+$ where $S_+ := \bigoplus_{i>0} S_i$. Then $S_{\mathfrak{M}}$ is regular if and only if R is regular and M a free R-module.*

PROOF. There are natural isomorphism

$$\mathfrak{M}/\mathfrak{M}^2 \cong \mathfrak{m}/\mathfrak{m}^2 \oplus S_+/\mathfrak{m}S_+ + S_+^2 \cong \mathfrak{m}/\mathfrak{m}^2 \oplus M/\mathfrak{m}M.$$

Consider M as a subset of S by the canonical injection $M \to S$.

If $S_{\mathfrak{M}}$ is regular it follows from Nakayama's lemma that each minimal system of generators of M is part of a regular system of parameters of $S_{\mathfrak{M}}$. Then $R = S_{\mathfrak{M}}/S_+ \cdot S_{\mathfrak{M}} = S_{\mathfrak{M}}/MS_{\mathfrak{M}}$ is a regular local ring too. Moreover,

$(S_+)_{\mathfrak{M}}/(S_+)^2_{\mathfrak{M}} \cong M$ is a free R-module as $(S_+)_{\mathfrak{M}}$ is generated by an $S_{\mathfrak{M}}$-regular sequence.

Conversely, if R is regular and M free, then it is clear that $S_{\mathfrak{M}}$ is a regular local ring, S being a polynomial algebra over R.

PROOF OF 1.3. For $x \in X$, $y = f(x)$ the $\mathcal{O}_x$-module $\Omega^1_{\mathcal{O}_x/\mathcal{O}_y}$ is finitely generated and $\mathbf{S}_{\mathcal{O}_x}(\Omega^1_{\mathcal{O}_x/\mathcal{O}_y})$ is an $\mathcal{O}_x$-algebra of finite type. If X/Y is smooth at x then $\Omega^1_{\mathcal{O}_x/\mathcal{O}_y}$ is free of rank $\dim_x f$ [1], IV.17.10.2), and $\mathbf{S}_{\mathcal{O}_x}(\Omega^1_{\mathcal{O}_x/\mathcal{O}_y})$ is a polynomial algebra over $\mathcal{O}_x$. It follows that $T_{X/Y}$ is smooth at any point z lying over x. This shows that smoothness of X/Y implies the smoothness of $T_{X/Y}/Y$, and the assertion of the proposition about relative dimensions follows.

Let $\overline{\mathcal{O}_x} := \mathcal{O}_x/\mathfrak{m}_y\mathcal{O}_x$, and let $\overline{\mathfrak{m}_x}$ be its maximal ideal. If $T_{X/Y}$ is smooth over Y then with $S := \mathbf{S}_{\overline{\mathcal{O}_x}}(\Omega^1_{\overline{\mathcal{O}_x}/k(y)})$ and $\mathfrak{M} := \overline{\mathfrak{m}_x}S \oplus S_+$ the local ring $S_{\mathfrak{M}}$ is regular, hence by 1.4 $\overline{\mathcal{O}_x}$ is regular and $\Omega^1_{\overline{\mathcal{O}_x}/k(y)}$ is free. Again S is a polynomial algebra over $\overline{\mathcal{O}_x}$, and the smoothness of $T_{X/Y}/Y$ implies the smoothness of X/Y.

EXAMPLE 1.5. Let K be a field, and let $X \subset \mathbf{A}^n_K$ be a closed subscheme defined by an ideal $I_X = (f_1, \dots, f_m)$ in the polynomial ring $K[X_1, \dots, X_n]$. Set $B := K[X_1, \dots, X_n]/I_X = K[x_1, \dots, x_n]$, where x_i is the image of X_i in B $(i = 1, \dots, n)$. Then

$$\Omega^1_{B/K} = \bigoplus_{k=1}^{n} B dX_k / \langle \{ \sum_{k=1}^{n} \frac{\partial f_i}{\partial x_k} dX_k \}_{i=1,\dots,m} \rangle,$$

where $\frac{\partial f_i}{\partial x_k}$ is the image of the partial derivative $\frac{\partial f_i}{\partial X_k}$ in B. Therefore

$$\mathbf{S}_B(\Omega^1_{B/K}) = K[X_1, \dots, X_n, Y_1, \dots, Y_n]/(\{f_i, \sum_{k=1}^{n} \frac{\partial f_i}{\partial X_k} Y_k\}_{i=1,\dots,m})$$

and $T_{X/K} = \operatorname{Spec} \mathbf{S}_B(\Omega^1_{B/K})$ is isomorphic to the closed subscheme of $\mathbf{A}^{2n}_K$ defined by the ideal $J_X = (\{f_i, \sum_{k=1}^{n} \frac{\partial f_i}{\partial X_k} Y_k\}_{i=1,\dots,m})$ in $K[X_1, \dots, X_n, Y_1, \dots, Y_n]$. By 1.3 this scheme is smooth over K if and only if X/K is.

In the following sections we are interested in the properties of $T_{X/Y}$ when the fibres of the scheme X/Y have singularities.

2. Dimension formulas. Asssume X/Y is locally of finite type. For $x \in X$, $y = f(x)$ let $\alpha_x : \mathrm{pr}^{-1}(x) \to T_{X/Y}$ be the natural morphism. We identify $\mathrm{im}\alpha_x$ set-theoretically with $\mathrm{pr}^{-1}(x)$. Since

$$\mathrm{pr}^{-1}(x) = \mathrm{Spec}\,\mathbf{S}_{k(x)}(\Omega^1_{\mathcal{O}_x/\mathcal{O}_y} \otimes_{\mathcal{O}_x} k(x)) \cong \mathbf{A}^{\mu_x}_{k(x)} \text{ with } \mu_x = \mu(\Omega^1_{\mathcal{O}_x/\mathcal{O}_y})$$

is an irreducible subset of $T_{X/Y}$ its topological closure $\overline{\mathrm{pr}^{-1}(x)}$ has a unique generic point, which we denote by $t(x)$.

