UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXXVIII
2000

THE STATIONARY NAVIER-STOKES EQUATIONS – APPLICATION OF THE IMPLICIT FUNCTION THEOREM TO THE PROBLEM OF STABILITY

BY ELŻBIETA MOTYL

Abstract. We prove that generically with respect to the pair (viscosity, external forces), the numerical method for the stationary Navier-Stokes equations introduced by K.Holly is stable in the sense that the set of all exact solutions is the limit of the sequence of the sets of approximate solutions in the Hausdorff metric over H_0^1. Then also the problem of uniqueness of the N-S equations is equivalent to the problem of uniqueness of almost all approximate equations.

INTRODUCTION

The Navier-Stokes equations for incompressible fluid flow have been the subject of intense investigation. Since a long time, it has been known that every internal approximation of the space V of all divergence-free vector fields enables us to look for a stationary velocity of the fluid with the aid of the Galerkin method without any limitations on viscosity and external forces. However, there arises the problem of the approximation of V. Fortin's result asserts that this approximation cannot be constructed with the aid of continuous piecewise linear functions (splines).

F.Thomasset solved that problem in the two-dimensional case with the aid of splines of degree 5 using the so called stream functions (see [**11**], [**32**]). Similarly, the upwind method introduced by M.Fortin concerns two-dimensional domains. This method is adequate for a description of turbulent flows which are observed when the viscosity of the fluid is small in comparison to the other parameters.

This research was supported by the KBN grant No.2 PO3A 055 10.

In general, no satisfactory approximation of V based on the elementary functions or their splines has been found in the case $n = 3$. The ideas of Thomasset and Fortin cannot be repeated for lack of a counterpart of the stream functions for $n = 3$. There are some ideas to look for the base vector fields of the form

$$\mathrm{rot} u \quad (u \in W_0^{1,2}) \qquad \text{or} \qquad \nabla\varphi \times \nabla\psi \quad (\varphi, \psi \in W_0^{2,4}(\Omega)),$$

But (not to mention the non-uniqueness of these representations) they lead to the appoximation of the Sobolev space $W_0^{2,4}(\Omega)$, which must be rather complicated for $n = 3$.

In some very special cases, elementary functions may be used, e.g., applying the Chebyshev polynomials one constructs the internal approximation of the space of horizontally periodic solenoidal vector fields on the domain $\mathbb{R}^2 \times]-1, 1[$ (see [**27**]).

Recently, K. Urban constructed a Riesz basis for the space of divergence-free vector fields on $\mathbb{R}^n$ with the aid of wavelets. So far, this construction does not cover bounded domains.

Difficulties with the approximation are an incentive to look for other methods. Generally, external approximations are more common in practice. They enable one to find an approximate velocity of the fluid where the incompressibility condition is satisfied with some error. The known modifications of the Uzawa and Arrow-Hurwitz algorithms, made by M. Fortin, R. Peyret and R. Temam, are noteworthy (see [**32**], [**11**]). However, they require strong limitations on viscosity and external forces. In the general case, they become divergent.

In 1996, K. Holly introduced a new numerical method of solving the stationary Navier-Stokes equations with homogeneous boundary condition in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \in \{2, 3, 4\}$) with the Lipschitz boundary $\partial\Omega$, i.e.

$$(N-S) \begin{cases} \sum_{i=1}^n v_i \frac{\partial v}{\partial x_i} = \nu\Delta v + f - \nabla p, \\ \mathrm{div} v = 0, \\ v_{|\partial\Omega} = 0, \end{cases}$$

Here the approach is different. The construction of the solution is based on the internal approximation of the whole Sobolev space H_0^1, which is in practice well approximated, e.g., by splines in the finite element method. Because this method has not been published, we describe it in Section 2.

The construction of the solutions is connected with the results of the paper [**15**]. To be more specific, inversion of the divdiv* operator is of fundamental importance. The method is split into two steps. In the first step the Navier-Stokes problem is approximated by some functional equation $(*)_s$ in H_0^1. The second step involves the discretization $(*)_{s,N}$ of the equation $(*)_s$ for fixed $s \in \mathbb{N}$

(see Th. 2.9). In practice, the computer scientist deals with the problem $(*)_{s,N}$, which reduces to some algebraic equation of degree 2 in a finite-dimensional subspace of H_0^1. Convergence of this method does not require any limitations on viscosity and external forces. Thus, it is adequate also for turbulent flows.

In this paper we are interested in the properties of the sequences of the sets of all solutions of the problems $(*)_s$, $(*)_{s,N}$ in the two- and three-dimensional cases. We prove that for the data (ν, f) = (viscosity, external forces) from some set $\mathcal{G}$, the set $\mathfrak{R}(\nu, f)$ – of all solutions of the Navier-Stokes problem, is the limit of the sequence $\left(\mathfrak{R}^s(\nu, f_s)\right)_s$ – of the sets of all solutions of the equations $(*)_s$ in the sense of the Hausdorff metric over H_0^1 (see Th.4.5). In turn, if we fix sufficiently large $s \in \mathbb{N}$, then the set $\mathfrak{R}^s(\nu, f_s)$ is the limit of the sequence $\left(\mathfrak{R}^{s,N}(\nu, f_s)\right)_N$ - of the sets of all solutions of the equations $(*)_{s,N}$ in the same metric space (see Th. 4.7). As a consequence, for $(\nu, f) \in \mathcal{G}$, each solution of the Navier-Stokes problem is approximated by the whole sequences of the solutions of the suitable approximate equations. In such case, we say that the method is stable.

Similar problems are investigated in [**13**], where the Galerkin equations induced by the Navier-Stokes problem on appropriate subspaces of solenoidal vector fields are considered.

Next, we consider generic properties of the stationary solutions of the N-S problem with respect to viscosity and external forces, simultaneously. We point out a set $\mathcal{G}$ such that for $(\nu, f) \in \mathcal{G}$:

- the Holly method is stable,
- the set of all solutions of the N-S problem and the sets of solutions of the equations $(*)_s$ and $(*)_{s,N}$ are finite and equinumerous for sufficiently large $s, N \in \mathbb{N}$.

In this way the problem of uniqueness has been reduced to the finite-dimensional space. Moreover, we prove that the set $\mathcal{G}$ is open and dense in $]0, \infty[\times(H_0^1)'$ (see Th. 3.12).

The openness and the density of the set $\mathcal{G}$ is a consequence of a more general result. In fact, we consider equation of the form

$$\mu u + T(u) = g$$

in an abstract Hilbert space H. We prove that there exists a set $\mathcal{O}$, open and dense in $]0, \infty[\times H$, such that for a given $(\mu, g) \in \mathcal{O}$, the set of all solutions of the above equation is finite (see Th. 3.6).

The N-S problem can be formulated as the equation of that form if we take as T an appropriate mapping induced by the nonlinear term and H is an appropriate Sobolev space.

The results concerning generic properties were inspired by the results of C. Foiaş, R. Temam, J.C. Saut and K. Holly for the N-S problem published in [4], [5], [6], [13], [30], [33].

The technique of analysis is closely related to those in the above cited papers. The crucial point is the application of some version of the implicit function theorem from [13] and the Smale theorem (see [31]).

1. Preliminaries

Definitions and notations from functional analysis. Suppose that $(X, |\cdot|_X), (Y, |\cdot|_Y)$ are real normed spaces. Then $K_X(x_0, r) := \{x \in X : |x - x_0|_X < r\}$ is the open ball with center at x_0 and radius r, and $\overline{K}_X(x_0, r) := \{x \in X : |x - x_0|_X \leq r\}$ is the closed ball. Moreover, $K_X(r) := K_X(0, r)$. If no confusion seems likely, we omit the index X.

The symbol $\mathcal{L}(X, Y)$ stands for the linear space of all continuous linear operators from X to Y. In this space, we consider the norm of an operator $A \in \mathcal{L}(X, Y)$ defined as

$$|A|_{\mathcal{L}(X,Y)} := \sup\{|A(x)|_Y, \quad x \in \overline{K}_X(1)\}.$$

$Epi(X, Y)$ – is the subspace of all *epimorphisms,* i.e., the family of all $A \in \mathcal{L}(X, Y)$ such that $A(X) = Y$ and $Mono(X, Y)$ – the subspace of all *monomorphisms,* i.e., the family of all injections in $\mathcal{L}(X, Y)$. Moreover,

$$Iso(X, Y) := \{A \in \mathcal{L}(X, Y) : \quad A \text{ is bijective and } A^{-1} \in \mathcal{L}(Y, X)\}$$

is the family of all *isomorphisms.* In particular, if $X = Y$, the space $\mathcal{L}(X, X) =: EndX$ is called the space of *endomorphisms* of X and $Iso(X, X) =: AutX$ is called the space of *authomorphisms* of X. Moreover, $EpiX := Epi(X, X)$ and $MonoX := Mono(X, X)$. If $Y = \mathbb{R}$, then $X' := \mathcal{L}(X, \mathbb{R})$ is called the *dual* space of X and its elements are called continuous linear functionals of X.

A sequence $(x_k)_{k\in\mathbb{N}} \in X^{\mathbb{N}}$ is weakly convergent to $x \in X$ if and only if for every $\xi \in X'$:

$$\xi(x_k) \to \xi(x) \quad \text{as } k \to \infty.$$

Here $\mathbb{N} := \{1, 2, \dots\}$.

Now, assume that $\big(X, (\cdot|\cdot)_X\big)$, $\big(Y, (\cdot|\cdot)_Y\big)$ are real Hilbert spaces. Let $A \in \mathcal{L}(X, Y)$. Then, for a fixed $y \in Y$, the functional

$$\xi : X \ni x \mapsto \big(A(x)|y\big)_Y \in \mathbb{R}$$

is continuous and linear. By virtue of the Riesz theorem (see Th. 43.1 in [16]), there exists the unique $A^*(y) \in X$ such that

$$\xi(x) = \big(x|A^*(y)\big)_X \quad \text{for any } x \in X.$$

The operator

$$A^* : Y \ni y \mapsto A^*(y) \in X$$

is called the *adjoint* operator of A. If $X = Y$ and $A = A^*$, then we say that A is *selfadjoint.*

Let us recall the Banach-Steinhaus theorem in the version used in this paper (see Th. 40.1 in [**16**]).

THEOREM 1.1. *Let (A_k) be a sequence of continuous linear operators from a Banach space X in a normed space Y. If for every $x \in X$, the sequence $(A_k(x))$ is bounded, then the sequence of norms $(|A_k|_{\mathcal{L}(X,Y)})$ is bounded.*

THEOREM 1.2. *(Th. 40.3 in* [**16**]*) Let (A_k) be a sequence of continuous linear operators from a normed space X in a Banach space Y. Assume that the sequence of norms $(|A_k|_{\mathcal{L}(X,Y)})$ is bounded. If the set $Q \subset X$ is dense in X and the sequence $(A_k(x))$ is convergent for every $x \in Q$, then it is convergent for every $x \in X$.*

We will use the following version of the Banach-Alaoglu theorem (see Th. 8.3 in [**25**]):

THEOREM 1.3. *Assume that H is a Hilbert space. Let $(x_k)_{k\in\mathbb{N}} \in H^{\mathbb{N}}$ be a bounded sequence. Then there exist $x \in H$ and an infinite subset $\mathcal{N} \subset \mathbb{N}$ such that*

$$x_k \to x \quad \textit{weakly in} \quad H \quad \textit{as} \quad \mathcal{N} \ni k \to \infty.$$

Other auxiliary theorems from functional analysis used in this paper are proved in Appendix.

The topology of a topological space Z is denoted by $topZ$. The symbol $cotopZ$ denotes the family of all closed subsets of Z, i.e.

$$cotopZ := \{Z \setminus \mathcal{O} : \mathcal{O} \in topZ\}.$$

The symbol id_X stands for the identity mapping on a set X; if no confusion seems likely, we omit the index X.

Fredholm mappings. Let X, Y be two real Banach spaces. An operator $L \in \mathcal{L}(X, Y)$ is called *a Fredholm operator* if

(a) $\dim \ker L < \infty$,
(b) $\mathrm{im} L := L(X) \in cotopY$,
(c) $\mathrm{codim} L := \dim Y/\mathrm{im} L < \infty$.

If L is Fredholm, then its *index* is defined as follows: $\mathrm{ind} L := \dim \ker L - \mathrm{codim} L$.

A $\mathcal{C}^1$-mapping $\mathcal{E} : X \to Y$ is called *a Fredholm mapping* if its Fréchet differential $d_x\mathcal{E} \in \mathcal{L}(X, Y)$ is a Fredholm operator for all $x \in X$. In such case,

the index of $d_x\mathcal{E}$ is independent of x (see Th. 1.1 in [**31**]), and, by definition

$$\mathrm{ind}\mathcal{E} := \mathrm{ind}\ d_x\mathcal{E}.$$

A mapping $T : X \to Y$ is *completely continuous* if it maps bounded subsets of X into relatively compact subsets of Y.

REMARK 1.4. *Let $A : X \to Y$ be a linear operator. Then the following conditions are equivalent*

(i) *A is completely continuous;*
(ii) *if $x_k \to x$ weakly in X, then $A(x_k) \to A(x)$ in Y.*

Let us collect some results which follow from the theory of completely continuous linear operators (see [**28**], Section IV).

REMARK 1.5. *If $A \in EndX$ is completely continuous, then for any $\lambda \neq 0$,*

$$\lambda\ \mathrm{id} + A : X \to X$$

is a Fredholm operator of index 0.

Thus

$$(\lambda\ \mathrm{id} + A) \in EpiX \Leftrightarrow (\lambda\ \mathrm{id} + A) \in MonoX \Leftrightarrow (\lambda\ \mathrm{id} + A) \in AutX. \tag{1.1}$$

LEMMA 1.6. *If a $\mathcal{C}^1$-mapping $T : X \to Y$ is completely continuous, then the Fréchet differential $d_xT \in \mathcal{L}(X, Y)$ is completely continuous for all $x \in X$.*

By virtue of Remark 1.5 and Lemma 1.6, we have

COROLLARY 1.7. *If a $\mathcal{C}^1$-mapping $T : X \to X$ is completely continuous, then*

$$\lambda\ \mathrm{id} + T : X \to X$$

is a Fredholm mapping of index 0 for $\lambda \neq 0$.

Let $\mathcal{E} : X \to Y$ be a $\mathcal{C}^1$-mapping. An element $x \in X$ is called *a regular point* of $\mathcal{E}$ if $d_x\mathcal{E} \in Epi(X, Y)$ and x is *singular* if it is not regular. The images of all singular points under $\mathcal{E}$ are called *the singular values* or *critical values* and their complements – the *regular values.*

S.Smale has proved the following infinite-dimensional version of the Sard theorem for Fredholm mappings. We recall this theorem in the version given by C. Foiaş and R. Temam in [**5**].

THEOREM 1.8. *(Smale) Let $\mathcal{E} : X \to Y$ be a Fredholm $\mathcal{C}^q$-mapping , where $q > \max(\mathrm{ind}\mathcal{E}, 0)$. Then the regular values of $\mathcal{E}$ form a dense G_δ subset of Y.*

If $g \in Y$ is a regular value of $\mathcal{E}$, then $\mathcal{E}^{-1}(\{g\})$ is empty or it is a manifold of dimension $\mathrm{ind}\mathcal{E}$.

If $g \in Y$ is a regular value of $\mathcal{E}$ and $\mathrm{ind}\mathcal{E} = 0$, then $\mathcal{E}^{-1}(\{g\})$ is discrete.

Let us recall that a G_δ set is a countable intersection of open sets.

The implicit function theorems.

THEOREM 1.9. *Let X, Y, Z be Banach spaces, $(x_0, y_0) \in X \times Y$ and $F : X \times Y \to Z$ be a $\mathcal{C}^1$-mapping satisfying the following conditions*

(i) $F(x_0, y_0) = 0$,

(ii) $d^{II}_{(x_0,y_0)}F := d_{y_0}F(x_0, \cdot) \in Iso(Y, Z)$,

Then there exist neighbourhoods $\mathcal{X} \in topX$ of x_0 and $\mathcal{Y} \in topY$ of y_0 such that the relation $\eta := \{F = 0\} \cap (\mathcal{X} \times \mathcal{Y})$ is a $\mathcal{C}^1$-mapping from $\mathcal{X}$ to $\mathcal{Y}$ and

$$d_x\eta = -\left(d^{II}_{(x,\eta(x))}F\right)^{-1} \circ \left(d^{I}_{(x,\eta(x))}F\right) \quad \text{for } x \in \mathcal{X}.$$

We will use the implicit function theorem in a more general version. K.Holly proved the following

THEOREM 1.10. (*Th. 1.15 in* [**13**]) *Consider a topological space X and Banach spaces Y, Z. Let $(x_0, y_0) \in X \times Y$. Assume that a mapping $F : X \times Y \to Z$ satisfies the following conditions*

(i) $F(x_0, y_0) = 0$,

(ii) *for all $(x, y) \in X \times Y$ there exists $d^{II}_{(x,y)}F := d_yF(x, \cdot) \in \mathcal{L}(Y, Z)$,*

(iii) $d^{II}_{(x_0,y_0)}F \in Iso(Y, Z)$,

(iv) *for all $y \in Y$, the mapping $F(\cdot, y) : X \to Z$ is continuous,*

(v) *the mapping $d^{II}F : X \times Y \ni (x, y) \mapsto d^{II}_{(x,y)}F \in \mathcal{L}(Y, Z)$ is continuous at (x_0, y_0).*

Then there exist neighbourhoods $\mathcal{X} \in topX$ of x_0 and $\mathcal{Y} \in topY$ of y_0 such that the relation $\eta := \{F = 0\} \cap (\mathcal{X} \times \mathcal{Y})$ is a continuous function from $\mathcal{X}$ to $\mathcal{Y}$.

