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LOCAL HOMEOMORPHISMS IN $\mathbf{R}^2$

BY ARMEN EDIGARIAN

Abstract. Using uniformization theorem for Riemann surfaces and some simple argument we show the Factorization Theorem for local homeomorphism in $\mathbb{R}^2$. We apply obtained result to extend Fessler's theorem related to the Markus-Yamabe Stability Conjecture.

We denote by E the unit disk in $\mathbb{C}$. The main purpose of the paper is to show the following extension of Fessler's theorem related to the Markus-Yamabe Stability Conjecture ([**1**], [**2**]).

THEOREM 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a local homeomorphism. Assume that there exists a compact set $K \subset \mathbb{R}^2$ such that f is a C^1 local diffeomorphism in $\mathbb{R}^2 \setminus K$ and $f'(x)v \neq \lambda v$ for all $x \in \mathbb{R}^2 \setminus K$, $v \in \mathbb{R}^2 \setminus \{0\}$ and $\lambda > 0$ (i.e. $f'(x)$ has no real positive eigenvalues for any x in some neighborhood of infinity). Then f is injective.*

The proof of Theorem 1 is based on Fessler's results and the following Factorization Theorem, which seems to be well-known.

FACTORIZATION THEOREM. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a local homeomorphism. Then there exists a homeomorphism $F : U \to \mathbb{R}^2$, where $U = \mathbb{C}$ or $U = E$ such that $\tilde{f} := f \circ F : U \to \mathbb{R}^2 \cong \mathbb{C}$ is a local biholomorphism.*

It is well known that the unit disk is C^∞-diffeomorphic to $\mathbb{R}^2$, so from the Factorization Theorem we obtain immediately following.

COROLLARY 2. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a local homeomorphism. Then there exists a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\tilde{f} := f \circ F : \mathbb{R}^2 \to \mathbb{R}^2$ is a C^∞ local diffeomorphism.*

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PROOF OF THE FACTORIZATION THEOREM. We identify $\mathbb{R}^2$ with $\mathbb{C}$. Then we conclude that $f : \mathbb{R}^2 \to \mathbb{C}$ is a local homeomorphism. Using f we may introduce a structure of one dimensional complex manifold on $\mathbb{R}^2$. Then $\mathbb{R}^2$ becomes a simply connected Riemann surface. By the uniformization theorem $\mathbb{R}^2$ is biholomorphically equivalent to $\mathbb{C}$ or to the unit disk E. □

To prove Theorem 1 we use some notions and results from [**1**].

For a curve $\alpha : I \to \mathbb{R}^2 \setminus \{0\}$ ($I = (a,b)$, $-\infty \le a < b \le +\infty$) we introduce the angle function $\angle\alpha : I \to \mathbb{R}$ as follows. We identify $\mathbb{R}^2$ with $\mathbb{C}$ then $\alpha : I \to \mathbb{C} \setminus \{0\}$. Recall that $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is a covering. Hence there exists a lifting $\tilde{\alpha} : I \to \mathbb{C}$ such that $\exp\tilde{\alpha} = \alpha$. We put $\angle\alpha := \operatorname{Im}\tilde{\alpha}$. Note that $\angle\alpha(s_2) - \angle\alpha(s_1)$, $s_1, s_2 \in I$, are uniquely determined.

Recall the following results of Fessler ([**1**]).

THEOREM 3. *Let $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ be a local diffeomorphism. Assume that f is not injective. Then for any compact set K there exists a curve $\gamma \in C^1(\mathbb{R}, \mathbb{R}^2 \setminus K)$ with the following properties*

1. γ is injective, proper and regular (i.e. $\gamma'(s) = (\gamma_1'(s), \gamma_2'(s)) \neq 0, s \in \mathbb{R}$).

2. There exists an $\epsilon > 0$ such that for every $s_1 \le 0$ and $s_2 \ge 1$

$$\angle(f \circ \gamma)'(s_2) - \angle(f \circ \gamma)'(s_1) \ge 3\pi + \epsilon.$$

THEOREM 4. *Let $\gamma \in C^1(\mathbb{R}, \mathbb{R}^2)$ be injective, proper and regular. Then for every $\epsilon > 0$ there exist $s_1 \le 0$ and $s_2 \ge 1$ such that*

$$\angle\gamma'(s_2) - \angle\gamma'(s_1) < \pi + \epsilon.$$

PROOF OF THEOREM 1. By Corollary 2 there exists a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\tilde{f} := f \circ F : \mathbb{R}^2 \to \mathbb{R}^2$ is a C^∞ local diffeomorphism. Assume that $\tilde{f}$ is not injective. Then according to Theorem 3 there is a curve $\gamma \in C^1(\mathbb{R}, \mathbb{R}^2 \setminus F^{-1}(K))$ with the following properties:

1. γ is injective, proper and regular.
2. There is an $\epsilon > 0$ such that for every $s_1 \le 0$ and $s_2 \ge 1$

$$\angle(\tilde{f} \circ \gamma)'(s_2) - \angle(\tilde{f} \circ \gamma)'(s_1) \ge 3\pi + \epsilon.$$

Let us consider the curve $\tilde{\gamma} := F \circ \gamma$. Then $\tilde{\gamma}$ is injective, proper and regular. According to Theorem 4 there exist $s_1 \le 0$ and $s_2 \ge 1$ such that

$$\angle\tilde{\gamma}'(s_2) - \angle\tilde{\gamma}'(s_1) < \pi + \epsilon.$$

Let us consider the function $\phi(s) := \angle(f \circ \tilde{\gamma})'(s) - \angle\tilde{\gamma}'(s)$. Then ϕ is a continuous function and $\phi(s_2) - \phi(s_1) \ge 2\pi$. Therefore, there exist $k \in \mathbb{Z}$ and $s_0 \in [s_1, s_2]$ such that $\phi(s_0) = 2\pi k$. So, there exists a $t_0 > 0$ such that $(f \circ \tilde{\gamma})'(s_0) = t_0\tilde{\gamma}'(s_0)$, which contradicts assumptions. □

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Instytut Matematyki
Uniwersytet Jagielloński
Reymonta 4/526
PL-30-056 Kraków
e-mail: edigaria@im.uj.edu.pl