Let $U = \mathrm{Spec}\,B$ be an affine neighbourhood of x such that $f(U)$ is contained in an affine subset $V = \mathrm{Spec}\,A$ of Y. Let $\mathfrak{p} \in \mathrm{Spec}\,B$ be the prime ideal corresponding to x and set $S := \mathbf{S}_B(\Omega^1_{B/A})$. Then $\mathrm{pr}^{-1}(x) \cong \mathrm{Spec}\,S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$ and $t(x)$ is the point of $T_{X/Y}$ corresponding to the homogeneous prime ideal

$$t(\mathfrak{p}) := \mathfrak{p}S_\mathfrak{p} \cap S$$

of S. Obviously $t(\mathfrak{p}) \cap B = \mathfrak{p}$, hence

$$\text{(1)} \qquad \mathrm{pr}(t(x)) = x \quad \text{and} \quad \mathrm{pr}(\overline{\mathrm{pr}^{-1}(x)}) = \overline{\{x\}}.$$

If x is a maximal point of X, i.e. $\mathfrak{p}$ a maximal ideal of B, then $t(\mathfrak{p}) = \mathfrak{p}S$ since $S/\mathfrak{p}S \to S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$ is injective, $S/\mathfrak{p}S$ being a polynomial algebra over $B/\mathfrak{p}$. In this case $\mathrm{pr}^{-1}(x)$ is already a closed subset of $T_{X/Y}$.

PROPOSITION 2.1. *Let Y be locally noetherian and of finite dimension. Assume that X/Y is locally of finite type. Then*

$$\dim \overline{\mathrm{pr}^{-1}(x)} = \dim \overline{\{x\}} + \mu_x.$$

PROOF. With the above notations we have to determine $\dim S/t(\mathfrak{p})$ where A is a noetherian ring of finite Krull dimension and B/A an algebra of finite type. Here $S/t(\mathfrak{p})$ is a graded integral domain of finite type over $B/\mathfrak{p}$ whose component of degree zero is $B/\mathfrak{p}$. With $L := Q(S/t(\mathfrak{p}))$ and $K := Q(B/\mathfrak{p})$ by a general dimension formula ([10], 1.1.2)

$$\dim S/t(\mathfrak{p}) = \dim B/\mathfrak{p} + \mathrm{Trdeg}(L/K).$$

Since $S/t(\mathfrak{p}) \otimes_{B/\mathfrak{p}} K = \mathbf{S}(\Omega^1_{B_\mathfrak{p}/A} \otimes_{B_\mathfrak{p}} K)$ is a polynomial algebra over K in $\mu_x = \mu(\Omega^1_{B_\mathfrak{p}/A})$ variables we have $\mu_x = \mathrm{Trdeg}(L/K)$. Since $\dim B/\mathfrak{p} = \dim \overline{\{x\}}$ the formula of the proposition follows.

PROPOSITION 2.2. *a) If z is the generic point of an irreducible component of $T_{X/Y}$, then $z = t(x)$ for some $x \in X$. b) If ξ is the generic point of an irreducible component of X, then $\overline{\{t(\xi)\}} = \overline{\mathrm{pr}^{-1}(\xi)}$ is an irreducible component of $T_{X/Y}$.*

PROOF. We have $T_{X/Y} = \bigcup_{x\in X} \mathrm{pr}^{-1}(x) = \bigcup_{x\in X} \overline{\mathrm{pr}^{-1}(x)}$.

a) Since $z \in \mathrm{pr}^{-1}(x)$ for some $x \in X$ it is clear that $\overline{\{z\}} = \overline{\mathrm{pr}^{-1}(x)}$, hence $z = t(x)$.

b) With the notations as above let $\mathfrak{p} \in \operatorname{Spec} B$ be the minimal prime corresponding to ξ. Let $\mathfrak{q} \subset t(\mathfrak{p})$ be a prime ideal of S. Then $\mathfrak{q} \cap B \subset t(\mathfrak{p}) \cap B = \mathfrak{p}$, hence $\mathfrak{q} \cap B = \mathfrak{p}$. The point of $T_{X/Y}$ corresponding to $\mathfrak{q}$ belongs to $\overline{\mathrm{pr}^{-1}(\xi)} = \overline{\{t(\xi)\}}$, i.e. $\mathfrak{q} \supset t(\mathfrak{p})$. This implies that $t(\mathfrak{p})$ is a minimal prime of S and $\overline{\{t(\xi)\}}$ an irreducible component of $T_{X/Y}$.

In general it is difficult to decide for which $x \in X$ the set $\overline{\mathrm{pr}^{-1}(x)}$ is an irreducible component of $T_{X/Y}$, see [7], section 3, and the later example 3.4. But for smooth points of X/Y we can apply the next lemma.

LEMMA 2.3. *For $x \in X$, $y = f(x)$ assume that $\Omega^1_{\mathcal{O}_x/\mathcal{O}_y}$ is a free $\mathcal{O}_x$-module. Then $\overline{\mathrm{pr}^{-1}(x)}$ is an irreducible component of $T_{X/Y}$ if and only if x is the generic point of an irreducible component of X.*

PROOF. Let U, V, S and $\mathfrak{p}$ be as above. Let ξ be the generic point of an irreducible component which contains x, and $\mathfrak{q} \in \operatorname{Spec} B$ the prime ideal corresponding to ξ. Set $\eta := f(\xi)$.