Sobolev spaces. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the Lipschitz boundary $\partial\Omega$ and let $Y \in \{\mathbb{R}, \mathbb{R}^n\}$. The symbol $\mathcal{D}(\Omega, Y)$ stands for the space of all *test functions* $\phi : \Omega \to Y$, i.e., $\mathcal{C}^\infty$-mappings with compact supports contained in Ω.

A function $u \in L^1_{loc}(\Omega, Y)$ has an α-derivative in the sense of Sobolev, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, if there exists a function $v \in L^1_{loc}(\Omega, Y)$ such that

$$\int_\Omega u(x) \cdot (D^\alpha\varphi)(x)dx = (-1)^{|\alpha|} \int_\Omega v(x) \cdot \varphi(x)dx \tag{1.2}$$

for all $\varphi \in \mathcal{D}(\Omega, \mathbb{R})$, where $|\alpha| := \sum_{i=1}^n \alpha_i$.
It is well known that there exists at most one element of $L^1_{loc}(\Omega, Y)$ satisfying (1.2). Therefore, we can define

$$D^\alpha u := v.$$

Consider the Hilbert space $L^2(\Omega, Y)$ with the scalar product

$$(u, w) \mapsto \int_\Omega u \cdot wdm := \int_\Omega u(x) \cdot w(x)dx,$$

where m is the Lebesgue measure in $\mathbb{R}^n$, $u(x) \cdot w(x)$ – the scalar product of $u(x), w(x)$ in Y and consider *the Sobolev space*

$$H^1(\Omega, Y) := \{u \in L^2(\Omega, Y) : \textit{ there exist } \frac{\partial u}{\partial x_i} \textit{ in the sense of Sobolev}$$
$$\textit{and } \frac{\partial u}{\partial x_i} \in L^2(\Omega, Y), \textit{ for each } 1 \leq i \leq n\},$$

which is a Hilbert space with the scalar product

$$(u, w) \mapsto (u|w)_{L^2(\Omega,Y)} + ((u|w)),$$

where $((u|w)) := \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} | \frac{\partial w}{\partial x_i}\right)_{L^2(\Omega,Y)}$. The bilinear form $((\cdot|\cdot))$ is called *the Dirichlet scalar product* of u and w.

The symbol $H_0^1(\Omega, Y)$ stands for the closure of $\mathcal{D}(\Omega, Y)$ in $H^1(\Omega, Y)$.

THEOREM 1.11. *(The Poincaré inequality) If an open set $G \subset \mathbb{R}^n$ satisfies the condition*

$$\textit{there exists } h \in \mathbb{R}^n \textit{ such that } G \cap (G + h) = \emptyset,$$

then $\|u\|_{L^2(G,\mathbb{R})} \leq |h| \|\nabla u\|_{L^2(G,\mathbb{R})}$ for all $u \in H_0^1(G, \mathbb{R})$.

In the case when G is bounded, this theorem is proved in [**16**] Section VI, §61, 61.3. In particular, the form $((\cdot|\cdot))$ is a scalar product in $H_0^1(\Omega, Y)$ inducing the topology inherited from $H^1(\Omega, Y)$.

For $p \in [1, \infty]$, let us consider the Banach space $L^p(\Omega, Y)$ with the norm

$$\|u\|_{L^p(\Omega,Y)} := \begin{cases} \left(\int_\Omega |u|^p dm\right)^{\frac{1}{p}} & \text{for } 1 \leq p < \infty, \\ esssup\{|u(x)|, x \in \Omega\} & \text{for } p = \infty, \end{cases}$$

where $|\cdot|$ is the norm in Y.

If $p, q \in]1, \infty[$, $\frac{1}{p} + \frac{1}{q} = 1$ and $u \in L^p(\Omega, Y), v \in L^q(\Omega, Y)$, then $u \cdot v \in L^1(\Omega, Y)$ and we have the *Hölder inequality*

$$\int_\Omega |u \cdot v| dm \leq \left(\int_\Omega |u|^p dm\right)^{\frac{1}{p}} \left(\int_\Omega |v|^q dm\right)^{\frac{1}{q}}.$$

From the Sobolev imbedding theorem (see Th. 5.4 in [**1**]), there follows that

$$\begin{aligned} &H_0^1(\Omega, Y) \subset L^{r(n)}(\Omega, Y) \\ &\textit{and the imbedding } \quad H_0^1(\Omega, Y) \hookrightarrow L^{r(n)}(\Omega, Y) \quad \textit{is continuous,} \end{aligned} \tag{1.3}$$

where

$$r(n) := \begin{cases} \frac{2n}{n-2} & \text{if } n \geq 3, \\ \textit{any number in }]1, \infty[& \text{if } n = 2. \end{cases}$$

In particular, *for* $n \in \{2,3,4\}$, *the imbedding*

$$\iota : H_0^1(\Omega, Y) \hookrightarrow L^4(\Omega, Y) \tag{1.4}$$

is well defined and continuous.

By virtue of the Rellich-Kondrashev theorem (see Th. 6.2 in [**1**]), *the imbedding*

$$H_0^1(\Omega, Y) \hookrightarrow L^2(\Omega, Y) \tag{1.5}$$

is completely continuous.

We will use the following abbreviations

$$\begin{aligned} \mathcal{D}(\Omega) &:= \mathcal{D}(\Omega, \mathbb{R}), & \mathcal{D} &:= \mathcal{D}(\Omega, \mathbb{R}^n), \\ L^2(\Omega) &:= L^2(\Omega, \mathbb{R}), & L^2 &:= L^2(\Omega, \mathbb{R}^n), \\ H^1(\Omega) &:= H^1(\Omega, \mathbb{R}), & H^1 &:= H^1(\Omega, \mathbb{R}^n), \\ H_0^1(\Omega) &:= H_0^1(\Omega, \mathbb{R}), & H_0^1 &:= H_0^1(\Omega, \mathbb{R}^n). \end{aligned}$$

Throughout this paper we consider the space H_0^1 *equipped with the Dirichlet scalar product* $((\cdot|\cdot))$.

2. The Holly method for the stationary Navier-Stokes equations

The stationary Navier-Stokes equations.
A steady in time flow of viscous incompressible fluid filling the domain Ω is described by the stationary Navier-Stokes equations

$$\partial^v v = \nu \Delta v + \tilde{f} - \nabla p, \tag{2.1}$$

$$\mathrm{div} v = 0, \tag{2.2}$$

where $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the *Laplace operator*, $\nabla := \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$ – *the gradient operator*. Moreover, for any differentiable vector fields

$$u = (u_1, \ldots, u_n) : \Omega \to \mathbb{R}^n, \qquad w = (w_1, \ldots, w_n) : \Omega \to \mathbb{R}^n,$$

divergence of u is the scalar function

$$\mathrm{div} u := \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} : \Omega \to \mathbb{R},$$

and by $\partial^u w$ we denote the vector field

$$\sum_{i=1}^n u_i \frac{\partial w}{\partial x_i} : \Omega \to \mathbb{R}^n.$$

Vector fields satysfying (2.2) are called *solenoidal* or *divergence-free.*

The number $\nu \in]0,\infty[$ (kinematic viscosity) and $\tilde{f} : \Omega \to \mathbb{R}^n$ (external forces) are given, while $v : \Omega \to \mathbb{R}^n$ (velocity) and $p : \Omega \to \mathbb{R}$ (pressure) are looked for.

We shall consider the homogeneous boundary condition

$$v_{|\partial\Omega} = 0, \tag{2.3}$$

Let $\mathcal{V} := \mathcal{D} \cap \{\mathrm{div} = 0\}$ denote the space of all divergence-free test vector fields on Ω, V denote its closure in the Hilbert space $\big(H_0^1, ((\cdot|\cdot))\big)$ and $V^\perp$ – its $((\cdot|\cdot))$-orthogonal complement in H_0^1. The symbols $P_V : H_0^1 \to V$, $P_{V^\perp} : H_0^1 \to V^\perp$ stand for the $((\cdot|\cdot))$-orthogonal projections onto V and $V^\perp$, respectively.

J.Leray stated the following

DEFINITION 2.1. Suppose that $n \in \{2,3,4\}$, $f \in (H_0^1)'$. A vector field $v \in V$ is a (weak) solution of the problem (2.1)–(2.3) if for all $\phi \in V$:

$$\int_\Omega (\partial^v v)\phi dm = -\nu((v|\phi)) + f(\phi). \tag{2.4}$$

In the classical paper [**18**], J. Leray proved that there exists at least one solution of the problem (2.1)–(2.3). J. Leray obtained a $\mathcal{C}^2$-solution assuming that external forces are of class $\mathcal{C}^2$. Generally, for given $(\nu, f) \in]0,\infty[\times(H_0^1)'$, J.L. Lions proved the existence of a weak solution with the aid of the Galerkin method (see [**21**], Sect.I, Th. 7.1).

However, from the numerical point of view, the Galerkin method requires the internal approximation of the space V. It is known that there exists such an approximation, since V is a separable Hilbert space. The incompressibility condition $div = 0$ makes it difficult to construct this approximation. This problem for arbitrary 3-dimensional domain has not been solved yet.

We present a new numerical method of solving the problem (2.1)–(2.3) introduced by K. Holly. The construction of the solution is based on the internal approximation of the whole H_0^1 space which is, in practice, well approximated, e.g., by splines in the finite element method.

The divergence operator. After the paper [**15**] let us consider the divergence operator

$$\mathrm{div} : H_0^1 \ni u \mapsto \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} \in \{\int = 0\}, \tag{2.5}$$

where $\{\int = 0\} := \{q \in L^2(\Omega) : \int_\Omega q dm = 0\}$ is a closed hyperplane in $L^2(\Omega)$. In particular, the pair $\big(\{\int = 0\}, (\cdot|\cdot)_{L^2(\Omega)}\big)$ is a Hilbert space. We recall some properties of this operator.

THEOREM 2.2. (*Th.1.7 in* [**15**]) *The operator* (2.5) *is well defined, linear, and continuous. Moreover,* $|\mathrm{div}|_{\mathcal{L}(H_0^1, L^2(\Omega))} = 1$.

Because of the regularity assumption on the boundary, we have

(2.6) $\ker \operatorname{div} := \{v \in H_0^1 : \operatorname{div} v = 0\} = V, \qquad \operatorname{im}\operatorname{div} := \operatorname{div}(H_0^1) = \{\int = 0\}$

(see [**32**] Th. 1.6).

Consider the operator

$$\operatorname{div}^* : \{\int = 0\} \to H_0^1 \tag{2.7}$$

adjoint to the divergence operator $\operatorname{div} : \big(H_0^1, ((\cdot|\cdot))\big) \to \big(\{\int = 0\}, (\cdot|\cdot)_{L^2(\Omega)}\big)$.

THEOREM 2.3. (*Th. 0.13 in* [**15**]) *The restriction of the divergence operator to $V^\perp$ is an isomorphism onto $\{\int = 0\}$ (i.e., it is a continuous linear bijection). Symbolically*

$$\operatorname{div}_{V^\perp} : V^\perp \tilde{\to} \{\int = 0\}.$$

Hence

$$\operatorname{div}^* : \{\int = 0\} \tilde{\to} V^\perp \tag{2.8}$$

and

$$\operatorname{div}\operatorname{div}^* : \{\int = 0\} \tilde{\to} \{\int = 0\}. \tag{2.9}$$

The von Neumann method allows to invert the authomorphism (2.9). In fact, we have

THEOREM 2.4. (*Th. 0.20 in* [**15**]) *For every $s \in \mathbb{N}$ we have*

$$\left|(\operatorname{div}\operatorname{div}^*)^{-1} - \sum_{j=0}^{s} (\mathrm{id} - \operatorname{div}\operatorname{div}^*)^j\right|_{End\{\int=0\}} \leq \frac{1}{\theta}(1-\theta)^{(s+1)}$$

where the constant $\theta = \theta(\Omega) \in]0,1[$ depends on Ω only.

The projection P_V can be represented with the aid of the authomorphism $(\operatorname{div}\operatorname{div}^*)^{-1}$. Because of the importance of this representation, we recall the reasoning leading to it (see. Remark 1.16 in [**15**]).

Let $u \in H_0^1$. Since $P_{V^\perp} u \in V^\perp$, by (2.8) there exists $q \in \{\int = 0\}$ such that $P_{V^\perp} u = \operatorname{div}^* q$. Applying the divergence operator, we obtain

$$\operatorname{div} P_{V^\perp} u = \operatorname{div}\operatorname{div}^* q. \tag{2.10}$$

Since $V \ni P_V u = u - P_{V^\perp} u$, $\operatorname{div} u = \operatorname{div} P_{V^\perp} u$. Hence, by (2.9),

$$(\operatorname{div}\operatorname{div}^*)^{-1} \operatorname{div} u = q.$$

Thus

$$P_{V^\perp} u = \operatorname{div}^* (\operatorname{div}\operatorname{div}^*)^{-1} \operatorname{div} u. \tag{2.11}$$

Because of the decomposition $u = P_V u + P_{V^\perp} u$, we obtain the desired representation

$$P_V u = u - \mathrm{div}^*(\mathrm{div}\mathrm{div}^*)^{-1}\mathrm{div}u. \tag{2.12}$$

Replacing in (2.11) the authomorphism $(\mathrm{div}\mathrm{div}^*)^{-1}$ with the finite sum of the von Neumann series, by Theorems 2.2 and 2.4, we obtain

COROLLARY 2.5. (*Corollary 0.21 in* [**15**]) *For every vector field* $u \in H_0^1$ *and for every integer* $s \in \mathbb{N}$, *we have*

$$\left\| P_{V^\perp} u - \mathrm{div}^* \sum_{j=0}^{s} (id - \mathrm{div}\mathrm{div}^*)^j \mathrm{div}u \right\|_{H_0^1} \leq \frac{||\mathrm{div}u||_{L^2(\Omega)}}{\theta}(1-\theta)^{s+1}.$$

The letter $\mathcal{R}$ stands for *the canonical Riesz isomorphism* in the space $\left(H_0^1, ((\cdot|\cdot))\right)$, i.e.

$$\mathcal{R} : H_0^1 \ni u \mapsto \left(H_0^1 \ni \phi \mapsto ((u|\phi)) \in \mathbb{R}\right) \in (H_0^1)'.$$

Discretization of the projection P_V and of the isomorphism $\mathcal{R}^{-1}$. The announced method requires some discretization of the operators div^*, P_V and $\mathcal{R}^{-1}$. Let the sequence $(H_N)_{N\in\mathbb{N}}$ of finite-dimensional subspaces of H_0^1 be *an internal approximation* of that space, i.e., for any $u \in H_0^1$

$$dist_{H_0^1}(u, H_N) = \|u - P_N(u)\|_{H_0^1} \to 0 \quad \text{as } N \to \infty,$$

where $P_N : H_0^1 \to H_N$ denotes the $((\cdot|\cdot))$ - orthogonal projection onto H_N.

Separability of the space H_0^1 yields the existence of that approximation. Approximations used in numerical practice do not usually form an increasing (in the sense of the inclusion) sequence of subspaces (e.g., subspaces spanned by splines in the finite element theory).

Let us fix $N \in \mathbb{N}$. Let $q \in \{\int = 0\}$. Then the field $\mathrm{div}_N^* q$ be defined as the $((\cdot|\cdot))$ – Riesz representation of the functional

$$H_N \ni \phi \mapsto \int_\Omega q\mathrm{div}\phi dm \in \mathbb{R}.$$

Thus, for any $\phi \in H_N$

$$((\mathrm{div}_N^* q|\phi)) = (q|\mathrm{div}\phi)_{L^2(\Omega)}.$$

In particular, putting $\phi := P_N\psi$, where $\psi \in H_0^1$, we obtain

$$\begin{aligned}((P_N\mathrm{div}_N^* q|\psi)) &= ((\mathrm{div}_N^* q|P_N\psi)) = (q|\mathrm{div}P_N\psi)_{L^2(\Omega)} \\ &= ((\mathrm{div}^* q|P_N\psi)) = ((P_N\mathrm{div}^* q|\psi)).\end{aligned}$$

Since $\mathrm{div}_N^* q \in H_N$, $P_N\mathrm{div}_N^* q = \mathrm{div}_N^* q$ and

$$\mathrm{div}_N^* = P_N \circ \mathrm{div}^*.$$

Hence $\mathrm{div}_N^* \in \mathcal{L}(\{\int = 0\}, H_0^1)$ and

$$\mathrm{div}_N^* \to \mathrm{div}^* \text{ pointwise as } N \to \infty. \tag{2.13}$$

Let the vectors $\overset{1}{w}, \ldots, \overset{r}{w}$, where $r = \dim H_N$, form a basis of the space H_N. Then

$$\mathrm{div}_N^* q = \sum_{j=1}^{r} \lambda_j \overset{j}{w}$$

for some coefficients $(\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r$. Multiplying the above equality by the vectors $\overset{i}{w}$ $(i = 1, \ldots, r)$, we infer that calculation of that coefficients reduces to solving the following linear system

$$\sum_{j=1}^{r} ((\overset{i}{w}|\overset{j}{w}))\lambda_j = \int_\Omega q \mathrm{div} \overset{i}{w} dm, \qquad i = 1, \ldots, r,$$

thus to the system with the Gram matrix of $\overset{1}{w}, \ldots, \overset{r}{w}$.