Then $S_\mathfrak{p} = \mathbf{S}_{\mathcal{O}_x}(\Omega^1_{\mathcal{O}_x/\mathcal{O}_y})$ and $S_\mathfrak{q} = \mathbf{S}_{\mathcal{O}_\xi}(\Omega^1_{\mathcal{O}_\xi/\mathcal{O}_\eta})$ are polynomial algebras over $B_\mathfrak{p}$ resp. $B_\mathfrak{q}$ in the same number of variables and the canonical map $S_\mathfrak{p}/\mathfrak{q}S_\mathfrak{p} \to S_\mathfrak{q}/\mathfrak{q}S_\mathfrak{q}$ is injective. This implies $\mathfrak{q}S_\mathfrak{q} \cap S_\mathfrak{p} = \mathfrak{q}S_\mathfrak{p} \subset \mathfrak{p}S_\mathfrak{p}$, hence $t(\mathfrak{q}) = \mathfrak{q}S_\mathfrak{q} \cap S \subset \mathfrak{p}S_\mathfrak{p} \cap S = t(\mathfrak{p})$ and

$$\overline{\mathrm{pr}^{-1}(\xi)} \supset \overline{\mathrm{pr}^{-1}(x)}$$

with equality if and only if $t(\xi) = t(x)$ i.e. if and only if $\xi = \mathrm{pr}(t(\xi)) = \mathrm{pr}(t(x)) = x$.

Prop. 2.1 and 2.2 immediately imply the following theorem which was first proved for fibre bundles of arbitrary coherent sheaves by Huneke-Rossi with analogous arguments ([7], Thm 2.6).

THEOREM 2.4. *Let Y be a noetherian scheme of finite dimension, and let X/Y be of finite type. Then*

$$\dim T_{X/Y} = \operatorname*{Max}_{x\in X}\{\dim \overline{\{x\}} + \mu(\Omega^1_{\mathcal{O}_x/\mathcal{O}_y})\}.$$

We call for a coherent $\mathcal{O}_X$-module $\mathcal{G}$

$$f_{\mathcal{G}} := \operatorname*{Max}_{x\in X}\{\dim \overline{\{x\}} + \mu(\mathcal{G}_x)\}$$

the *Forster number* of $\mathcal{G}$. If $X = \operatorname{Spec} B$ with a noetherian ring B of finite Krull dimension then by a theorem of Forster ([5], Satz 1) the B-module $\Gamma(X,\mathcal{G})$ can be generated with $f_{\mathcal{G}}$ global sections. Theorem 2.4 states that $\dim T_{X/Y} = f_{\Omega^1_{X/Y}}$. In the following we will describe $f_{\Omega^1_{X/Y}}$ more explicitely.

Let $(R,\mathfrak{m})$ and $(S,\mathfrak{n})$ be noetherian local rings with residue fields K resp. L. Let $R \to S$ be a local homomorphism and assume S/R is essentially of finite type. If $\operatorname{Char} K = p > 0$ we set $\mathfrak{m}' := \mathfrak{m} \cap R[S^p]$ and denote by $\deg_p(L/K)$ the p-degree of L/K. Further edim denotes the embedding dimension of noetherian local rings.

DEFINITION 2.5. The number

$$\operatorname{insep}(S/R) := \deg_p(L/K) - \operatorname{Trdeg}(L/K) - (\operatorname{edim} S/\mathfrak{m}S - \operatorname{edim} S/\mathfrak{m}'S)$$

is called the *inseparability* of S/R. We set $\operatorname{insep}(S/R) = 0$, if $\operatorname{Char} K = 0$.

LEMMA 2.6. *We always have* $\operatorname{insep}(S/R) \geq 0$, *with equality when* L/K *is a separable field extension.*

PROOF. It suffices to consider the case where $\operatorname{Char} K = p > 0$. By [8], 6.4 and 6.7a) there is a commutative diagram with exact rows

$$\begin{array}{ccccccccccc}
T & \to & T(L/\delta_K) & \to & \mathfrak{n}/\mathfrak{n}^2 + \mathfrak{m}S & \to & \Omega^1_{S/R}/\mathfrak{n}\Omega^1_{S/R} & \to & \Omega^1_{L/K} & \to & 0 \\
 & & & & \downarrow & & \| & & \| & & \\
 & & 0 & \to & \mathfrak{n}/\mathfrak{n}^2 + \mathfrak{m}'S & \to & \Omega^1_{S/R}/\mathfrak{n}\Omega^1_{S/R} & \to & \Omega^1_{L/K} & \to & 0.
\end{array}$$

Here $\dim_L T(L/\delta_K) = \deg_p(L/K) - \operatorname{Trdeg}(L/K)$ by Cartier's equality ([8], 5.12b)), hence the diagram shows

$$\operatorname{edim} S/\mathfrak{m}S \leq \operatorname{edim} S/\mathfrak{m}'S + (\deg_p(L/K) - \operatorname{Trdeg}(L/K)),$$

which implies $\operatorname{insep}(S/R) \geq 0$.

On the other hand, the separability of L/K implies $\operatorname{edim} S/\mathfrak{m}S = \operatorname{edim} S/\mathfrak{m}'S$ ([8], 6.5a)), hence $\operatorname{insep}(S/R) = 0$.

We call the number

$$\delta(S/R) := \operatorname{edim} S/\mathfrak{m}S - \dim S/\mathfrak{m}S$$

the *regularity defect* of S/R. The following is a reformulation of [8], 6.5b) resp. 6.7b):

PROPOSITION 2.7. *Under the assumptions of 2.6*

$$\mu(\Omega^1_{S/R}) = \delta(S/R) + \dim S/\mathfrak{m}S + \operatorname{Trdeg}(L/K) + \operatorname{insep}(S/R).$$

For X/Y as in Section 1 assume that X is locally of finite type over Y. For $x \in X$, $y = f(x)$ let

$$\delta_x(X/Y) := \operatorname{edim} \mathcal{O}_x/\mathfrak{m}_y\mathcal{O}_x - \dim \mathcal{O}_x/\mathfrak{m}_y\mathcal{O}_x$$

be the regularity defect of $\mathcal{O}_x/\mathcal{O}_y$. We call it also the regularity defect of X/Y at x. Since $\mathcal{O}_x/\mathfrak{m}_y\mathcal{O}_x$ is a localization of an affine $k(x)$-algebra we have

$$\operatorname{Trdeg}(k(x)/k(y)) = \dim_x(f^{-1}(y)) - \dim \mathcal{O}_x/\mathfrak{m}_y\mathcal{O}_x. \tag{2}$$

Therefore 2.7 implies

COROLLARY 2.8. $\dim_{k(x)} T_{X/Y}(x) = \mu(\Omega^1_{\mathcal{O}_x/\mathcal{O}_y})$

$$= \delta_x(X/Y) + \dim_x f^{-1}(y) + \operatorname{insep} \mathcal{O}_x/\mathcal{O}_y.$$

Now 2.1 and 2.4 yield

THEOREM 2.9. *Let Y be noetherian of finite dimension and X/Y of finite type.*
a) For all $x \in X$, $y = f(x)$

$$\dim \overline{\operatorname{pr}^{-1}(x)} = \delta_x(X/Y) + \dim \overline{\{x\}} + \dim_x f^{-1}(y) + \operatorname{insep} \mathcal{O}_x/\mathcal{O}_y,$$

b) $\dim T_{X/Y} = f_{\Omega^1_{X/Y}} = \operatorname*{Max}_{x\in X}\{\delta_x(X/Y) + \dim\overline{\{x\}} + \dim_x f^{-1}(y) + \operatorname{insep} \mathcal{O}_x/\mathcal{O}_y\}.$

Of course the irreducible components of $T_{X/Y}$ of maximal dimension are the sets $\overline{\operatorname{pr}^{-1}(x)}$ for those x for which the number $\delta_x(X/Y) + \dim\overline{\{x\}} + \dim_x f^{-1}(y) + \operatorname{insep} \mathcal{O}_x/\mathcal{O}_y$ is maximal.

Suppose now that X and Y are integral schemes and $f : X \to Y$ is dominant. Let $\mathcal{R}(Y) \subset \mathcal{R}(X)$ be the fields of rational functions on X resp. Y. Taking for x the generic point ξ of X, hence for y the generic point η of Y, we see that

$$\dim T_{X/Y} \geq \dim X + e + \operatorname{insep} \mathcal{R}(X)/\mathcal{R}(Y), \tag{3}$$

where the expression on the right is $\dim \overline{\operatorname{pr}^{-1}(\xi)}$ and $e := \dim f^{-1}(\eta)$ is the dimension of the generic fibre of X/Y. Since $\overline{\operatorname{pr}^{-1}(\xi)}$ is an irreducible component of $T_{X/Y}$ we obtain

COROLLARY 2.10. *If X and Y are integral schemes and f is dominant the following assertions are equivalent:*
a) $T_{X/Y}$ is equidimensional.
b) For all $x \in X$, $y = f(x)$

$$\delta_x(X/Y) + (\dim_x f^{-1}(y) - e) + \operatorname{insep} \mathcal{O}_x/\mathcal{O}_y \\ \leq \dim X - \dim \overline{\{x\}} + \operatorname{insep} \mathcal{R}(X)/\mathcal{R}(Y),$$

where the equality sign holds for exactly those $x \in X$ for which $\overline{\mathrm{pr}^{-1}(x)}$ is an irreducible component of $T_{X/Y}$.

Here $\dim_x f^{-1}(y) \geq e$ for all $x \in X$ ([1], IV, 13.1.6). If X is equicodimensional and catenarian, then $\dim X - \dim \overline{\{x\}}$ can be replaced by $\dim \mathcal{O}_x$, and if in this case f is an equidimensional morphism (i.e. $\dim_x f^{-1}(y) = e$ for all $x \in X$) then the formula of 2.10b) becomes

$$\delta_x(X/Y) + \operatorname{insep} \mathcal{O}_x/\mathcal{O}_y \leq \dim \mathcal{O}_x + \operatorname{insep} \mathcal{R}(X)/\mathcal{R}(Y). \tag{4}$$

3. Tangent bundles of algebraic K-schemes. Let X be an algebraic scheme over a field K. We write $\delta_x(X)$ for the regularity defect $\delta_x(X/K) = \delta(\mathcal{O}_x/K)$ at $x \in X$. Then, since $\dim \overline{\{x\}} = \dim_x X - \dim \mathcal{O}_x$, we have by 2.9b)

$$\dim T_{X/K} = \operatorname*{Max}_{x \in X}\{2 \dim_x X + \delta_x(X) + \operatorname{insep} \mathcal{O}_x/K - \dim \mathcal{O}_x\} \tag{5}$$

and, when X is equidimensional,

$$\dim T_{X/K} = 2 \dim X + \operatorname*{Max}_{x \in X}\{\delta_x(X) + \operatorname{insep} \mathcal{O}_x/K - \dim \mathcal{O}_x\}, \tag{6}$$

where this maximum is greater or equal 0. Thus if X is equidimensional $\dim T_{X/K} = 2 \dim X$ holds if and only if

$$\delta_x(X) + \operatorname{insep} \mathcal{O}_x/K \leq \dim \mathcal{O}_x.$$

In particular, for the generic points ξ of the irreducible components of X it is necessary that $\mathcal{O}_\xi$ is a separable extension field of K.