The choice of the above mentioned splines leads to band matrices, thus to the matrices which are easily numerically invertible.

We move to the discretization of the isomorphism $\mathcal{R}^{-1}$. For the functional $l \in (H_0^1)'$, let $\mathcal{R}_N^{-1}(l)$ denote the $((\cdot|\cdot))$ - Riesz representation of the restriction $l_{|H_N}$ of l to H_N. In particular, $\mathcal{R}_N^{-1} = P_N \circ \mathcal{R}^{-1}$. Thus $\mathcal{R}_N^{-1} \in \mathcal{L}((H_0^1)', H_0^1)$ and

$$\mathcal{R}_N^{-1} \to \mathcal{R}^{-1} \quad \text{pointwise as } N \to \infty. \tag{2.14}$$

Again, computation of $\mathcal{R}_N^{-1}(l)$ reduces to solving the linear system on the coefficients $(\mu_1, \ldots, \mu_r) \in \mathbb{R}^r$ of the expansion of $\mathcal{R}_N^{-1}(l)$ with respect to the basis $\overset{1}{w}, \ldots, \overset{r}{w}$. This system is of the form

$$\sum_{j=1}^{r} ((\overset{i}{w}|\overset{j}{w}))\mu_j = l(\overset{i}{w}), \qquad i = 1, \ldots, r.$$

Thus, solving that system reduces to inverting the same Gram matrix.

Let us fix $s, N \in \mathbb{N}$. K.Holly introduced the following operators

$$P_{V\perp}^s := \mathrm{div}^* \circ \Big(\sum_{j=0}^{s} (\mathrm{id} - \mathrm{div}\mathrm{div}^*)^j\Big) \circ \mathrm{div}$$

$$P_{V\perp}^{s,N} := \mathrm{div}_N^* \circ \Big(\sum_{j=0}^{s} (\mathrm{id} - \mathrm{div}\mathrm{div}_N^*)^j\Big) \circ \mathrm{div} \circ P_N,$$

$$P_V^s := \mathrm{id} - P_{V\perp}^s, \qquad P_V^{s,N} := \mathrm{id} - P_{V\perp}^{s,N}.$$

Directly from the above definitions, it follows that $P_V^s\psi = \psi$ for any $\psi \in V$. Moreover, each subspace H_N is invariant with respect to the discretization $P_V^{s,N}$ of the projection P_V, i.e.

$$P_V^{s,N}(H_N) \subset H_N. \tag{2.15}$$

We collect key properties of the operators P_V^s and $P_V^{s,N}$ in the following

REMARK 2.6.

(a) *The operators $P_V^s, P_V^{s,N}$ are selfadjoint endomorphisms of the space H_0^1 for any $s, N \in \mathbb{N}$,*

(b) $|P_V^s - P_V|_{EndH_0^1} \leq \frac{1}{\theta}(1-\theta)^{s+1}$ *and* $P_V \leq P_V^s \leq \mathrm{id}_{H_0^1}$ *for any* $s \in \mathbb{N}$,

(c) *for fixed* $s \in \mathbb{N}, P_V^{s,N} \to P_V^s$ *pointwise as* $N \to \infty$,

(d) $||\mathrm{div} P_V^s u||_{L^2(\Omega)} \leq \frac{1}{\theta}(1-\theta)^{s+1}||\mathrm{div} u||_{L^2(\Omega)} \leq \frac{1}{\theta}(1-\theta)^{s+1}||u||_{H_0^1}$ *for any* $u \in H_0^1$. *(θ is the constant from Th. 2.4)*

PROOF. Assertion (a) follows directly from the definitions of that operators. Because of (2.12), we have

$$P_V^s - P_V = P_{V^\perp} - P_{V^\perp}^s = \mathrm{div}^*\left((\mathrm{div}\mathrm{div}^*)^{-1} - \sum_{j=0}^{s}(\mathrm{id} - \mathrm{div}\mathrm{div}^*)^j\right)\mathrm{div} \tag{2.16}$$

Hence, by Theorem 2.2 and Theorem 2.4, we obtain

$$\begin{aligned} &|P_V^s - P_V|_{EndH_0^1} \\ &\leq \left|(\mathrm{div}\mathrm{div}^*)^{-1} - \sum_{j=0}^{s}(\mathrm{id} - \mathrm{div}\mathrm{div}^*)^j\right|_{End\{f=0\}} \leq \frac{1}{\theta}(1-\theta)^{(s+1)}. \end{aligned}$$

The second part of assertion (b) is a consequence of the inequality $0 \leq \mathrm{div}\mathrm{div}^* \leq \mathrm{id}$ (see Corollary 1.8 in **[15]**).

Assertion (c) follows from (2.13) and Remark A.4 (see Appendix).

Since $\mathrm{div}(P_V u) = 0$, (2.16), Theorem 2.2 and Corollary 2.5 yield

$$\begin{aligned} &||\mathrm{div} P_V^s(u)||_{L^2(\Omega)} = ||\mathrm{div}(P_V^s - P_V)(u)||_{L^2(\Omega)} \leq ||(P_V^s - P_V)(u)||_{H_0^1} \\ &\leq \frac{1}{\theta}(1-\theta)^{(s+1)}||\mathrm{div} u||_{L^2(\Omega)} \leq \frac{1}{\theta}(1-\theta)^{(s+1)}||u||_{H_0^1}, \end{aligned}$$

which proves (d). □

The acceleration functional. In the sequel we assume that $n \in \{2, 3, 4\}$.

LEMMA 2.7.

a) For any $u, w \in H_0^1$ the functional

$$\mathcal{A}_{u,w} : H_0^1 \ni \phi \mapsto \int_\Omega \left(\partial^u w + \frac{\mathrm{div}\, u}{2} w\right)\phi dm \in \mathbb{R} \tag{2.17}$$

is well-defined, linear, and continuous. Moreover, for any $\phi \in H_0^1$

$$\mathcal{A}_{u,w}(\phi) = -\mathcal{A}_{u,\phi}(w); \tag{2.18}$$

in particular

$$\mathcal{A}_{u,w}(w) = 0. \tag{2.19}$$

b) The mapping

$$\mathcal{A} : H_0^1 \times H_0^1 \ni (u,w) \mapsto \mathcal{A}_{u,w} \in (H_0^1)' \tag{2.20}$$

is bilinear and continuous.

The functional defined by (2.17) is called *the acceleration functional.*

PROOF. By the Schwarz and the Hölder inequalities, we obtain

$$|\mathcal{A}_{u,w}(\phi)| = \Big|\int_\Omega \big(\partial^u w + \frac{\mathrm{div} u}{2} w\big)\phi dm\Big| \le \int_\Omega |(\partial^u w)\phi| dm + \frac{1}{2}\int_\Omega |(\mathrm{div} u) w\phi| dm$$

$$\le \int_\Omega \sum_{i=1}^n |u_i \frac{\partial w}{\partial x_i}\phi| dm + \frac{1}{2}\int_\Omega |(\mathrm{div} u) w\phi| dm \le \int_\Omega |u| (\sum_{i=1}^n |\frac{\partial w}{\partial x_i}|^2)^{\frac{1}{2}} |\phi| dm$$

$$+ \frac{1}{2}||\mathrm{div} u||_{L^2}||w||_{L^4}||\phi||_{L^4} \le ||u||_{L^4}||w||_{H_0^1}||\phi||_{L^4} + \frac{1}{2}||u||_{H_0^1}||w||_{L^4}||\phi||_{L^4}.$$

To obtain the last inequality, we applied Theorem 2.2. Because of the imbedding (1.4), we have

$$|\mathcal{A}_{u,w}(\phi)| \le \frac{3}{2}|\iota| ||u||_{H_0^1}||w||_{H_0^1}||\phi||_{L^4} \le \frac{3}{2}|\iota|^2 ||u||_{H_0^1}||w||_{H_0^1}||\phi||_{H_0^1}. \tag{2.21}$$

Thus $\mathcal{A}_{u,w} \in (H_0^1)'$ and

$$|\mathcal{A}_{u,w}|_{(H_0^1)'} \le \frac{3}{2}|\iota|^2 ||u||_{H_0^1}||w||_{H_0^1}. \tag{2.22}$$

The symbol $|\iota|$ stands for the norm of the imbedding (1.4) in the space $\mathcal{L}(H_0^1, L^4)$.

Integrating by parts, we can write the functional (2.17) in the form

$$\mathcal{A}_{u,w}(\phi) = \int_\Omega \big(\partial^u w + \frac{\mathrm{div} u}{2} w\big)\phi dm = \int_\Omega (\partial^u w)\phi dm + \frac{1}{2}\int_\Omega \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} w\phi dm$$

$$= \int_\Omega (\partial^u w)\phi dm - \frac{1}{2}\int_\Omega \sum_{i=1}^n u_i \frac{\partial w}{\partial x_i}\phi dm - \frac{1}{2}\int_\Omega \sum_{i=1}^n u_i \frac{\partial \phi}{\partial x_i} w dm$$

$$= \frac{1}{2}\int_\Omega \big((\partial^u w)\phi - (\partial^u \phi) w\big) dm.$$

As a consequence we obtain (2.18) and (2.19). Assertion b) follows immediately from estimate (2.22). □

We will prove that the acceleration functional has the following property

$$\begin{aligned} &\textit{if} \quad \overset{k}{u} \to u, \quad \overset{k}{w} \to w \quad \textit{weakly} \quad \textit{in} \quad H_0^1, \\ &\textit{then} \quad \mathcal{A}_{\overset{k}{u},\overset{k}{w}} \to \mathcal{A}_{u,w} \quad \textit{pointwise on } H_0^1 \quad \textit{as} \quad k \to \infty. \end{aligned} \tag{2.23}$$

PROOF. By (2.22),

$$\left|\mathcal{A}_{\overset{k}{u},\overset{k}{w}}\right|_{(H_0^1)'} \leq \frac{3}{2}|\iota|^2 \|\overset{k}{u}\|_{H_0^1} \|\overset{k}{w}\|_{H_0^1}.$$

Since weakly convergent sequence is bounded, the sequence of norms $\left(\left|\mathcal{A}_{\overset{k}{u},\overset{k}{w}}\right|_{(H_0^1)'}\right)$ is bounded. By the density of $\mathcal{D}$ in H_0^1, Theorem 1.2 yields that it is sufficient to prove that

$$\mathcal{A}_{\overset{k}{u},\overset{k}{w}}(\phi) \to \mathcal{A}_{u,w}(\phi) \quad \text{as } k \to \infty \tag{2.24}$$

for any $\phi \in \mathcal{D}$. Let us fix $\phi \in \mathcal{D}$.

Since $\overset{k}{u} \to u$ weakly in H_0^1, $\overset{k}{u} \to u$ in L^2 by (1.5) and Remark 1.4. Hence

$$\begin{aligned} \|\overset{k}{u}_i\phi - u_i\phi\|_{L^2}^2 &= \int_\Omega |\overset{k}{u}_i - u_i|^2|\phi|^2 dm \leq \|\phi\|_{L^\infty}^2 \int_\Omega |\overset{k}{u}_i - u_i|^2 dm \\ &= \|\phi\|_{L^\infty}^2 \|\overset{k}{u}_i - u_i\|_{L^2(\Omega)}^2 \to 0 \quad \text{as} \quad k \to \infty \end{aligned}$$

for any $i \in \{1, \ldots, n\}$. Since $\overset{k}{w} \to w$ weakly in H_0^1, in particular $\frac{\partial \overset{k}{w}}{\partial x_i} \to \frac{\partial w}{\partial x_i}$ weakly in L^2. Thus by Remark A.1

$$\left(\overset{k}{u}_i\phi \middle| \frac{\partial \overset{k}{w}}{\partial x_i}\right)_{L^2} \to \left(u_i\phi \middle| \frac{\partial w}{\partial x_i}\right)_{L^2} \quad \text{as} \quad k \to \infty$$

for $i \in \{1, \ldots, n\}$, and hence

$$\int_\Omega (\partial^{\overset{k}{u}} \overset{k}{w})\phi dm \to \int_\Omega (\partial^u w)\phi dm \quad \text{as} \quad k \to \infty. \tag{2.25}$$

Similar reasoning yields that $\operatorname{div}\overset{k}{u} \to \operatorname{div} u$ weakly in $L^2(\Omega)$ and $\overset{k}{w}\phi \to w\phi$ in L^2. Thus, applying Remark A.1 once again, we infer that

$$\int_\Omega (\operatorname{div}\overset{k}{u})\overset{k}{w}\phi dm \to \int_\Omega (\operatorname{div} u) w\phi dm \quad \text{as} \quad k \to \infty. \tag{2.26}$$

Jointly, (2.25) and (2.26) yield (2.24), which ends the proof. □

With the aid of the acceleration functional we can rewrite identity (2.4) as an operator equation. In fact, we have

REMARK 2.8. *Suppose* $\nu \in]0,\infty[, f \in (H_0^1)', v \in H_0^1$. *Then the following conditions are equivalent*

(i) $v \in V$ *and* $\int_\Omega (\partial^v v)\phi dm = -\nu((v|\phi)) + f(\phi)$ *for any* $\phi \in V$,
(ii) $\nu v = P_V \mathcal{R}^{-1}(f - \mathcal{A}_{v,v})$.

PROOF. To prove the implication (i) $\Rightarrow$ (ii), let $\psi \in H_0^1$. Putting $\phi = P_V\psi$ in identity (i), we have

$$0 = \int_\Omega (\partial^v v)\phi dm - f(\phi) + \nu((v|\phi)) = ((\mathcal{R}^{-1}(\mathcal{A}_{v,v} - f)|\phi)) + ((\nu v|\phi))$$
$$= ((P_V\mathcal{R}^{-1}(\mathcal{A}_{v,v} - f)|\psi)) + ((\nu v|\psi)) = ((P_V\mathcal{R}^{-1}(\mathcal{A}_{v,v} - f) + \nu v|\psi)).$$

By arbitrariness of ψ, we obtain (ii).

To prove that (i) follows from (ii), let us remark that v is an element of V, since v belongs to the image of the projection P_V. Scalar multiplication of the equation (ii) by $\phi \in V$ yields

$$\nu((v|\phi)) = ((P_V\mathcal{R}^{-1}(f - \mathcal{A}_{v,v})|\phi)) = ((\mathcal{R}^{-1}(f - \mathcal{A}_{v,v})|\phi))$$
$$= f(\phi) - \mathcal{A}_{v,v}(\phi) = f(\phi) - \int_\Omega (\partial^v v)\phi dm.$$

□

Discretization of the Navier-Stokes problem. Construction of the approximate solutions is based on the formulation of the Navier-Stokes problem as the operator equation from Remark 2.8 (ii). Replacing, in this equation, the projection P_V by the operator P_V^s and external forces f – by approximate forces, we obtain the first approximation of the considered problem. However, this is still an equation in the space H_0^1; thus in an infinite-dimensional space. The second step involves the discretization. This time, the operator P_V^s is replaced by the operator $P_V^{s,N}$ and the isomorphism $\mathcal{R}^{-1}$ – by the operator $\mathcal{R}_N^{-1}$. This procedure leads to the equation in the space H_N , thus in a finite-dimensional one.

Now, we will formulate the main theorem of this section.

THEOREM 2.9. *(Holly) Assume that* $(H_N)_{N\in\mathbb{N}}$ *is an internal approximation of* H_0^1. *Let* $\nu \in]0,\infty[$ *and* $(f_s)_{s\in\mathbb{N}} \to f$ *in* $(H_0^1)'$ *as* $s \to \infty$. *Then*

a) for any pair $(s,N) \in \mathbb{N}^2$:

$$\text{if} \quad w \in H_N, \quad \|w\|_{H_0^1} > \frac{1}{\nu}|f_s|_{(H_0^1)'}|P_V^{s,N}|_{EndH_0^1},$$
$$\text{then} \quad ((\nu w - P_V^{s,N}\mathcal{R}_N^{-1}(f_s - \mathcal{A}_{w,P_V^{s,N}w})|w)) > 0;$$

b) if for a fixed $s \in \mathbb{N}$ vector fields $\overset{s,N}{v}$, $(N \in \mathbb{N})$ satisfy succesively the equations

$$((*)_{s,N}) \qquad \nu w = P_V^{s,N}\mathcal{R}_N^{-1}(f_s - \mathcal{A}_{w,P_V^{s,N}w})$$

and $\overset{s,N}{v} \to \overset{s}{v}$ weakly in H_0^1 as $\mathcal{N} \ni N \to \infty$ for some infinite subset $\mathcal{N} \subset \mathbb{N}$, then $\lim_{\mathcal{N}\ni N\to\infty} \| \overset{s,N}{v} - \overset{s}{v}\|_{H_0^1} = 0$, and $\overset{s}{v}$ is a solution of the equation

$$((*)_s) \qquad \nu u = P_V^s\mathcal{R}^{-1}(f_s - \mathcal{A}_{u,P_V^s u});$$

c) every solution $\overset{s}{v}$ of the equation $(*)_s$ satisfies the estimate

$$\|\overset{s}{v}\|_{H_0^1} \leq \frac{1}{\nu}|f_s|_{(H_0^1)'};$$

d) if vector fields $\overset{s}{v} \in H_0^1$ $(s \in \mathbb{N})$ satisfy succesively the equations $(*)_s$ and $\overset{s}{v} \to v$ weakly in H_0^1 as $\mathcal{S} \ni s \to \infty$ for some infinite subset $\mathcal{S} \subset \mathbb{N}$, then $\lim_{\mathcal{S}\ni s\to\infty} \|\overset{s}{v} - v\|_{H_0^1} = 0$ and

$$(*) \qquad \nu v = P_V\mathcal{R}^{-1}(f - \mathcal{A}_{v,v}).$$

PROOF. To prove assertion a), let us fix $w \in H_N$ such that $\|w\|_{H_0^1} > \frac{1}{\nu}|f_s|_{(H_0^1)'}|P_V^{s,N}|_{EndH_0^1}$. Then, by selfadjointness of the operator $P_V^{s,N}$ (Remark. 2.6 a)) and of the projection P_N, (2.15) and (2.19), we have

$$\begin{aligned}
&((P_V^{s,N}\mathcal{R}_N^{-1}(f_s - \mathcal{A}_{w,P_V^{s,N}w})|w)) = ((P_N\mathcal{R}^{-1}(f_s - \mathcal{A}_{w,P_V^{s,N}w})|P_V^{s,N}w)) \\
&= ((\mathcal{R}^{-1}(f_s - \mathcal{A}_{w,P_V^{s,N}w})|P_V^{s,N}w)) = f_s(P_V^{s,N}w) - \mathcal{A}_{w,P_V^{s,N}w}(P_V^{s,N}w) \\
&\leq |f_s|_{(H_0^1)'}|P_V^{s,N}|_{EndH_0^1}\|w\|_{H_0^1} < \nu\|w\|_{H_0^1}^2,
\end{aligned}$$

which proves a).

For a fixed vector field $\phi \in H_0^1$, we apply the functional $((\cdot|\phi))$ to the equation $(*)_{s,N}$ and we obtain

$$(2.27) \qquad \nu((\overset{s,N}{v}|\phi)) = f_s(P_N P_V^{s,N}\phi) - \mathcal{A}_{\overset{s,N}{v},P_V^{s,N}\overset{s,N}{v}}(P_N P_V^{s,N}\phi).$$

By Remark A.4 and Remark 2.6 c), $P_N P_V^{s,N}\phi \to P_V^s\phi$ in H_0^1 as $N \to \infty$. Remark A.2 yields that

$$(2.28) \qquad P_V^{s,N}\overset{s,N}{v} \to P_V^s\overset{s}{v} \quad \text{weakly in } H_0^1 \text{ as } \mathcal{N} \ni N \to \infty,$$

and hence, by (2.23)

$$\mathcal{A}_{\overset{s,N}{v},P_V^{s,N}\overset{s,N}{v}} \to \mathcal{A}_{\overset{s}{v},P_V^s\overset{s}{v}} \quad \text{pointwise as } \mathcal{N} \ni N \to \infty.$$

Jointly, by Remark A.3

$$\mathcal{A}_{\overset{s,N}{v},P_V^{s,N}\overset{s,N}{v}}(P_N P_V^{s,N}\phi) \to \mathcal{A}_{\overset{s}{v},P_V^s\overset{s}{v}}(P_V^s\phi) \quad \text{as } \mathcal{N} \ni N \to \infty. \tag{2.29}$$

Passing to the limit $\mathcal{N} \ni N \to \infty$ in (2.27) gives

$$\nu((\overset{s}{v}|\phi)) = ((P_V^s\mathcal{R}^{-1}(f_s - \mathcal{A}_{\overset{s}{v},P_V^s\overset{s}{v}}|\phi)). \tag{2.30}$$

Since ϕ is arbitrary, the field $\overset{s}{v}$ is a solution of the equation $(*)_s$.

To prove that $\lim_{\mathcal{N}\ni N\to\infty} \| \overset{s,N}{v} - \overset{s}{v}\|_{H_0^1} = 0$, it is sufficient to show that

$$\| \overset{s,N}{v}\|_{H_0^1} \to \|\overset{s}{v}\|_{H_0^1} \quad \text{as } \mathcal{N} \ni N \to \infty, \tag{2.31}$$

since $\| \overset{s,N}{v} - \overset{s}{v}\|_{H_0^1}^2 = \| \overset{s,N}{v}\|_{H_0^1}^2 - 2((\overset{s,N}{v}|\overset{s}{v})) + \|\overset{s}{v}\|_{H_0^1}^2$ and the subsequence $(\overset{s,N}{v})_{N\in\mathcal{N}}$ is weakly convergent. Putting $\phi := \overset{s,N}{v}$ in (2.27) and taking into account (2.15) and (2.19), we obtain

$$\| \overset{s,N}{v}\|_{H_0^1}^2 = \frac{1}{\nu} f_s(P_V^{s,N}\overset{s,N}{v}). \tag{2.32}$$

Passing to the limit $\mathcal{N} \ni N \to \infty$, by (2.28) we conclude that

$$\| \overset{s,N}{v}\|_{H_0^1}^2 \to \frac{1}{\nu} f_s(P_V^s\overset{s}{v}) \quad \text{as } \mathcal{N} \ni N \to \infty.$$

Let us remark that scalar multiplication in H_0^1 by $\overset{s}{v}$ of the equation $(*)_s$, by (2.19) yields

$$\|\overset{s}{v}\|_{H_0^1}^2 = \frac{1}{\nu}((P_V^s\mathcal{R}^{-1}f_s|\overset{s}{v})) = \frac{1}{\nu} f_s(P_V^s\overset{s}{v}), \tag{2.33}$$

which ends the proof of the convergence (2.31), and at the same time of assertion b).

Due to Remark 2.6 b), $0 \le P_V^s \le id$, thus $|P_V^s|_{EndH_0^1} \le 1$. Hence, by (2.33), we obtain estimate from c).

We move to the proof of assertion d). For a given $\phi \in H_0^1$, we apply the functional $((\cdot|\phi))$ to the equation $(*)_s$ and obtain

$$\nu((\overset{s}{v}|\phi)) = f_s(P_V^s\phi) - \mathcal{A}_{\overset{s}{v},P_V^s\overset{s}{v}}(P_V^s\phi). \tag{2.34}$$

A reasoning similar to that in the proof of statement b) yields that $P_V^s\overset{s}{v} \to P_V v$ weakly in H_0^1 and $\mathcal{A}_{\overset{s}{v},P_V^s\overset{s}{v}}(P_V^s\phi) \to \mathcal{A}_{v,P_V v}(P_V\phi)$ as $\mathcal{S} \ni s \to \infty$. Thus, passing to the limit $\mathcal{S} \ni s \to \infty$ in (2.34) gives

$$\nu((v|\phi)) = ((P_V\mathcal{R}^{-1}(f - \mathcal{A}_{v,P_V v})|\phi)), \tag{2.35}$$

and by the arbitrariness of ϕ

$$\nu v = P_V \mathcal{R}^{-1}(f - \mathcal{A}_{v,P_V v}).$$

In particular, the field v is an element of the space V, and hence

$$\nu v = P_V \mathcal{R}^{-1}(f - \mathcal{A}_{v,v}).$$

Putting $\phi := v$ in (2.35), by (2.19), we conclude

$$\|v\|^2_{H^1_0} = \frac{1}{\nu}((P_V \mathcal{R}^{-1} f|v)) = \frac{1}{\nu} f(P_V v) = \frac{1}{\nu} f(v). \tag{2.36}$$

Since $P^s_V \overset{s}{v} \to P_V v$ weakly in H^1_0 as $\mathcal{S} \ni s \to \infty$ and $f_s \to f$ in $(H^1_0)'$ as $s \to \infty$, $f_s(P^s_V \overset{s}{v}) \to f(P_V v)$ as $\mathcal{S} \ni s \to \infty$. Hence, by (2.33) and (2.36), $\|\overset{s}{v}\|^2_{H^1_0} \to \|v\|^2_{H^1_0}$ as $\mathcal{S} \ni s \to \infty$. At the same time $\overset{s}{v} \to v$ weakly in H^1_0 as $\mathcal{S} \ni s \to \infty$. Thus $\lim_{\mathcal{S} \ni s \to \infty} \|\overset{s}{v} - v\|_{H^1_0} = 0$. □

Statement a) of the above theorem, by the Brouwer fix point theorem in the version proved by J.L.Lions (see Lemma 4.3 in [**21**]), guarantees the existence of a solution of the equation$(*)_{s,N}$. In numerical practice, the computer scientist deals just with this equation. Solving this equation reduces to solving the system of algebraic equations of degree two on the coefficients of the expansion of $\overset{s,N}{v}$ with respect to the basis of the space H_N.

By Theorem 2.9 a)

$$\|\overset{s,N}{v}\|_{H^1_0} \leq \frac{1}{\nu} |f_s|_{(H^1_0)'} |P^{s,N}_V|_{EndH^1_0}.$$

Hence, by Remark 2.6 c) and by the Banach-Steinhaus theorem, the sequence $(\overset{s,N}{v})_{N \in \mathbb{N}}$ is bounded in H^1_0. Thus, the Banach-Alaoglu theorem yields that the sequence $(\overset{s,N}{v})_{N \in \mathbb{N}}$ contains a subsequence weakly convergent in H^1_0. By Theorem 2.9 b), this subsequence is convergent in the norm in H^1_0 and the limit is a solution of the equation $(*)_s$.

Similarly, by statement c), the sequence $(\overset{s}{v})_{s \in \mathbb{N}}$ is bounded in H^1_0, and again by the Banach-Alaoglu theorem, it contains a subsequence weakly convergent in H^1_0. Hence, by d), this subsequence is convergent in the sense of norm to a solution of the Navier-Stokes equation.

There arise the questions:

1) Is every solution of the Navier-Stokes problem attainable by the Holly method?
2) Are the subsequences $(\overset{s,N}{v})_{N \in \mathcal{N}}$, $(\overset{s}{v})_{s \in \mathcal{S}}$ the whole sequences?
   In other words: are the subsets $\mathcal{N}$ and $\mathcal{S}$ in Theorem 2.9 b),d) equal to the set $\mathbb{N}$?

The above questions will be considered in connection with the problem of the stability of the presented method.

3. Generic properties

General case. In this section we assume that $(H, (\cdot|\cdot))$ is a real Hilbert space; the norm induced by the scalar product $(\cdot|\cdot)$ is denoted by $|\cdot|$. Let $T : H \to H$ be a $\mathcal{C}^1$-mapping satisfying the following conditions

(3.1) *if* $\overset{k}{u} \to u$ *weakly in* H, *then* $T(\overset{k}{u}) \to T(u)$ *in* H *as* $k \to \infty$,

(3.2) *for any pair* $(\mu, g) \in]0, \infty[\times H$
the set $\mathfrak{R}(T, \mu, g) := \{u \in H : \mu u + T(u) = g\}$ *is nonempty,*

(3.3) *there exists a continuous function* $\kappa :]0, \infty[\times[0, \infty[\to [0, \infty[$ *such that for any* $u \in \mathfrak{R}(T, \mu, g)$ *the following estimation holds:* $|u| \leq \kappa(\mu, |g|)$.

Let us remark that in (3.2) we assume the existence of a solution of the equation

$$\mu u + T(u) = g \tag{3.4}$$

for a given $(\mu, g) \in]0, \infty[\times H$.

We will investigate the properties of the set $\mathfrak{R}(T, \mu, g)$ of all solutions of the equation (3.4). We will point some set $\mathcal{O}$ open and dense in $]0, \infty[\times H$ such that for $(\mu, g) \in \mathcal{O}$, the set $\mathfrak{R}(T, \mu, g)$ is finite.

The density of the set $\mathcal{O}$ will be obtained by an application of the Smale theorem (see Theorem 1.8) while its openness follows from the implicit function theorem (see Theorem 1.9).

From the condition (3.1), it follows that the mapping T is completely continuous. Let us write this property as the following

Remark 3.1. *The mapping* T *is completely continuous.*

Proof. Let us fix a bounded subset $B \subset H$. Let $(\overset{k}{w})_{k\in\mathbb{N}}$ be a sequence of elements from $T(B)$. We will show that $(\overset{k}{w})$ contains a convergent subsequence.

Indeed, since $\overset{k}{w} \in T(B)$, $\overset{k}{w} = T(\overset{k}{v})$ for some $\overset{k}{v} \in B$. Hence, by the Banach-Alaoglu theorem

there exist an infinite subset $\mathcal{N} \subset \mathbb{N}$ and an element $v \in H$

such that $\overset{k}{v} \to v$ weakly in H as $\mathcal{N} \ni k \to \infty$.

Then, by (3.1),

$$T(\overset{k}{v}) \to T(v) \text{ in } H \text{ as } \mathcal{N} \ni k \to \infty,$$

which ends the proof. □

By Remark 1.4, for linear mappings, the condition (3.1) is equivalent to the complete continuity of T. In general, there exist (nonlinear) completely continuous mappings that do not satisfy the condition (3.1).

EXAMPLE 3.2. Suppose that the space H is separable. Let us fix $e \in H \setminus \{0\}$. Then, the mapping

$$T : H \ni x \mapsto |x|^2 e \in H$$

is completely continuous.

Let $\{h_k, k \in \mathbb{N}\}$ be any complete orthonormal family. Then $h_k \to 0$ weakly in H as $k \to \infty$. Indeed, let us fix a functional $\xi \in H'$. By the Riesz theorem, there exists a vector $x_\xi \in H$ such that $\xi = (x_\xi|\cdot)$. The vector x_ξ can be written in the form

$$x_\xi = \sum_{i=1}^{\infty} \lambda_i h_i$$

for some coefficients $\lambda_i \in \mathbb{R}$ $(i \in \mathbb{N})$. Then

$$\begin{aligned} \xi(h_k) = (x_\xi|h_k) &= \Big(\sum_{i=1}^{\infty} \lambda_i h_i | h_k\Big) \\ &= \sum_{i=1}^{\infty} \lambda_i (h_i|h_k) = \lambda_k |h_k|^2 = \lambda_k \to 0 \quad \text{as } k \to \infty \end{aligned}$$

while

$$T(h_k) = e \nrightarrow 0 \quad \text{as } k \to \infty.$$

□

REMARK 3.3. *For a pair $(\mu, g) \in]0, \infty[\times H$, the set $\mathfrak{R}(T, \mu, g)$ is a compact subset of H.*

PROOF. Let us fix $(\mu, g) \in]0, \infty[\times H$. It is sufficient to show that any sequence $(\overset{k}{u})_{k \in \mathbb{N}} \in \big(\mathfrak{R}(T, \mu, g)\big)^{\mathbb{N}}$ contains a subsequence convergent to some element of $\mathfrak{R}(T, \mu, g)$. By (3.3), the sequence $(\overset{k}{u})_{k \in \mathbb{N}}$ is bounded. Thus, by the Banach-Alaoglu theorem

(3.5) there exist an infinite subset $\mathcal{N} \subset \mathbb{N}$ and an element $u \in H$ such that $\overset{k}{u} \to u$ weakly in H as $\mathcal{N} \ni k \to \infty$.

We will show that $u \in \mathfrak{R}(T, \mu, g)$. Indeed, scalar multiplication of the equality

$$\mu \overset{k}{u} + T(\overset{k}{u}) = g \tag{3.6}$$

by an arbitrary $\phi \in H$ gives

$$\mu(\overset{k}{u}|\phi) + \big(T(\overset{k}{u})|\phi\big) = (g|\phi). \tag{3.7}$$

Passing to the limit $\mathcal{N} \ni k \to \infty$, by (3.1), and (3.5) implies

$$\mu(u|\phi) + \big(T(u)|\phi\big) = (g|\phi).$$

By the arbitrariness of ϕ, the above equality means

$$\mu u + T(u) = g,$$

which proves that $u \in \mathfrak{R}(T, \mu, g)$.