EXAMPLE 3.1. Let X be a zero-dimensional algebraic K-scheme with points $z_1, \ldots, z_s$, hence with irreducible components $X_i = \{z_i\}$ $(i = 1, \ldots, s)$. Then

$$T_{X/K} = T_{X_1/K} \cup \cdots \cup T_{X_s/K},$$

where $T_{X_i/K} = \overline{\mathrm{pr}^{-1}(z_i)}$ are the irreducible components of $T_{X/K}$. They have dimension

$$\operatorname{edim} \mathcal{O}_{z_i} + \operatorname{insep} \mathcal{O}_{z_i}/K \qquad (i = 1, \dots, s).$$

If $\operatorname{Char} K = p > 0$ a field K_0 with $K^p \subset K_0 \subset K$, $[K : K_0] < \infty$ is called *admissible* for an algebraic K-scheme X ([8],6.19) if for all $x \in X$

$$(7) \qquad \mu(\Omega^1_{\mathcal{O}_x/K_0}) = \operatorname{edim} \mathcal{O}_x + \operatorname{Trdeg} k(x)/K + \deg_p(K/K_0).$$

LEMMA 3.2. *A field K_0 with $K^p \subset K_0 \subset K$, $[K : K_0] < \infty$ is admissible for X if and only if*

$$\operatorname{insep} \mathcal{O}_x/K_0 = \deg_p(K/K_0)$$

for all $x \in X$.

PROOF. Compare the general formula ([8], 6.7b)

$$(8) \qquad \mu(\Omega^1_{\mathcal{O}_x/K_0}) = \operatorname{edim} \mathcal{O}_x/\mathfrak{m}'\mathcal{O}_x + \deg_p(k(x)/K_0),$$

where $\mathfrak{m}' := \mathfrak{m}_x \cap K_0[\mathcal{O}_x^p]$ with formula (7). Then, since $\operatorname{Trdeg} k(X)/K_0 = \operatorname{Trdeg} k(X)/K$, the assertion follows from the definition of the inseparability.

By [8], 6.24 for each algebraic K-scheme X there exists an admissible field K_0. We set $K_0 = K$ when K has characteristic 0. The reason for introducing the notion of admissible field is that assertions about Kaehler differentials which can be made for a perfect ground field K remain true, if X/K is replaced by X/K_0.

If we write $r = \deg_p(K/K_0)$ when $\operatorname{Char} K = p > 0$ and $r = 0$ when $\operatorname{Char} K = 0$, then we obtain by (5) and (6)

$$(5') \qquad \dim T_{X/K_0} = \operatorname*{Max}_{x \in X}\{2 \dim_x X + \delta_x(X) - \dim \mathcal{O}_x\} + r$$

and when X is equidimensional

$$(6') \qquad \dim T_{X/K_0} = 2 \dim X + r + \operatorname*{Max}_{x \in X}\{\delta_x(X) - \dim \mathcal{O}_x\}.$$

Here $\dim T_{X/K_0} = 2 \dim X + r$ if and only if $\delta_x(X) \leq \dim \mathcal{O}_x$ for all $x \in X$, i.e. if the singularities of X are not "too bad" in terms of the regularity defect.

PROPOSITION 3.3. *Assume X is reduced and equidimensional, and let K_0 be admissible for X/K. Then the following assertions are equivalent:*

a) T_{X/K_0} is equidimensional.

b) For all $x \in X$ we have $\delta_x(X) \leq \dim \mathcal{O}_x$, where equality holds for exactly those $x \in X$ for which $\overline{\mathrm{pr}^{-1}(x)}$ is an irreducible component of X.

Moreover, T_{X/K_0} is regular (i.e. all its local rings are regular) if and only if X is.

PROOF. If ξ is the generic point of an irreducible component of X, then $\overline{\mathrm{pr}^{-1}(\xi)}$ is an irreducible component of T_{X/K_0} (2.2). By 2.1

$$\dim \overline{\mathrm{pr}^{-1}(\xi)} = 2\dim X + r$$

and (6') shows that a) and b) are equivalent.

By [8], 7.5 the local ring $\mathcal{O}_x$ of $x \in X$ is regular if and only if $\Omega^1_{\mathcal{O}_x/K_0}$ is free. The assertion about regularity follows from lemma 1.4.

EXAMPLE 3.4. Let $X = \operatorname{Spec} B$ where B is a K-algebra of finite type and a domain with quotient field L. Let $x \in X$ be a closed point and $\mathfrak{p}$ the maximal ideal of B corresponding to X. Assume x is the only singularity of X. Further let ξ be the generic point of X, and let K_0 be an admissible field for X.

If $\delta_x(X) > \dim \mathcal{O}_x$, then T_{X/K_0} is not equidimensional by 3.3. Assume now that $\delta_x(X) \leq \dim \mathcal{O}_x$. Write $S := \mathbf{S}_B(\Omega^1_{B/K_0})$. Then $t(x)$ corresponds to the prime ideal $\mathfrak{p}S$ and $t(\xi)$ to the kernel of $S \to S \otimes_K L$ which is also the torsion of S as a B-module.