We will check that $\lim_{\mathcal{N}\ni k\to\infty} |\overset{k}{u} - u| = 0$. Since

$$|\overset{k}{u} - u|^2 = (\overset{k}{u} - u|\overset{k}{u} - u) = |\overset{k}{u}|^2 - 2(u|\overset{k}{u}) + |u|^2,$$

it is sufficient to show that $|\overset{k}{u}|^2 \to |u|^2$ as $\mathcal{N} \ni k \to \infty$. Putting $\phi := \overset{k}{u}$ in (3.7), we obtain

$$\mu|\overset{k}{u}|^2 + \big(T(\overset{k}{u})|\overset{k}{u}\big) = (g|\overset{k}{u}).$$

Passing to the limit $\mathcal{N} \ni k \to \infty$, by (3.5), (3.1) and Remark A.1, we obtain

$$|\overset{k}{u}|^2 = -\frac{1}{\mu}\big(T(\overset{k}{u})|\overset{k}{u}\big) + \frac{1}{\mu}(g|\overset{k}{u}) \to -\frac{1}{\mu}\big(T(u)|u\big) + \frac{1}{\mu}(g|u) = |u|^2 \text{ as } \mathcal{N} \ni k \to \infty,$$

which ends the proof. □

We will use the following symbol

$$\mathcal{E}_\mu : H \ni u \mapsto \mu u + T(u) \in H \qquad \text{for fixed } \mu \in]0, \infty[.$$

Let us remark that for any $(\mu, g) \in]0, \infty[\times H$:

$$\mathfrak{R}(T, \mu, g) = \mathcal{E}_\mu^{-1}(\{g\}). \tag{3.8}$$

LEMMA 3.4. *Assume that $(\overset{0}{\mu}, \overset{0}{g}) \in]0, \infty[\times H$, $\overset{0}{u} \in \mathfrak{R}(T, \overset{0}{\mu}, \overset{0}{g})$ and the Fréchet differential $d_{\overset{0}{u}}\mathcal{E}_{\overset{0}{\mu}}$ is an epimorphism of H. Then, there exist a neighbourhood $\mathcal{X} \in top(]0, \infty[\times H)$ of $(\overset{0}{\mu}, \overset{0}{g})$ and a neighbourhood $\mathcal{Y} \in topH$ of $\overset{0}{u}$ such that*

(i) $\#\big(\mathcal{Y} \cap \mathfrak{R}(T, \mu, g)\big) = 1$ *for* $(\mu, g) \in \mathcal{X}$,

(ii) *the function* $\mathcal{X} \ni (\mu, g) \mapsto \overset{\mu,g}{u} \in \mathcal{Y}$ *is of class* $\mathcal{C}^1$, *where* $\{\overset{\mu,g}{u}\} := \mathcal{Y} \cap \mathfrak{R}(T, \mu, g)$.

PROOF. We apply the implicit function theorem (Theorem 1.9) to the mapping

$$F : (]0, \infty[\times H) \times H \ni \big((\mu, g), u\big) \mapsto \mu u + T(u) - g \in H$$

with respect to the point $\big((\overset{0}{\mu}, \overset{0}{g}), \overset{0}{u}\big)$. Directly from the assumptions, it follows that

$$F\big((\overset{0}{\mu}, \overset{0}{g}), \overset{0}{u}\big) = \mathcal{E}_{\overset{0}{\mu}}(\overset{0}{u}) - \overset{0}{g} = 0,$$

the mapping F is of class $\mathcal{C}^1$ and for $\big((\mu, g), u\big) \in (]0, \infty[\times H) \times H$:

$$d^{II}_{((\mu,g),u)} F := d_u F\big((\mu, g), \cdot\big) = d_u \mathcal{E}_\mu = \mu \mathrm{id} + d_u T. \tag{3.9}$$

It remains to check that

$$d^{II}_{((\overset{0}{\mu}, \overset{0}{g}), \overset{0}{u})} F \in AutH. \tag{3.10}$$

Due to Remark 3.1, the mapping T is completely continuous. Thus, by Corollary 1.7

$$\mathcal{E}_\mu \text{ is the Fredholm mapping of index 0.} \tag{3.11}$$

Hence, by (1.1) and (3.9), for any $u \in H$

$$d_u \mathcal{E}_\mu \in EpiH \Leftrightarrow d_u \mathcal{E}_\mu \in MonoH \Leftrightarrow d_u \mathcal{E}_\mu \in AutH. \tag{3.12}$$

Since $d_{\overset{0}{u}} \mathcal{E}_{\overset{0}{\mu}} \in EpiH$, by (3.12) and (3.9), (3.10) holds.

Now, from the implicit function theorem, it follows that there exist a neighbourhoods $\mathcal{X} \in top(]0, \infty[\times H)$ of $(\overset{0}{\mu}, \overset{0}{g})$ and $\mathcal{Y} \in topH$ of $\overset{0}{u}$ such that the relation $\eta := \{F = 0\} \cap (\mathcal{X} \times \mathcal{Y})$ is a $\mathcal{C}^1$ - function from $\mathcal{X}$ to $\mathcal{Y}$. In particular, for any pair $(\mu, g) \in \mathcal{X}$ there exists the unique $\overset{\mu,g}{u} \in \mathcal{Y}$ such that $\eta(\mu, g) = \overset{\mu,g}{u}$. Since $\eta \subset \{F = 0\}$, $F\big((\mu, g), \overset{\mu,g}{u}\big) = 0$, or equivalently $\overset{\mu,g}{u} \in \mathcal{Y} \cap \mathfrak{R}(T, \mu, g)$. Moreover, the function $\mathcal{X} \ni (\mu, g) \mapsto \overset{\mu,g}{u} \in \mathcal{Y}$ is of class $\mathcal{C}^1$. □

LEMMA 3.5. *Assume that $\overset{0}{\mu} \in]0, \infty[$ and $H \ni \overset{0}{g}$ is a regular value of the mapping $\mathcal{E}_{\overset{0}{\mu}}$. Then there exists a neighbourhood $\mathcal{X} \in top(]0, \infty[\times H)$ of $(\overset{0}{\mu}, \overset{0}{g})$ such that*

$$\#\mathfrak{R}(T, \mu, g) = \#\mathfrak{R}(T, \overset{0}{\mu}, \overset{0}{g}) < \infty \quad \textit{for any } (\mu, g) \in \mathcal{X}.$$

(For the set X, the symbol $\#X$ denotes its cardinality).

PROOF. Due to (3.11), $\mathcal{E}_{\overset{0}{\mu}} : H \to H$ is a Fredholm mapping of index 0. By virtue of the Smale theorem (see Theorem 1.8), the set $\mathfrak{R}(T, \overset{0}{\mu}, \overset{0}{g}) = \mathcal{E}^{-1}_{\overset{0}{\mu}}(\{\overset{0}{g}\})$ is discrete. On the other hand, by Remark 3.3, this set is compact. Thus, it is finite.

Let $u \in \mathfrak{R}(T, \overset{0}{\mu}, \overset{0}{g})$. According to Lemma 3.4, there exist (dependent on u) neighbourhoods: $\mathcal{X}(u) \in top(]0, \infty[\times H)$ – of $(\overset{0}{\mu}, \overset{0}{g})$ and $\mathcal{Y}(u) \in topH$ – of u such that

(i) $\#\big(\mathcal{Y}(u) \cap \mathfrak{R}(T,\mu,g)\big) = 1 \quad$ for $(\mu,g) \in \mathcal{X}(u)$,

(ii) the function $\mathcal{X}(u) \ni (\mu,g) \mapsto \overset{\mu,g}{u} \in \mathcal{Y}(u)$ is of class $\mathcal{C}^1$, where $\{\overset{\mu,g}{u}\} := \mathcal{Y}(u) \cap \mathfrak{R}(T,\mu,g)$.

Since the set $\mathfrak{R}(T,\overset{0}{\mu},\overset{0}{g})$ is finite, there exists $\delta > 0$ such that the balls $\{\overline{K}_H(u,\delta)$, $u \in \mathfrak{R}(T,\overset{0}{\mu},\overset{0}{g})\}$ are pairwise disjoint and $\overline{K}_H(u,\delta) \subset \mathcal{Y}(u)$ $(u \in \mathfrak{R}(T,\overset{0}{\mu},\overset{0}{g}))$. By the continuity of the function in (ii), we infer that there exists a neighbourhood $\mathcal{X}_1(u) \subset \mathcal{X}(u)$ of $(\overset{0}{\mu},\overset{0}{g})$ w $]0,\infty[\times H$ such that $\overset{\mu,g}{u} \in \overline{K}_H(u,\delta)$ for $(\mu,g) \in \mathcal{X}_1(u)$.

Thus, putting $\mathcal{X}_1 := \bigcap\{\mathcal{X}_1(u), u \in \mathfrak{R}(T,\overset{0}{\mu},\overset{0}{g})\}$, we infer that if $(\mu,g) \in \mathcal{X}_1$ then the function

$$\mathfrak{R}(T,\overset{0}{\mu},\overset{0}{g}) \ni u \mapsto \overset{\mu,g}{u} \in \big(\mathcal{Y}(u) \cap \mathfrak{R}(T,\mu,g)\big)$$

is an injection and, as a consequence,

$$\#\mathfrak{R}(T,\overset{0}{\mu},\overset{0}{g}) \le \#\mathfrak{R}(T,\mu,g).$$

We assert that there exists a neighbourhood $\mathcal{X} \in top]0,\infty[\times H$ of $(\overset{0}{\mu},\overset{0}{g})$ such that

$$\#\mathfrak{R}(T,\overset{0}{\mu},\overset{0}{g}) = \#\mathfrak{R}(T,\mu,g) \quad \text{for } (\mu,g) \in \mathcal{X}.$$

Suppose that the opposite of our claim holds. We can then find a sequence $(\overset{k}{\mu},\overset{k}{g}) \in \mathcal{X}_1^{\mathbb{N}}$ convergent to $(\overset{0}{\mu},\overset{0}{g})$ in $]0,\infty[\times H$ such that

$$\#\mathfrak{R}(T,\overset{0}{\mu},\overset{0}{g}) < \#\mathfrak{R}(T,\overset{k}{\mu},\overset{k}{g}) \quad \text{for } k \in \mathbb{N}.$$

The subset $\{\overset{\overset{k}{\mu},\overset{k}{g}}{u}, u \in \mathfrak{R}(T,\overset{0}{\mu},\overset{0}{g})\}$ has exactly $\#\mathfrak{R}(T,\overset{0}{\mu},\overset{0}{g})$ elements. Therefore, it is not the whole $\mathfrak{R}(T,\overset{k}{\mu},\overset{k}{g})$.(Recall that $\overset{\overset{k}{\mu},\overset{k}{g}}{u}$ is the unique element of the set $\mathfrak{R}(T,\overset{k}{\mu},\overset{k}{g})$ which is in a suitable neighbourhood of the solution $u \in \mathfrak{R}(T,\overset{0}{\mu},\overset{0}{g})$). Let us select a sequence

$$\overset{k}{a} \in \mathfrak{R}(T,\overset{k}{\mu},\overset{k}{g}) \setminus \{\overset{\overset{k}{\mu},\overset{k}{g}}{u}, \quad u \in \mathfrak{R}(T,\overset{0}{\mu},\overset{0}{g})\}.$$

It is clear that

$$\overset{k}{a} \notin \bigcup\{\mathcal{Y}(u), \quad u \in \mathfrak{R}(T,\overset{0}{\mu},\overset{0}{g})\}. \tag{3.13}$$

Since $\overset{k}{a} \in \mathfrak{R}(T,\overset{k}{\mu},\overset{k}{g})$,

$$|\overset{k}{a}| \le \kappa(\overset{k}{\mu},|\overset{k}{g}|) \le \max\{\kappa(\overset{l}{\mu},|\overset{l}{g}|), \quad l = 0,1,2,\dots\} < \infty \quad \text{for } k \in \mathbb{N},$$

by (3.3) and by the compactness of the set $\{(\overset{0}{\mu}, |\overset{0}{g}|), (\overset{l}{\mu}, |\overset{l}{g}|), l = 1, 2, \dots\}$. Then, the Banach-Alaoglu theorem yields that

$$\text{there exist an infinite subset } \mathcal{N} \subset \mathbb{N} \text{ and an element } a \in H \text{ such that } \overset{k}{a} \to a \text{ weakly in } H \text{ as } \mathcal{N} \ni k \to \infty. \tag{3.14}$$

On the other hand, multiplying the equality

$$\overset{k}{\mu}\overset{k}{a} + T(\overset{k}{a}) = \overset{k}{g} \tag{3.15}$$

scalarly in H by any $\phi \in H$, we obtain

$$\overset{k}{\mu}(\overset{k}{a}|\phi) + \big(T(\overset{k}{a})|\phi\big) = (\overset{k}{g}|\phi). \tag{3.16}$$

By virtue of (3.14) and (3.1), the passage to the limit $\mathcal{N} \ni k \to \infty$ in (3.16) is possible and we find at the limit

$$\overset{0}{\mu}(a|\phi) + \big(T(a)|\phi\big) = (\overset{0}{g}|\phi).$$

Since ϕ is arbitrary,

$$\overset{0}{\mu}a + T(a) = \overset{0}{g}$$

or equivalently

$$a \in \mathfrak{R}(T, \overset{0}{\mu}, \overset{0}{g}) \tag{3.17}$$

We will show that

$$|\overset{k}{a} - a| \to 0 \quad \text{as } \mathcal{N} \ni k \to \infty. \tag{3.18}$$

Indeed, by the equality

$$|\overset{k}{a} - a|^2 = |\overset{k}{a}|^2 - 2(\overset{k}{a}|a) + |a|^2$$

and by (3.14), it is sufficient to show that

$$|\overset{k}{a}|^2 \to |a|^2 \quad \text{as } \mathcal{N} \ni k \to \infty. \tag{3.19}$$

Condition (3.19) can be obtained by putting $\phi := \overset{k}{a}$ in (3.16) and using (3.14) and (3.1).

As a consequence of (3.13), (3.17) and (3.18), we obtain the contradiction

$$\bigcup\{\mathcal{Y}(u),\ u \in \mathfrak{R}(T, \overset{0}{\mu}, \overset{0}{g})\} \not\ni \ a \in \mathfrak{R}(T, \overset{0}{\mu}, \overset{0}{g}) \subset \bigcup\{\mathcal{Y}(u),\ u \in \mathfrak{R}(T, \overset{0}{\mu}, \overset{0}{g})\},$$

which completes the proof. □

Now, we are in a position to formulate the main result of this subsection. Let us introduce

$$\mathcal{O} := \{(\mu, g) \in]0, \infty[\times H : g \textit{ is a regular value of the mapping } \mathcal{E}_\mu : H \to H\}.$$

THEOREM 3.6. *The set $\mathcal{O}$ is open and dense in* $]0,\infty[\times H$. *Moreover,* $\#\mathfrak{R}(T,\mu,g)<\infty$ *for* $(\mu,g)\in\mathcal{O}$.

PROOF. We begin with the openness of $\mathcal{O}$. Let us fix $(\overset{0}{\mu},\overset{0}{g})\in\mathcal{O}$. Directly from the definition of $\mathcal{O}$ and from (3.8), it follows that the differential $d_u\mathcal{E}_{\overset{0}{\mu}}\in EpiH$ for any $u\in\mathfrak{R}(T,\overset{0}{\mu},\overset{0}{g})$. Then, by (3.12), $d_u\mathcal{E}_{\overset{0}{\mu}}\in AutH$. Let us fix $u\in\mathfrak{R}(T,\overset{0}{\mu},\overset{0}{g})$.

It is well known that the subspace $AutH$ is open in the space $EndH$. Therefore,

$$\text{there exists a neighbourhood } \mathcal{W}(u)\in top(EndH) \text{ of } d_u\mathcal{E}_{\overset{0}{\mu}} \text{ such that } \mathcal{W}(u)\subset AutH. \tag{3.20}$$

Since the mapping T is of class $\mathcal{C}^1$, the mapping

$$(]0,\infty[\times H)\times H\ni\big((\mu,g),u\big)\mapsto d_u\mathcal{E}_\mu\in EndH$$

is continuous at $\big((\overset{0}{\mu},\overset{0}{g}),u\big)$. Hence

$$\begin{aligned}&\text{there exist a neighbourhood } \mathcal{X}_1(u)\in top(]0,\infty[\times H) \text{ of } (\overset{0}{\mu},\overset{0}{g})\\ &\text{and a neighbourhood } \mathcal{U}(u)\in topH \text{ of } u \text{ such that}\\ &d_w\mathcal{E}_\mu\in\mathcal{W}(u) \text{ for } \big((\mu,g),w\big)\in\mathcal{X}_1(u)\times\mathcal{U}(u).\end{aligned} \tag{3.21}$$

According to Lemma 3.5

$$\begin{aligned}&\text{there exists a neighbourhood } \mathcal{X}_2\in \mathrm{top}(]0,\infty[\times H) \text{ of } (\overset{0}{\mu},\overset{0}{g}) \text{ such that}\\ &\#\mathfrak{R}(T,\mu,g)=\#\mathfrak{R}(T,\overset{0}{\mu},\overset{0}{g})<\infty \text{ for } (\mu,g)\in\mathcal{X}_2,\end{aligned}$$

and, from the proof of this lemma, we know that

$$\mathfrak{R}(T,\mu,g)=\{\overset{\mu,g}{u},\quad u\in\mathfrak{R}(T,\overset{0}{\mu},\overset{0}{g})\}. \tag{3.22}$$

In turn, by virtue of Lemma 3.4 (ii), the functions

$$\mathcal{X}_2\ni(\mu,g)\mapsto\overset{\mu,g}{u}\in\mathcal{Y}(u),\quad u\in\mathfrak{R}(T,\overset{0}{\mu},\overset{0}{g})$$

are continuous. Thus,

$$\begin{aligned}&\text{there exists a neighbourhood } \mathcal{X}_3(u)\in top(]0,\infty[\times H) \text{ of } (\overset{0}{\mu},\overset{0}{g})\\ &\text{such that } \overset{\mu,g}{u}\in\mathcal{U}(u) \text{ for } (\mu,g)\in\mathcal{X}_3(u).\end{aligned} \tag{3.23}$$

Putting $\mathcal{X} := \mathcal{X}_2 \cap \bigcap\{\mathcal{X}_1(u) \cap \mathcal{X}_3(u), u \in \mathfrak{R}(T, \overset{0}{\mu}, \overset{0}{g})\}$, we infer from (3.8), (3.21), (3.22), (3.23) that for $(\mu, g) \in \mathcal{X}$

$$\mathcal{E}_\mu^{-1}(\{g\}) = \mathfrak{R}(T, \mu, g) = \{ \overset{\mu,g}{u}, u \in \mathfrak{R}(T, \overset{0}{\mu}, \overset{0}{g})\}$$
$$\text{and } d_{\overset{\mu,g}{u}}\mathcal{E}_\mu \in \mathcal{W}(u) \subset AutH.$$

This means that $\mathcal{X}$ is the neighbourhood of $(\overset{0}{\mu}, \overset{0}{g})$ in $]0, \infty[\times H$ contained in $\mathcal{O}$.

We move to a proof of the density. First, we establish that for a fixed $\mu \in]0, \infty[$

$$\text{(3.24)} \qquad \begin{aligned} &\text{the set } \mathcal{O}_\mu := \{g \in H : g \textit{ is a regular value} \\ &\textit{of the mapping } \mathcal{E}_\mu\} \text{ is dense in } H. \end{aligned}$$

Indeed, according to (3.11), $\mathcal{E}_\mu$ is a Fredholm mapping. Thus, by virtue of the Smale theorem, statement (3.24) holds.

To prove the density of the set $\mathcal{O}$ in $]0, \infty[\times H$, it is sufficient to show that

$$A \cap \mathcal{O} \neq \emptyset \quad \text{for any } \emptyset \neq A \in top(]0, \infty[\times H).$$

We may assume that $A =]a, b[\times \mathcal{C}$ for some $a, b \in \mathbb{R}$ such that $0 < a < b$ and for some nonempty set $\mathcal{C} \in topH$. Let us choose $\mu \in]a, b[$. By (3.24), the intersection $\mathcal{C} \cap \mathcal{O}_\mu$ is nonempty. Let us choose $g \in \mathcal{C} \cap \mathcal{O}_\mu$. In particular, the pair $(\mu, g) \in]a, b[\times \mathcal{C}$. Since $g \in \mathcal{O}_\mu$, g is a regular value of the mapping $\mathcal{E}_\mu$. According to the definition of the set $\mathcal{O}$, this means that $(\mu, g) \in \mathcal{O}$. In conclusion, $(\mu, g) \in \mathcal{O} \cap (]a, b[\times \mathcal{C})$. □

Application to the Navier-Stokes problem. *In the sequel, we assume that* $n \in \{2, 3\}$. In Section 4, we will prove that the Holly method is stable whenever the data (viscosity, external forces) belong to some set $\mathcal{G}$. In this subsection, we prove that this set $\mathcal{G}$ is open and dense in $]0, \infty[\times (H_0^1)'$. This result follows from Theorem 3.6.

The restriction on the dimension n is caused by the fact that we apply the theory of Fredholm mappings. The crucial point is that then not only is the imbedding (1.4) continuous but it is also completely continuous. We prove this property in the sequel (see the proof of Lemma 3.7).

If $n \in \{2, 3\}$, then the sequence in the second part of the implication (2.23) is convergent in the norm. In fact, we have the following

LEMMA 3.7. *If* $\overset{k}{u} \to u, \overset{k}{w} \to w$ *weakly in* H_0^1, *then* $\mathcal{A}_{\overset{k}{u},\overset{k}{w}} \to \mathcal{A}_{u,w}$ *in* $(H_0^1)'$ *as* $k \to \infty$.

PROOF. It is easy to see that

$$\sum_{i=1}^{n} \overset{1}{v}_i \frac{\partial \overset{2}{v}}{\partial x_i} + \frac{\operatorname{div} \overset{1}{v}}{2} \overset{2}{v} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\overset{1}{v}_i \overset{2}{v}) - \frac{\operatorname{div} \overset{1}{v}}{2} \overset{2}{v}.$$

for any fields $\overset{1}{v}, \overset{2}{v} \in H_0^1$.

Let $\phi \in H_0^1$. Using the above equality and the integration by parts formula, we obtain

$$\begin{aligned}
&\mathcal{A}_{\overset{k}{u},\overset{k}{w}}(\phi) - \mathcal{A}_{u,w}(\phi) \\
&= \int_\Omega \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\overset{k}{u}_i \overset{k}{w}) - \frac{\operatorname{div} \overset{k}{u}}{2} \overset{k}{w} \right) \phi dm - \int_\Omega \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(u_i w) - \frac{\operatorname{div} u}{2} w \right) \phi dm \\
&= \int_\Omega \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\overset{k}{u}_i \overset{k}{w} - u_i w) \right) \phi dm - \frac{1}{2} \int_\Omega \left((\operatorname{div} \overset{k}{u}) \overset{k}{w} - (\operatorname{div} u) w \right) \phi dm \\
&= - \int_\Omega \sum_{i=1}^{n} \left(\overset{k}{u}_i (\overset{k}{w} - w) + (\overset{k}{u} - u)_i w \right) \frac{\partial \phi}{\partial x_i} dm \\
&\quad - \frac{1}{2} \int_\Omega \left((\operatorname{div} \overset{k}{u})(\overset{k}{w} - w) + \big(\operatorname{div}(\overset{k}{u} - u)\big) w \right) \phi dm \\
&= - \int_\Omega \sum_{i=1}^{n} \overset{k}{u}_i (\overset{k}{w} - w) \frac{\partial \phi}{\partial x_i} dm - \int_\Omega \sum_{i=1}^{n} (\overset{k}{u} - u)_i w \frac{\partial \phi}{\partial x_i} dm \\
&\quad - \frac{1}{2} \int_\Omega (\operatorname{div} \overset{k}{u})(\overset{k}{w} - w) \phi dm - \frac{1}{2} \int_\Omega \operatorname{div}(\overset{k}{u} - u) w \phi dm \\
&= - \int_\Omega \sum_{i=1}^{n} \overset{k}{u}_i (\overset{k}{w} - w) \frac{\partial \phi}{\partial x_i} dm - \frac{1}{2} \int_\Omega (\operatorname{div} \overset{k}{u})(\overset{k}{w} - w) \phi dm \\
&\quad + \frac{1}{2} \int_\Omega \sum_{i=1}^{n} (\overset{k}{u} - u)_i \frac{\partial w}{\partial x_i} \phi dm - \frac{1}{2} \int_\Omega \sum_{i=1}^{n} (\overset{k}{u} - u)_i w \frac{\partial \phi}{\partial x_i} dm.
\end{aligned}$$

To obtain the last equality, we integrated by parts $\int_\Omega \operatorname{div}(\overset{k}{u} - u) w \phi dm$. The Hölder inequality and the continuity of the imbedding (1.4) yields the estimation

$$\begin{aligned}
&|\mathcal{A}_{\overset{k}{u},\overset{k}{w}}(\phi) - \mathcal{A}_{u,w}(\phi)| \\
&\leq \left(\frac{3}{2} |\iota| \|\overset{k}{u}\|_{H_0^1} \|\overset{k}{w} - w\|_{L^4} + |\iota| \|w\|_{H_0^1} \|\overset{k}{u} - u\|_{L^4} \right) \|\phi\|_{H_0^1}.
\end{aligned}$$

Hence

$$
\begin{aligned}
&\left|\mathcal{A}_{\overset{k}{u},\overset{k}{w}} - \mathcal{A}_{u,w}\right|_{(H_0^1)'} \\
&\leq \frac{3}{2}|\iota|\|\overset{k}{u}\|_{H_0^1}\|\overset{k}{w} - w\|_{L^4} + |\iota|\|w\|_{H_0^1}\|\overset{k}{u} - u\|_{L^4}.
\end{aligned} \tag{3.25}
$$

Since $n \in \{2,3\}$, by (1.3), the imbedding

$$H_0^1 \hookrightarrow L^6 \tag{3.26}$$

is continuous. From the Hölder inequality, it follows that

$$\|v\|_{L^4} \leq \left(\|v\|_{L^2}\right)^{1/4}\left(\|v\|_{L^6}\right)^{3/4} \tag{3.27}$$

for any $v \in H_0^1$. Hence, we infer that

(3.28) the imbedding $\iota : H_0^1 \hookrightarrow L^4$ is completely continuous.

Indeed, by virtue of the Remark 1.4, it is sufficient to check that

$$\text{if } \overset{k}{v} \to 0 \text{ weakly in } H_0^1, \text{ then } \iota(\overset{k}{v}) \to 0 \text{ in } L^4 \text{ as } k \to \infty.$$

By the continuity of the imbedding (3.26), the sequence $(\|\overset{k}{v}\|_{L^6})$ is bounded and by complete continuity of the imbedding (1.5), $\lim_{k\to\infty}\|\overset{k}{v}\|_{L^2} = 0$. Hence, by the inequality (3.27),

$$\lim_{k\to\infty}\|\iota(\overset{k}{v})\|_{L^4} = \lim_{k\to\infty}\|\overset{k}{v}\|_{L^4} = 0.$$

By virtue of (3.28) and estimation (3.25), we conclude

$$\left|\mathcal{A}_{\overset{k}{u},\overset{k}{w}} - \mathcal{A}_{u,w}\right|_{(H_0^1)'} \to 0 \quad \text{as } k \to \infty,$$

since $\|\overset{k}{u} - u\|_{L^4} \to 0$, $\|\overset{k}{w} - w\|_{L^4} \to 0$ as $k \to \infty$ and the sequence $(\|\overset{k}{u}\|_{H_0^1})_{k\in\mathbb{N}}$ is bounded. □

Since the mapping $\mathcal{A}$ is bilinear, we have the following

COROLLARY 3.8. *For any $u, w \in H_0^1$, the Fréchet differential*

$$d_{(u,w)}\mathcal{A} : H_0^1 \times H_0^1 \to (H_0^1)'$$

is completely continuous.

PROOF. Let the sequence $\left((\overset{1}{h}_k, \overset{2}{h}_k)\right)_{k\in\mathbb{N}}$ tend to $(0,0)$ weakly in $H_0^1 \times H_0^1$ as $k \to \infty$. According to (2.20), $\mathcal{A}$ is bilinear and continuous. Hence, by Lemma

3.7, we obtain

$$\begin{aligned}(d_{(u,w)}\mathcal{A})(\overset{1}{h}_k, \overset{2}{h}_k) &= \mathcal{A}(u, \overset{2}{h}_k) + \mathcal{A}(\overset{1}{h}_k, w)\\ &= \mathcal{A}_{u,\overset{2}{h}_k} + \mathcal{A}_{\overset{1}{h}_k,w} \to 0 \quad \text{in} \quad (H_0^1)' \quad \text{as } k \to \infty.\end{aligned}$$

□

REMARK. The complete continuity of the differential $d_{(u,w)}\mathcal{A}$ is also a consequence of (3.28) and of Lemma 1.6.

The property of the operator $\mathcal{A}$ expressed in Lemma 3.7 is of fundamental importance. We will use it while investigating the problem of stability as well as that of genericity.

Referring to Theorem 2.9, we introduce the following notations

$\mathfrak{R}(\nu, f)$ – the set of all solutions of (*),

$\mathfrak{R}^s(\nu, f_s)$ – the set of all solutions of $(*)_s$ for fixed $s \in \mathbb{N}$,

$\mathfrak{R}^{s,N}(\nu, f_s)$ – the set of all solutions of $(*)_{s,N}$ for fixed $(s, N) \in \mathbb{N}^2$.

REMARK 3.9.

(1) *The set $\mathfrak{R}(\nu, f)$ is a compact subset of the sphere*

$$S(c, \|c\|_{H_0^1}) := \{v \in V : \|v - c\|_{H_0^1} = \|c\|_{H_0^1}\},$$

where $c = \frac{1}{2\nu} P_V \mathcal{R}^{-1} f$.

(2) *The set $\mathfrak{R}^s(\nu, f_s)$ is a compact subset of the sphere*

$$S(\overset{s}{c}, \|\overset{s}{c}\|_{H_0^1}) := \{u \in H_0^1 : \|u - \overset{s}{c}\|_{H_0^1} = \|\overset{s}{c}\|_{H_0^1}\},$$

where $\overset{s}{c} = \frac{1}{2\nu} P_V^s \mathcal{R}^{-1} f_s$.

(3) *The set $\mathfrak{R}^{s,N}(\nu, f_s)$ is a compact subset of the sphere*

$$S(\overset{s,N}{c}, \|\overset{s,N}{c}\|_{H_0^1}) := \{w \in H_N : \|w - \overset{s,N}{c}\|_{H_0^1} = \|\overset{s,N}{c}\|_{H_0^1}\},$$

where $\overset{s,N}{c} = \frac{1}{2\nu} P_V^{s,N} \mathcal{R}_N^{-1} f_s$.

PROOF. By Theorem 2.9 d), the set $\mathfrak{R}(\nu, f)$ is a nonempty subset of V. Let $v \in \mathfrak{R}(\nu, f)$. Due to (2.36),

$$\|v\|_{H_0^1}^2 = \frac{1}{\nu}((P_V \mathcal{R}^{-1} f | v)) = ((2c|v)).$$

Hence

$$\|v\|_{H_0^1} \le \frac{1}{\nu} \|P_V \mathcal{R}^{-1} f\|_{H_0^1} = \|2c\|_{H_0^1} \tag{3.29}$$

and

$$\|v-c\|^2_{H^1_0} = \|v\|^2_{H^1_0} - 2((c|v)) + \|c\|^2_{H^1_0} = \|c\|^2_{H^1_0},$$

which means that $\mathfrak{R}(\nu, f) \subset S(c, \|c\|_{H^1_0})$. To prove the compactness of the set $\mathfrak{R}(\nu, f)$, we apply Remark 3.3 with

$$H := V, \quad T : V \ni \phi \mapsto P_V \mathcal{R}^{-1} \mathcal{A}_{\phi,\phi} \in V, \quad \mu := \nu, \quad g := P_V \mathcal{R}^{-1} f.$$

Then $\mathfrak{R}(\nu, f) = \mathfrak{R}(T, \mu, g)$. The mapping T is of class $\mathcal{C}^1$ (and even of class $\mathcal{C}^\infty$) as a polynomial of degree 2 of the space V. It remains to check the assumptions (3.1) and (3.3). The assumption (3.1) follows from Lemma 3.7 and the continuity of the operators P_V and $\mathcal{R}^{-1}$. By (3.29), the estimation (3.3) holds if we take

$$\kappa :]0, \infty[\times [0, \infty[\ni (\mu, r) \mapsto \frac{r}{\mu} \in [0, \infty[.$$

Let us move to statement (2). By Theorem 2.9 b), the set $\mathfrak{R}^s(\nu, f_s)$ is a nonempty subset of the space H^1_0. Let $u \in \mathfrak{R}^s(\nu, f_s)$. Due to (2.33),

$$\|u\|^2_{H^1_0} = \frac{1}{\nu}((P^s_V \mathcal{R}^{-1} f_s|u)) = ((2\overset{s}{c}|u)).$$

Hence

$$\|u\|_{H^1_0} \le \|P^s_V \mathcal{R}^{-1} f_s\|_{H^1_0} = \|2\overset{s}{c}\|_{H^1_0} \tag{3.30}$$

and

$$\|u - \overset{s}{c}\|^2_{H^1_0} = \|\overset{s}{c}\|^2_{H^1_0},$$

which means that $\mathfrak{R}^s(\nu, f_s) \subset S(\overset{s}{c}, \|\overset{s}{c}\|_{H^1_0})$. Again, the compactness of $\mathfrak{R}^s(\nu, f_s)$ follows from Remark 3.3 if we take

$$H := H^1_0, \quad T : H^1_0 \ni \phi \mapsto P^s_V \mathcal{R}^{-1} \mathcal{A}_{\phi, P^s_V \phi} \in H^1_0, \quad \mu := \nu, \quad g := P^s_V \mathcal{R}^{-1} f_s,$$

$$\kappa :]0, \infty[\times [0, \infty[\ni (\mu, r) \mapsto \frac{r}{\mu} \in [0, \infty[.$$

Then $\mathfrak{R}^s(\nu, f_s) = \mathfrak{R}(T, \mu, g)$. Lemma 3.7, the estimation (3.30) and the continuity of the operators P^s_V, $\mathcal{R}^{-1}$ yield that the conditions (3.1),(3.3) are fulfilled.