Thus we have

$$\overline{\mathrm{pr}^{-1}(x)} \subset \overline{\mathrm{pr}^{-1}(\xi)},$$

if and only if for all $i \in \mathbf{N}_+$ the torsion of $\mathbf{S}_i(\Omega^1_{B/K_0})$ is contained in $\mathfrak{p}\mathbf{S}_i(\Omega^1_{B/K_0})$. Since $\Omega^1_{\mathcal{O}_z/K_0}$ is free for each $z \in X$, $z \neq x$, this is equivalent to

$$\operatorname{Torsion} \mathbf{S}_i(\Omega^1_{\mathcal{O}_x/K_0}) \subset \mathfrak{m}_x \mathbf{S}_i(\Omega^1_{\mathcal{O}_x/K_0}).$$

In order that T_{X/K_0} be equidimensional this condition must be satisfied when $\delta_x(X) < \dim \mathcal{O}_x$. Unfortunately, questions about the torsion of differential modules and their symmetric powers are in general difficult to decide.

4. Irreducibility of the tangent bundle. For results about irreduciblity and integrality of symmetric algebras, see [9], [10]. Here we want to show

THEOREM 4.1. *Let X and Y are integral schemes where Y is noetherian and $f : X \to Y$ is of finite type, dominant and equidimensional. Further assume that X is equicodimensional and catenarian, i.e.* $\dim X = \dim \overline{\{x\}} + \dim \mathcal{O}_x$ *for all $x \in X$. Then the following conditions are equivalent:*

a) $T_{X/Y}$ is irreducible.

b) $T_{X/Y}$ is equidimensional and for all $x \in X$ which are different from the generic point of X

$$\delta_x(X/Y) + \operatorname{insep} \mathcal{O}_x/\mathcal{O}_y < \dim \mathcal{O}_x + \operatorname{insep} \mathcal{R}(X)/\mathcal{R}(Y).$$

PROOF. Let ξ be the generic point of X. Since $\overline{\mathrm{pr}^{-1}(\xi)}$ is always an irreducible component of $T_{X/Y}$ the tangent bundle is irreducible if and only if $T_{X/Y} = \overline{\mathrm{pr}^{-1}(\xi)}$. The assertion follows from 2.10 in connection with formula (4).

COROLLARY 4.2. *Let X be an integral algebraic scheme over a field K, and let $K_0 \subset K$ be admissible for X/K. Then T_{X/K_0} is irreducible if and only if T_{X/K_0} is equidimensional and*

$$\delta_x(X) < \dim \mathcal{O}_x$$

for each singular point x of X.

PROOF. Being an integral algebraic K-scheme X is certainly equicodimensional and catenarian. Further $\operatorname{insep} \mathcal{O}_x/K_0 = \operatorname{insep} \mathcal{R}(X)/K_0$ by 3.2. The assertion follows from 4.1 since condition 4.1b) is obviously satisfied for regular points x of X which are different from the generic point of X.

COROLLARY 4.3. *Under the assumptions of 4.2 let* $\dim X = 1$. *Then the following assertions are equivalent:*

a) T_{X/K_0} is irreducible.

b) X is regular.

c) $\mathbf{S}_i(\Omega^1_{\mathcal{O}_X/K_0})$ is a torsion free $\mathcal{O}_X$-module for all $i \in \mathbf{N}_+$.

d) T_{X/K_0} is integral.

PROOF. a)$\to$b). When T_{X/K_0} is irreducible, then $\delta_x(X) < \dim \mathcal{O}_x \le 1$ implies that all $x \in X$ are regular points.

b)$\to$c) follows from the fact that $\Omega^1_{\mathcal{O}_X/K_0}$ is a locally free $\mathcal{O}_X$-module ([8], 7.5).

c)$\to$d) follows from 2.1.

REMARK 4.4. If K is a perfect field Berger's problem ([3]) asks whether it is enough to require instead of c) that $\Omega^1_{X/K}$ be torsion free in order that X is regular. A survey with positive results to this question is given in [4].

5. Tangent bundles of local complete intersections. Let $f : X \to Y$ be a morphism of schemes where X is locally noetherian. We say that X/Y is *locally a complete intersection*, if X/Y is flat and for each $x \in X$, $y = f(x)$ the local ring $\mathcal{O}_x/\mathfrak{m}_y\mathcal{O}_x$ is a complete intersection, i.e. its completion is a homomorphic image of a regular local ring modulo an ideal generated by a regular sequence. In this section we consider the following situation.

ASSUMPTIONS 5.1. Y is a regular noetherian scheme, X is reduced and X/Y is locally of finite type and locally a complete intersection. For each generic point ξ of an irreducible component of X and $\eta = f(\xi)$ the field extension $\mathcal{O}_\xi/\mathcal{O}_\eta$ is separable.

Notice that η is the generic point of an irreducible component of Y, since X/Y is flat. Hence $\mathcal{O}_\eta$ is a field.

Let $y \in f(X)$ and let $x \in X$ be a point which is closed on the fibre $f^{-1}(y)$. Set $\overline{\mathcal{O}_x} := \mathcal{O}_x/\mathfrak{m}_y\mathcal{O}_x$. Then we have a presentation

$$\overline{\mathcal{O}_x} = k(y)[X_1, \dots, X_n]_\mathfrak{m}/(\overline{f_1}, \dots, \overline{f_m})_\mathfrak{m} \tag{9}$$

with a maximal ideal $\mathfrak{m}$ of $k(y)[X_1, \dots, X_n]$ and a regular sequence $\{\overline{f_1}, \dots, \overline{f_m}\}$ in $k(y)[X_1, \dots, X_n]_\mathfrak{m}$ consisting of polynomials $\overline{f_i} \in k(y)[X_1, \dots, X_n]$. If x is a specialization of $x' \in f^{-1}(y)$, hence $\overline{\mathcal{O}_{x'}}$ is a localization of $\overline{\mathcal{O}_x}$, then

$$n - m = \dim \overline{\mathcal{O}_{x'}} + \operatorname{Trdeg} k(x')/k(y) \tag{10}$$

in particular, $n - m = \dim \overline{\mathcal{O}_x}$.