To prove (3), let us remark that, by Theorem 2.9 a), the set $\mathfrak{R}^{s,N}(\nu, f_s)$ is a subset of the space H_N. Let $w \in H_N$. Due to (2.32)

$$\|w\|^2_{H^1_0} = \frac{1}{\nu} f_s(P^{s,N}_V w) = \frac{1}{\nu}((P^{s,N}_V \mathcal{R}^{-1}_N f_s|w)) = ((2\overset{s,N}{c}|w)).$$

Hence

$$\|w\|_{H^1_0} \le \frac{1}{\nu}\|P^{s,N}_V \mathcal{R}^{-1}_N f_s\|_{H^1_0} = \|2\overset{s,N}{c}\|_{H^1_0} \tag{3.31}$$

and

$$\|w - \overset{s,N}{c}\|^2_{H^1_0} = \|\overset{s,N}{c}\|^2_{H^1_0},$$

which means that $\mathfrak{R}^{s,N}(\nu, f_s) \subset S(\overset{s,N}{c}, \| \overset{s,N}{c} \|_{H_0^1})$. Again, the compactness of the set $\mathfrak{R}^{s,N}(\nu, f_s)$ follows from Remark 3.3. This time, we put

$$H := H_N, \quad T : H_N \ni \phi \mapsto P_V^{s,N} \mathcal{R}_N^{-1} \mathcal{A}_{\phi, P_V^{s,N}\phi} \in H_N, \quad \mu := \nu,$$

$$g := P_V^{s,N} \mathcal{R}_N^{-1} f_s, \quad \kappa :]0, \infty[\times [0, \infty[\ni (\mu, r) \mapsto \frac{r}{\mu} \in [0, \infty[.$$

Then $\mathfrak{R}^{s,N}(\nu, f_s) = \mathfrak{R}(T, \mu, g)$. To check the conditions (3.1), (3.3), it is sufficient to refer to Lemma 3.7, estimation (3.31) and the continuity of the operators $P_V^{s,N}$ and $\mathcal{R}_N^{-1}$. □

LEMMA 3.10. *The set*

$$\mathcal{G}_1 := \{(\nu, c) \in]0, \infty[\times V : c \text{ is a regular value of the mapping } V \ni \phi \mapsto \nu\phi + P_V \mathcal{R}^{-1} \mathcal{A}_{\phi,\phi} \in V\}$$

is open and dense in $]0, \infty[\times V$. *Moreover, if* $(\nu, c) \in \mathcal{G}_1$ *then the set* $\{v \in V : \nu v + P_V \mathcal{R}^{-1} \mathcal{A}_{v,v} = c\}$ *is finite.*

PROOF. The statement follows from Theorem 3.6 if we take

$$H := V, \quad T : V \ni \phi \mapsto P_V \mathcal{R}^{-1} \mathcal{A}_{\phi,\phi} \in V, \quad \mu := \nu, \quad g := c.$$

It follows from the proof of Remark 3.9 (1) that T is a $\mathcal{C}^1$ - mapping satisfying the conditions (3.1)-(3.3). □

Now, we prove some simple topological result which we will need in the proof of the main theorem of this subsection.

REMARK 3.11. *Let* X, Y *be topological spaces. Assume that* $P : X \to Y$ *is an open mapping and* Q *is a dense subset of* Y. *Then* $P^{-1}(Q)$ *is a dense subset of* X.

PROOF. It is sufficient to show that for any nonempty set $A \in topX$, the intersection $A \cap P^{-1}(Q)$ is nonempty. Since P is open, $P(A)$ is a nonempty open subset of Y. Then, the density of the set Q in the space Y yields that $P(A) \cap Q$ is nonempty. Hence, there exists an element $a \in A$ such that $P(a) \in Q$, or equivalently, $a \in A \cap P^{-1}(Q)$. □

In Section 4, we will prove that the Holly method is stable if the data (viscosity, external forces) belong to the set

$$\mathcal{G} := \{(\nu, f) \in]0, \infty[\times (H_0^1)' : P_V \mathcal{R}^{-1} f \text{ is a regular value of the mapping } V \ni \phi \mapsto \nu\phi + P_V \mathcal{R}^{-1} \mathcal{A}_{\phi,\phi} \in V\}.$$

The main result concerning the properties of the set $\mathcal{G}$ is the following theorem inspired by the results of K.Holly, C.Foiaş and R.Temam (see. [**13**], [**5**]). This

theorem asserts generic properties of the set of solutions of the Navier-Stokes problem with respect to the viscosity ν and external forces f, simultaneously.

THEOREM 3.12. *The set $\mathcal{G}$ is open and dense in $]0,\infty[\times(H_0^1)'$. Moreover, if $(\nu,f)\in\mathcal{G}$, then the set $\mathfrak{R}(\nu,f)$ is finite.*

PROOF. Let us remark that

$$\mathcal{G}=(\mathrm{id}\times P_V\mathcal{R}^{-1})^{-1}(\mathcal{G}_1). \tag{3.32}$$

Indeed,

$$\begin{aligned}&(\nu,f)\in\mathcal{G}\Leftrightarrow(\nu,P_V\mathcal{R}^{-1}f)\in\mathcal{G}_1\Leftrightarrow(\mathrm{id}\times P_V\mathcal{R}^{-1})(\nu,f)\in\mathcal{G}_1\\&\Leftrightarrow(\nu,f)\in(\mathrm{id}\times P_V\mathcal{R}^{-1})^{-1}(\mathcal{G}_1).\end{aligned}$$

The mapping

$$(\mathrm{id}\times P_V\mathcal{R}^{-1}):]0,\infty[\times(H_0^1)'\to]0,\infty[\times V \tag{3.33}$$

is open and continuous (its openness follows from the openness of the projection P_V). According to Lemma 3.10, the set $\mathcal{G}_1$ is dense in $]0,\infty[\times V$. The openness of the mapping (3.33) and (3.32) yield, by Remark 3.11, that $\mathcal{G}$ is dense in $]0,\infty[\times(H_0^1)'$. Since $\mathcal{G}_1$ is open in $]0,\infty[\times V$ and the mapping (3.33) is continuous, we infer from (3.32) that $\mathcal{G}$ is also open in $]0,\infty[\times(H_0^1)'$.

To prove the second part of the statement, it is sufficient to apply the second part of Lemma 3.10 with $c:=P_V\mathcal{R}^{-1}f$. □

4. THE PROBLEM OF STABILITY

We investigate properties of the sequences of the sets $\big(\mathfrak{R}^s(\nu,f_s)\big)_{s\in\mathbb{N}}$, $\big(\mathfrak{R}^{s,N}(\nu,f_s)\big)_{N\in\mathbb{N}}$ in the case $n\in\{2,3\}$. We prove that for $(\nu,f)\in\mathcal{G}$, the sequence $\big(\mathfrak{R}^s(\nu,f_s)\big)_{s\in\mathbb{N}}$ is convergent to the set $\mathfrak{R}(\nu,f)$ in the Hausdorff metric over H_0^1. Next, if we fix a sufficiently large $s\in\mathbb{N}$, then the sequence $\big(\mathfrak{R}^{s,N}(\nu,f_s)\big)_{N\in\mathbb{N}}$ is convergent to the set $\mathfrak{R}^s(\nu,f_s)$ as $N\to\infty$ in the same metric space, for the same data $(\nu,f)\in\mathcal{G}$.

Let us recall that, if (X,ϱ) is a metric space and A,B are nonempty, bounded, and closed subsets of X, then their *Hausdorff distance* is defined as follows

$$d(A,B):=\sup\{dist(a,B),dist(b,A):\quad a\in A,b\in B\},$$

where for $Y\subset X,x\in X$: $dist(x,Y):=\inf\{\varrho(x,y):\quad y\in Y\}$ is the distance between x and the set Y.

In the sequel, we consider *the Hausdorff metric on the family of all nonempty, bounded, and closed subsets of the space H_0^1.*

Let us remark that

$$d(A,B) \leq \varepsilon \Leftrightarrow (A \subset B + \overline{K}(\varepsilon) \text{ and } B \subset A + \overline{K}(\varepsilon)),$$

where $\overline{K}(\varepsilon) := \overline{K}_{H_0^1}(0,\varepsilon)$.

In this section, we make the same assumptions as in Theorem 2.9, i.e., $(\nu, f) \in]0,\infty[\times(H_0^1)'$, $f_s \to f$ in $(H_0^1)'$ and the sequence $(H_N)_{N\in\mathbb{N}}$ is an internal approximation of the space H_0^1.

The following two results are prototypes of the main results.

THEOREM 4.1. *If* $\#\mathfrak{R}(\nu,f) = 1$, *then* $\lim_{s\to\infty}\mathfrak{R}^s(\nu,f_s) = \mathfrak{R}(\nu,f)$ *in the Hausdorff metric; in particular,* $\lim_{s\to\infty} diam\mathfrak{R}^s(\nu,f_s) = 0$, *(*$diam\mathfrak{R}^s(\nu,f_s)$ *denotes the diameter of the set* $\mathfrak{R}^s(\nu,f_s)$*).*

PROOF. Let v be the unique solution of the equation (*).
Let $(\overset{1}{v}, \overset{2}{v}, \dots) \in \times_{s=1}^{\infty}\mathfrak{R}^s(\nu,f_s)$. We will show that $\lim_{s\to\infty}\|\overset{s}{v} - v\|_{H_0^1} = 0$.

Assume, contrary to our claim, that $\overset{s}{v} \nrightarrow v$ in H_0^1 as $s \to \infty$. Then

$$\text{there exist } \varepsilon_0 > 0 \text{ and an infinite subset } \mathcal{S} \subset \mathbb{N} \text{ such that } \|\overset{s}{v} - v\|_{H_0^1} \geq \varepsilon_0 \text{ for } s \in \mathcal{S}. \tag{4.1}$$

Since, by Theorem 2.9 c), the sequence $(\overset{s}{v})_{s\in\mathcal{S}}$ is bounded, the Banach-Alaoglu theorem yields that

$$\text{there exist } u \in H_0^1 \text{ and an infinite subset } \mathcal{S}_1 \subset \mathcal{S} \text{ such that } \overset{s}{v} \to u \text{ weakly in } H_0^1 \text{ as } \mathcal{S}_1 \ni s \to \infty.$$

Then, by Theorem 2.9 d), $u \in \mathfrak{R}(\nu,f)$ and $\|\overset{s}{v} - u\|_{H_0^1} \to 0$ as $\mathcal{S}_1 \ni s \to \infty$. Since $\#\mathfrak{R}(\nu,f) = 1$, $u = v$. This contradicts (4.1). Thus

$$\text{for every } (\overset{1}{v}, \overset{2}{v}, \dots) \in \times_{s=1}^{\infty}\mathfrak{R}^s(\nu,f_s) : \lim_{s\to\infty}\|\overset{s}{v} - v\|_{H_0^1} = 0. \tag{4.2}$$

Let $\varepsilon > 0$. By (4.2), we infer that for a sufficiently large $s \in \mathbb{N}$:

$$\{v\} \subset \mathfrak{R}^s(\nu,f_s) + \overline{K}(\varepsilon) \quad \text{and} \quad \mathfrak{R}^s(\nu,f_s) \subset \{v\} + \overline{K}(\varepsilon).$$

□

If we fix $s \in \mathbb{N}$ and let $N \to \infty$, then we get a similar result.

THEOREM 4.2. *If* $\#\mathfrak{R}^s(\nu,f_s) = 1$, *then* $\lim_{N\to\infty}\mathfrak{R}^{s,N}(\nu,f_s) = \mathfrak{R}^s(\nu,f_s)$ *in the Hausdorff metric; in particular,* $\lim_{N\to\infty} diam\mathfrak{R}^{s,N}(\nu,f_s) = 0$.

PROOF. Let $\overset{s}{v}$ be the unique solution of the equation $(*)_s$. Let $(\overset{1}{u}, \overset{2}{u}, \dots) \in \times_{N=1}^{\infty}\mathfrak{R}^{s,N}(\nu,f_s)$. We will show that $\lim_{N\to\infty}\|\overset{N}{u} - \overset{s}{v}\|_{H_0^1} = 0$.

Suppose, contrary to our claim, that $\overset{N}{u} \nrightarrow \overset{s}{v}$ in H_0^1 as $N \to \infty$. Then

$$\text{there exist } \varepsilon_0 > 0 \text{ and an infinite subset } \mathcal{N} \subset \mathbb{N} \tag{4.3}$$
$$\text{such that } \| \overset{N}{u} - \overset{s}{v} \|_{H_0^1} \geq \varepsilon_0 \text{ for } N \in \mathcal{N}.$$

By (3.31)

$$\| \overset{N}{u} \|_{H_0^1} \leq \frac{1}{\nu} |P_V^{s,N}|_{EndH_0^1} |f_s|_{(H_0^1)'}.$$

Thus, by Remark 2.6 (c), the sequence $(\overset{N}{u})_{N\in\mathbb{N}}$ is bounded. By virtue of the Banach-Alaoglu theorem

$$\text{there exist } u \in H_0^1 \text{ and an infinite subset } \mathcal{N}_1 \subset \mathcal{N}$$
$$\text{such that } \overset{N}{u} \to u \text{ weakly in } H_0^1 \text{ as } \mathcal{N}_1 \ni N \to \infty.$$

Then, by Theorem 2.9 b), $u \in \mathfrak{R}^s(\nu, f_s)$ and $\| \overset{N}{u} - u \|_{H_0^1} \to 0$ as $\mathcal{N}_1 \ni N \to \infty$. Since $\#\mathfrak{R}^s(\nu, f_s) = 1$, $u = \overset{s}{v}$, which contradictes (4.3). Thus,

$$\text{for any } (\overset{1}{u}, \overset{2}{u}, \dots) \in \times_{N=1}^{\infty} \mathfrak{R}^{s,N}(\nu, f_s) : \lim_{N\to\infty} \| \overset{N}{u} - \overset{s}{v} \|_{H_0^1} = 0, \tag{4.4}$$

and hence, we infer that for any $\varepsilon > 0$:

$$\{\overset{s}{v}\} \subset \mathfrak{R}^{s,N}(\nu, f_s) + \overline{K}(\varepsilon) \quad \text{and} \quad \mathfrak{R}^{s,N}(\nu, f_s) \subset \{\overset{s}{v}\} + \overline{K}(\varepsilon)$$

for almost all $N \in \mathbb{N}$. □

For the operator $L \in EndH_0^1$, we denote

$$\mathcal{A}^L := \mathcal{A} \circ (\mathrm{id} \vartriangle L) : H_0^1 \to (H_0^1)',$$

i.e., $\mathcal{A}^L(u) = \mathcal{A}\big(u, L(u)\big) = \mathcal{A}_{u,L(u)}$ for any $u \in H_0^1$. For fixed $\nu \in]0, \infty[$

$$\mathcal{K}_\nu := (\nu \mathrm{id} + P_V \circ \mathcal{R}^{-1} \circ \mathcal{A}^{P_V}) : H_0^1 \to H_0^1,$$

and for fixed $s, N \in \mathbb{N}$

$$\mathcal{K}_\nu^s := (\nu \mathrm{id} + P_V^s \circ \mathcal{R}^{-1} \circ \mathcal{A}^{P_V^s}) : H_0^1 \to H_0^1,$$
$$\mathcal{K}_\nu^{s,N} := (\nu \mathrm{id} + P_V^{s,N} \circ \mathcal{R}_N^{-1} \circ \mathcal{A}^{P_V^{s,N}}) : H_0^1 \to H_0^1.$$

Let us remark that

$$\mathfrak{R}(\nu, f) = (\mathcal{K}_\nu)^{-1}(\{P_V \mathcal{R}^{-1} f\}), \tag{4.5}$$
$$\mathfrak{R}^s(\nu, f_s) = (\mathcal{K}_\nu^s)^{-1}(\{P_V^s \mathcal{R}^{-1} f_s\}), \tag{4.6}$$
$$\mathfrak{R}^{s,N}(\nu, f_s) = (\mathcal{K}_\nu^{s,N})^{-1}(\{P_V^{s,N} \mathcal{R}_N^{-1} f_s\}). \tag{4.7}$$

Indeed, $v \in \mathfrak{R}(\nu, f)$ if and only if

$$\nu v + P_V \mathcal{R}^{-1} \mathcal{A}_{v, P_V v} = P_V \mathcal{R}^{-1} f,$$

which is equivalent to (4.5). Equalities (4.6), (4.7) can be checked in a similar way.

An application of the implicit function theorem (Theorem 1.10) to $X = \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, $Y = Z = H_0^1$,

$$F : \overline{\mathbb{N}} \times H_0^1 \ni (s,u) \mapsto \begin{cases} \mathcal{K}_\nu^s(u) - P_V^s \mathcal{R}^{-1} f_s & \text{for} \quad s < \infty, \\ \mathcal{K}_\nu(u) - P_V \mathcal{R}^{-1} f & \text{for} \quad s = \infty \end{cases} \tag{4.8}$$

with respect to $(x_0, y_0) := (\infty, v)$, where $v \in \mathfrak{R}(\nu, f)$, yields the following result.

LEMMA 4.3. *If $v \in \mathfrak{R}(\nu, f)$ and $d_v\mathcal{K}_\nu \in EpiH_0^1$, then there exists a neighbourhood $\mathcal{Y} \in topH_0^1$ of v such that*

(i) $\#\big(\mathcal{Y} \cap \mathfrak{R}^s(\nu, f_s)\big) = 1$ *for almost all* $s \in \mathbb{N}$,

(ii) $\lim_{s\to\infty} \|\overset{s}{v} - v\|_{H_0^1} = 0$, *where* $\{\overset{s}{v}\} = \mathcal{Y} \cap \mathfrak{R}^s(\nu, f_s)$.