Since $\mathcal{O}_y$ is regular we have

$$\mathcal{O}_x = \mathcal{O}_y[X_1, \dots, X_n]_\mathfrak{M}/(f_1, \dots, f_m)_\mathfrak{M}$$

with a regular sequence $\{f_1, \dots, f_m\}$ in $\mathcal{O}_y[X_1, \dots, X_n]_\mathfrak{M}$, the f_i being liftings of the $\overline{f_i}$ in $\mathcal{O}_y[X_1, \dots, X_n]$ and $\mathfrak{M}$ a maximal ideal of $\mathcal{O}_y[X_1, \dots, X_n]$ lying over $\mathfrak{m}_y$. For a generic point ξ of an irreducible component of X containing x and $\eta = f(\xi)$ we obtain

$$n - m = \operatorname{Trdeg} \mathcal{O}_\xi/\mathcal{O}_\eta. \tag{11}$$

Set $I := (f_1, \dots, f_m)_{\mathfrak{M}}$. Then the sequence

$$0 \to I/I^2 \to \Omega^1_{\mathcal{O}_y[X_1,\dots,X_n]_{\mathfrak{M}}/\mathcal{O}_y}/I\Omega^1_{\mathcal{O}_y[X_1,\dots,X_n]_{\mathfrak{M}}/\mathcal{O}_y} \to \Omega^1_{\mathcal{O}_x/\mathcal{O}_y} \to 0$$

is exact since I/I^2 is a free $\mathcal{O}_x$-module of rank m and the localization of the sequence with respect to the minimal primes of $\mathcal{O}_x$ are exact by the separability assumption of 5.1. Thus we have an exact sequence

$$0 \to \mathcal{O}_x^m \xrightarrow{A} \mathcal{O}_x^n \to \Omega^1_{\mathcal{O}_x/\mathcal{O}_y} \to 0,$$

where $A = [a_{ik}]_{\substack{i=1,\dots,m\\k=1,\dots,n}}$ is the Jacobian matrix of $f_1, \dots, f_m$, i.e. $a_{ik} = \frac{\partial f_i}{\partial x_k}$ is the image of the partial derivative in $\mathcal{O}_x$. Let $\ell_i := \sum_{k=1}^{n} a_{ik} Y_k$ $(i = 1, \dots, m)$. Then

$$S := \mathbf{S}_{\mathcal{O}_x}(\Omega^1_{\mathcal{O}_x/\mathcal{O}_y}) = \mathcal{O}_x[Y_1, \dots, Y_n]/(\ell_1, \dots, \ell_m). \tag{12}$$

The next theorem follows from a result of Avramov ([2], Prop.1). For more in this direction, see also [12].

THEOREM 5.2. *Under the assumptions 5.1 the following assertions are equivalent:*

a) $T_{X/Y}/Y$ is locally a complete intersection.

b) For all $x \in X$, $y = f(x)$

$$\delta_x(X/Y) + \operatorname{insep} \mathcal{O}_x/\mathcal{O}_y \leq \dim \mathcal{O}_x/\mathfrak{m}_y\mathcal{O}_x.$$

PROOF. Assume $T_{X/Y}/Y$ is locally a complete intersection. Then $T_{X/Y}$ is flat over Y and all localizations of $S/\mathfrak{m}_y S$ are complete intersections. Since $\mathcal{O}_y$ is regular, the ideal $\mathfrak{m}_y S$ is generated by a regular sequence in S, hence also the localizations of S are complete intersections.

For x as above let $\mathfrak{P} := \mathfrak{m}_x \oplus S_+$ and $\mathfrak{Q} := \mathfrak{m}_x + (Y_1, \dots, Y_n)$, its preimage in $\mathcal{O}_x[Y_1, \dots, Y_n]$. Then since

$$S_{\mathfrak{P}} = \mathcal{O}_x[Y_1, \dots, Y_n]_{\mathfrak{Q}}/(\ell_1, \dots, \ell_m)_{\mathfrak{Q}}$$

is a complete intersection, the ideal $(\ell_1, \dots, \ell_m)_{\mathfrak{Q}}$ is generated by a regular sequence of $\mathcal{O}_x[Y_1, \dots, Y_n]_{\mathfrak{Q}}$. Moreover, $S_{\mathfrak{P}}$ is equidimensional.

Let pr_x be the projection of the tangent bundle of $\operatorname{Spec} \mathcal{O}_x/\operatorname{Spec} \mathcal{O}_y$. Then for the generic point ξ of an irreducible component of $\operatorname{Spec} \mathcal{O}_x$ we have by 2.1 $\dim \operatorname{pr}_x^{-1}(\xi) = \dim \overline{\{\xi\}} + \operatorname{Trdeg} \mathcal{O}_\xi/\mathcal{O}_\eta = \dim \mathcal{O}_x + n - m$ with $\eta = f(\xi)$. Thus

$(\ell_1,\dots,\ell_m)_{\mathfrak{Q}}$ is an ideal of height m and $\{\ell_1,\dots,\ell_m\}$ a regular sequence in $\mathcal{O}_x[Y_1,\dots,Y_n]_{\mathfrak{Q}}$. Let $\overline{\ell_i} = \sum_{k=1}^{n} \overline{a_{ik}} Y_k$ be the image of ℓ_i in $\overline{\mathcal{O}_x}[Y_1,\dots,Y_n]_{\overline{\mathfrak{Q}}}$ where $\overline{\mathfrak{Q}}$ denotes the image of $\mathfrak{Q}$ in $\overline{\mathcal{O}_x}[Y_1,\dots,Y_n]$ and $\overline{a_{ik}}$ the image of a_{ik} in $\overline{\mathcal{O}_x}$. It follows that $\{\overline{\ell_1},\dots,\overline{\ell_m}\}$ is a regular sequence in $\overline{\mathcal{O}_x}[Y_1,\dots,Y_n]_{\overline{\mathfrak{Q}}}$. But then $\{\overline{\ell_1},\dots,\overline{\ell_m}\}$ is already a regular sequence in $\overline{\mathcal{O}_x}[Y_1,\dots,Y_n]$, the $\overline{\ell_i}$ being homogeneous polynomials of positive degree.

Conversely, assume that this is the case. Then for each $x' \in f^{-1}(y)$ of which x is a specialization and each $z \in T_{X/Y}$ lying over x' the local ring $\mathcal{O}_z$ of z on $T_{X/Y}$ is flat over $\mathcal{O}_y$ and $\mathcal{O}_z/\mathfrak{m}_y\mathcal{O}_z$ is a complete intersection.

Thus in order that $T_{X/Y}/Y$ be locally a complete intersection it is necessary and sufficient that for each pair (x,y) as in formula (9) the sequence $\{\overline{\ell_1},\dots,\overline{\ell_m}\}$ is regular in $\overline{\mathcal{O}_x}[Y_1,\dots,Y_n]$. Remember that

$$\Omega^1_{\overline{\mathcal{O}_x}/k(y)} \cong \bigoplus_{k=1}^{n} \overline{\mathcal{O}_x} \cdot Y_k / \langle \overline{\ell_1},\dots,\overline{\ell_m}\rangle .$$

Let $\overline{A} = [\overline{a_{ik}}]_{\substack{i=1,\dots,m\\ k=1,\dots,n}}$ and let I_s be the ideal generated by the s-minors of $\overline{A}$, in other words: the Fitting ideal of order $n-s$ of $\Omega^1_{\overline{\mathcal{O}_x}/k(y)}$ or Kaehler different $\vartheta_{n-s}(\overline{\mathcal{O}_x}/k(y))$.

By [2], Prop.1 the following assertions are equivalent:

1) $\{\overline{\ell_1},\dots,\overline{\ell_m}\}$ is a regular sequence in $\overline{\mathcal{O}_x}[Y_1,\dots,Y_n]$.

2) $(I_s)_{\mathfrak{p}} = (\overline{\mathcal{O}_x})_{\mathfrak{p}}$ for each $\mathfrak{p} \in \operatorname{Spec}\overline{\mathcal{O}_x}$ with $\dim(\overline{\mathcal{O}_x})_{\mathfrak{p}} = m - s$ $(s = 1,\dots,m)$.

Notice that the height and the grade of an ideal in $\overline{\mathcal{O}_x}$ are the same, since $\overline{\mathcal{O}_x}$ is a Cohen-Macaulay ring. It is easily seen ([8], D.9) that 1) and 2) are also equivalent to

3) $\mu((\Omega^1_{\overline{\mathcal{O}_x}/k(y)})_{\mathfrak{p}}) \le n - m + \dim(\overline{\mathcal{O}_x})_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec}\overline{\mathcal{O}_x}$ with $\dim(\overline{\mathcal{O}_x})_{\mathfrak{p}} < m$.

Let $x' \in X$ be the point corresponding to $\mathfrak{p} \in \operatorname{Spec}\overline{\mathcal{O}_x}$. Then $(\overline{\mathcal{O}_x})_{\mathfrak{p}} = \overline{\mathcal{O}_{x'}}$ and by 2.8 and (10)

$$\begin{aligned}\mu((\Omega^1_{\overline{\mathcal{O}_x}/k(y)})_{\mathfrak{p}}) &= \mu(\Omega^1_{\mathcal{O}_{x'}/\mathcal{O}_y})\\ &= \delta_{x'}(X/Y) + \dim \mathcal{O}_{x'}/\mathfrak{m}_y\mathcal{O}_{x'} + \operatorname{Trdeg} k(x')/k(y) + \operatorname{insep}\mathcal{O}_{x'}/\mathcal{O}_y\\ &= \delta_{x'}(X/Y) + n - m + \operatorname{insep}\mathcal{O}_{x'}/\mathcal{O}_y .\end{aligned}$$

Since $\mu(\Omega^1_{\mathcal{O}_{x'}/\mathcal{O}_y}) \le n$ by (12) we have for all x' with $\dim\overline{\mathcal{O}_{x'}} \ge m$,

$$\delta_{x'}(X/Y) + \operatorname{insep}\mathcal{O}_{x'}/\mathcal{O}_y \le n - (n-m) = m \le \dim\overline{\mathcal{O}_{x'}} .$$

For x' with $\dim \overline{\mathcal{O}_{x'}} < m$ the condition

$$\delta_{x'}(X/Y) + \operatorname{insep} \mathcal{O}_{x'}/\mathcal{O}_y \leq \dim \overline{\mathcal{O}_{x'}}$$

is equivalent to 3). That this condition holds for all $x' \in X$, $y = f(x')$ is therefore equivalent to $T_{X/Y}/Y$ being locally a complete intersection.

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Fakultät für Mathematik
Universität Regensburg
Universitäts Str. 31
D 93-040 Regensburg