PROOF. Assume for a moment that the mapping (4.8) satisfies the assumptions of Theorem 1.10. Then there exist neighbourhoods: $\mathcal{X} \in top\overline{\mathbb{N}}$ – of infinity and $\mathcal{Y} \in topH_0^1$ – of v such that the relation $\eta := \{F = 0\} \cap (\mathcal{X} \times \mathcal{Y})$ is a continuous function from $\mathcal{X}$ to $\mathcal{Y}$ and $\eta(\infty) = v$. In particular, for every $s \in \mathcal{X}$ there exists the unique $\overset{s}{v} \in \mathcal{Y}$ such that $\eta(s) = \overset{s}{v}$. Since $\eta \subset \{F = 0\}$, $F(s, \overset{s}{v}) = 0$, and hence, by (4.6), $\overset{s}{v} \in \mathfrak{R}^s(\nu, f_s)$. The continuity of the function η at infinity means that $\lim_{s\to\infty} \|\overset{s}{v} - v\|_{H_0^1} = 0$.

Thus, it is sufficient to verify the assumptions of Theorem 1.10. Since $v \in \mathfrak{R}(\nu, f)$, $F(\infty, v) = 0$. By Remark 2.6 (b), we infer that the mapping $F(\cdot, u) : \overline{\mathbb{N}} \to H_0^1$ is continuous for any $u \in H_0^1$. Moreover, for $(s,u) \in \overline{\mathbb{N}} \times H_0^1$

$$d^{II}_{(s,u)} F = \begin{cases} d_u \mathcal{K}_\nu^s & \text{for} \quad s < \infty, \\ d_u \mathcal{K}_\nu & \text{for} \quad s = \infty. \end{cases} \tag{4.9}$$

By virtue of Lemma 3.7 and of the continuity of P_V and $\mathcal{R}^{-1}$, the following conclusion holds

$$\text{if } \overset{k}{u} \to u \text{ weakly in } H_0^1, \text{ then } P_V \mathcal{R}^{-1} \mathcal{A}_{\overset{k}{u}, P_V \overset{k}{u}} \to P_V \mathcal{R}^{-1} \mathcal{A}_{u, P_V u} \text{ in } H_0^1 \tag{4.10}$$

as $k \to \infty$. Thus, the mapping $P_V \circ \mathcal{R}^{-1} \mathcal{A}^{P_V} H_0^1 \to H_0^1$ satisfies the condition (3.1) and hence, by Remark 3.1, it is completely continuous. Hence, by virtue of Corollary 1.7,

$$\mathcal{K}_\nu \text{ is a Fredholm mapping of index } 0. \tag{4.11}$$

Since $d_v\mathcal{K}_\nu \in EpiH_0^1$, $d_v\mathcal{K}_\nu \in AutH_0^1$, by (1.1).

It remains to check that the differential

$$d^{II}F : \overline{\mathbb{N}} \times H_0^1 \ni (s,u) \mapsto d^{II}_{(s,u)} F \in EndH_0^1$$

is continuous at (∞, v). Let $(\overset{k}{s}, \overset{k}{u}) \to (\infty, v)$ in $\overline{\mathbb{N}} \times H_0^1$ as $k \to \infty$. Until the end of this proof, we use the notation $P_V^\infty := P_V$. Then, by (4.9)

$$d^{II}_{(\overset{k}{s},\overset{k}{u})} F = d_{\overset{k}{u}} \mathcal{K}_\nu^{\overset{k}{s}} = \nu id + P_V^{\overset{k}{s}} \circ \mathcal{R}^{-1} \circ d_{\overset{k}{u}} \mathcal{A}^{P_V^{\overset{k}{s}}}.$$

Since, by Remark 2.6 (b), $P_V^{\overset{k}{s}} \to P_V$ in $EndH_0^1$ as $k \to \infty$, the proof will be complete if we show that

$$d_{\overset{k}{u}} \mathcal{A}^{P_V^{\overset{k}{s}}} \to d_v \mathcal{A}^{P_V} \quad \text{in} \quad \mathcal{L}\big(H_0^1, (H_0^1)'\big) \quad \text{as} \quad k \to \infty. \tag{4.12}$$

Let $h \in H_0^1$. By Lemma 2.7 b) and by (2.22), we obtain

$$\begin{aligned}
&\big|d_{\overset{k}{u}} \mathcal{A}^{P_V^{\overset{k}{s}}}(h) - d_v \mathcal{A}^{P_V}(h)\big|_{(H_0^1)'} \\
&= |\mathcal{A}(\overset{k}{u}, P_V^{\overset{k}{s}} h) + \mathcal{A}(h, P_V^{\overset{k}{s}} \overset{k}{u}) - \mathcal{A}(v, P_V h) - \mathcal{A}(h, P_V v)|_{(H_0^1)'} \\
&\le \big|\mathcal{A}_{\overset{k}{u}-v, P_V^{\overset{k}{s}} h}\big|_{(H_0^1)'} + \big|\mathcal{A}_{v,(P_V^{\overset{k}{s}}-P_V)h}\big|_{(H_0^1)'} + \big|\mathcal{A}_{h, P_V^{\overset{k}{s}} \overset{k}{u} - P_V v}\big|_{(H_0^1)'} \\
&\le \frac{3}{2}|\iota|^2 \Big(\|\overset{k}{u} - v\|_{H_0^1} + \|v\|_{H_0^1} |P_V^{\overset{k}{s}} - P_V|_{EndH_0^1} + \|P_V^{\overset{k}{s}} \overset{k}{u} - P_V v\|_{H_0^1} \Big) \|h\|_{H_0^1}
\end{aligned}$$

for any $k \in \mathbb{N}$. Hence

$$\begin{aligned}
&|d_{\overset{k}{u}} \mathcal{A}^{P_V^{\overset{k}{s}}} - d_v \mathcal{A}^{P_V}|_{\mathcal{L}\left(H_0^1, (H_0^1)'\right)} \\
&\le \frac{3}{2}|\iota|^2 \Big(\|\overset{k}{u} - v\|_{H_0^1} + \|v\|_{H_0^1} |P_V^{\overset{k}{s}} - P_V|_{EndH_0^1} + \|P_V^{\overset{k}{s}} \overset{k}{u} - P_V v\|_{H_0^1} \Big), \quad k \in \mathbb{N}.
\end{aligned}$$

Passing to the limit as $k \to \infty$, by Remark 2.6 (b) and by Remark A.3, we obtain (4.12). □

REMARK 4.4. *If $c \in V$, then the following conditions are equivalent*

(i) *c is a regular value of the mapping $\mathcal{K}_\nu : H_0^1 \to H_0^1$,*
(ii) *c is a regular value of the mapping $(\mathcal{K}_\nu)_{|V} : V \to V$.*

PROOF. Since $\mathcal{K}_\nu(V) \subset V$, the mapping $(\mathcal{K}_\nu)_{|V}$ is well defined. Moreover, for $u \in V$

$$d_u(\mathcal{K}_\nu)_{|V} = (d_u \mathcal{K}_\nu)_{|V}. \tag{4.13}$$

According to (4.11), $\mathcal{K}_\nu, (\mathcal{K}_\nu)_{|V}$ are Fredholm mappings of index 0. Hence, by (1.1), the Fréchet differentials of these mappings are epimorphisms if and only if they are monomorphisms. Since $c \in V$,

$$\{u \in H_0^1 : \mathcal{K}_\nu(u) = c\} = \{v \in V : (\mathcal{K}_\nu)_{|V}(v) = c\}. \tag{4.14}$$

Let $u \in (\mathcal{K}_\nu)^{-1}(\{c\})$.

To prove that (ii) follows from (i), it is sufficient to check that

$$d_u(\mathcal{K}_\nu)_{|V} \in MonoV.$$

Let $h \in V$ be such that $\big(d_u(\mathcal{K}_\nu)_{|V}\big)(h) = 0$. By (4.13), $(d_u\mathcal{K}_\nu)(h) = 0$. Since c is a regular value of $\mathcal{K}_\nu$, $d_u\mathcal{K}_\nu \in MonoH_0^1$, and hence $h = 0$.

Now, suppose that c is a regular value of the restriction $(\mathcal{K}_\nu)_{|V}$. Let $h \in H_0^1$ be such that $(d_u\mathcal{K}_\nu)(h) = 0$. Hence

$$\nu h = -(P_V \circ \mathcal{R}^{-1} \circ d_u\mathcal{A}^{P_V})(h).$$

In particular, $h \in V$, and by (4.13), $\big(d_u(\mathcal{K}_\nu)_{|V}\big)(h) = 0$. Since $d_u(\mathcal{K}_\nu)_{|V}$ is a monomorphism, $h = 0$. □

The main result concerning stability as $s \to \infty$ is the following

THEOREM 4.5. *If $(\nu, f) \in \mathcal{G}$, then*

(i) $\lim_{s\to\infty} \mathfrak{R}^s(\nu, f_s) = \mathfrak{R}(\nu, f)$ *in the Hausdorff metric;*

(ii) *for almost all $s \in \mathbb{N}$:* $\#\mathfrak{R}^s(\nu, f_s) = \#\mathfrak{R}(\nu, f) < \infty$.

PROOF. Let $(\nu, f) \in \mathcal{G}$. Let $v \in \mathfrak{R}(\nu, f)$. According to the definition of the set $\mathcal{G}$ together with (4.5), $d_v(\mathcal{K}_\nu)_{|V} \in EpiV$. By Remark 4.4 and by (4.14), $d_v\mathcal{K}_\nu \in EpiH_0^1$. From Lemma 4.3, it follows that there exist $s(v) \in \mathbb{N}$ and a neighbourhood $\mathcal{Y}(v) \in topH_0^1$ – of v such that

(i) $\#\big(\mathcal{Y}(v) \cap \mathfrak{R}^s(\nu, f_s)\big) = 1 \quad \text{for} \quad s > s(v)$,

(ii) $\lim_{s\to\infty} \|\overset{s}{v} - v\|_{H_0^1} = 0, \quad \text{where} \quad \{\overset{s}{v}\} = \mathcal{Y}(v) \cap \mathfrak{R}^s(\nu, f_s)$.

According to Theorem 3.12, the set $\mathfrak{R}(\nu, f)$ is finite. Thus, if $s > s_* := \max\{s(v), \}$, $v \in \mathfrak{R}(\nu, f)$, then $\#\big(\mathcal{Y}(v) \cap \mathfrak{R}^s(\nu, f_s)\big) = 1$ for every $v \in \mathfrak{R}(\nu, f)$. Moreover, there exists $\delta > 0$ such that the balls $\{\overline{K}(v, \delta), v \in \mathfrak{R}(\nu, f)\}$ (in the space H_0^1) are pairwise disjoint and $\overline{K}(v, \delta) \subset \mathcal{Y}(v)$ for every $v \in \mathfrak{R}(\nu, f)$.

Since, by (ii) and finiteness of the set $\mathfrak{R}(\nu, f)$,

$$\lim_{s\to\infty} \max\{\|\overset{s}{v} - v\|_{H_0^1}, \quad v \in \mathfrak{R}(\nu, f)\} = 0, \tag{4.15}$$

there exists $s(\delta) \in \mathbb{N}$ such that $s(\delta) > s_*$ and for any $s > s(\delta)$

$$\max\{\|\overset{s}{v} - v\|_{H_0^1}, v \in \mathfrak{R}(\nu, f)\} \le \delta.$$

Hence, if $s > s(\delta)$, then the function

$$\mathfrak{R}(\nu, f) \ni v \mapsto \overset{s}{v} \in \mathfrak{R}^s(\nu, f_s)$$

is an injection. In particular

$$\#\mathfrak{R}(\nu, f) \le \mathfrak{R}^s(\nu, f_s).$$

Suppose that the set

$$\mathcal{S} := \{s \in \mathbb{N} : s > s(\delta) \quad \text{and} \quad \#\mathfrak{R}(\nu, f) < \#\mathfrak{R}^s(\nu, f_s)\}$$

is infinite. Let $s \in \mathcal{S}$. The set $\{\overset{s}{v}, v \in \mathfrak{R}(\nu, f)\}$ has exactly $\#\mathfrak{R}(\nu, f)$ elements. Thus, it is not the whole set $\mathfrak{R}^s(\nu, f_s)$. Let us select $\overset{s}{a} \in \mathfrak{R}^s(\nu, f_s) \backslash \{\overset{s}{v}, v \in \mathfrak{R}(\nu, f)\}$. It is clear that $\overset{s}{a} \notin \bigcup_{v \in \mathfrak{R}(\nu,f)} \mathcal{Y}(v)$. According to Theorem 2.9 c), the sequence $(\overset{s}{a})_{s \in \mathcal{S}}$ is bounded. Thus, the Banach-Alaoglu theorem yields that

$$\begin{aligned} &\text{there exist } a \in H_0^1, \text{ and an infinite subset } \mathcal{S}_1 \subset \mathcal{S} \\ &\text{such that } \overset{s}{a} \to a \text{ weakly in } H_0^1 \text{ as } \mathcal{S}_1 \ni s \to \infty. \end{aligned}$$

Then, by Theorem 2.9 d), $a \in \mathfrak{R}(\nu, f)$ and $\overset{s}{a} \to a$ in H_0^1 as $\mathcal{S}_1 \ni s \to \infty$. This leads to the contradiction

$$\bigcup_{v \in \mathfrak{R}(\nu,f)} \mathcal{Y}(v) \not\ni \quad a \in \mathfrak{R}(\nu, f) \subset \bigcup_{v \in \mathfrak{R}(\nu,f)} \mathcal{Y}(v).$$

Thus, $\#\mathcal{S} < \infty$ and for any $s > s_{**} := \max\{s(\delta), \sup \mathcal{S}\}$:

$$\#\mathfrak{R}^s(\nu, f_s) = \#\mathfrak{R}(\nu, f).$$

Let us now prove statement (i). Let $\varepsilon > 0$. Then, by (4.15), there exists $\overline{s} \in \mathbb{N}$ such that $\overline{s} > s_{**}$ and

$$\max\{\|\overset{s}{v} - v\|_{H_0^1}, \quad v \in \mathfrak{R}(\nu, f)\} \le \varepsilon \quad \text{for } s > \overline{s}. \tag{4.16}$$

Let $s > \overline{s}$. From (4.16), it follows that

$$\mathfrak{R}(\nu, f) \subset \mathfrak{R}^s(\nu, f_s) + \overline{K}(\varepsilon). \tag{4.17}$$

We will show that

$$\mathfrak{R}^s(\nu, f_s) \subset \mathfrak{R}(\nu, f) + \overline{K}(\varepsilon). \tag{4.18}$$

Indeed, let $u \in \mathfrak{R}^s(\nu, f_s) = \{\overset{s}{v}, v \in \mathfrak{R}(\nu, f)\}$. Then $u = \overset{s}{v}$ for some $v \in \mathfrak{R}(\nu, f)$. Therefore, by (4.16), $\|u - v\|_{H_0^1} = \|\overset{s}{v} - v\|_{H_0^1} \le \varepsilon$. Hence $u \in \{v\} + \overline{K}(\varepsilon) \subset \mathfrak{R}(\nu, f) + \overline{K}(\varepsilon)$.

The inclusions (4.17), (4.18) mean that the Hausdorff distance between $\mathfrak{R}^s(\nu, f_s)$ and $\mathfrak{R}(\nu, f)$ is not greater than ε (for $s > \overline{s}$). □

Now, let us fix $s \in \mathbb{N}$ and let $N \to \infty$. An application of Theorem 1.10 (the implicit function theorem), this time to $X = \overline{\mathbb{N}}$, $Y = Z = H_0^1$,

$$F : \overline{\mathbb{N}} \times H_0^1 \ni (N, u) \mapsto \begin{cases} \mathcal{K}_\nu^{s,N}(u) - P_V^{s,N}\mathcal{R}_N^{-1} f_s & \text{for } N < \infty, \\ \mathcal{K}_\nu^s(u) - P_V^s \mathcal{R}^{-1} f_s & \text{for } N = \infty \end{cases} \tag{4.19}$$

with respect to the point $(x_0, y_0) = (\infty, \overset{s}{v})$, where $\overset{s}{v} \in \mathfrak{R}^s(\nu, f_s)$, leads to the following result.

LEMMA 4.6. *If* $(\nu, f_s) \in]0, \infty[\times(H_0^1)'$, $\overset{s}{v} \in \mathfrak{R}^s(\nu, f_s)$ *and* $d_{\overset{s}{v}}\mathcal{K}_\nu^s \in EpiH_0^1$, *then there exists a neighbourhood* $\mathcal{Y} \in topH_0^1$ *of* $\overset{s}{v}$ *such that*

(i) $\#\big(\mathcal{Y} \cap \mathfrak{R}^{s,N}(\nu, f_s)\big) = 1$, *for almost all* $N \in \mathbb{N}$,

(ii) $\lim_{N\to\infty} \| \overset{s,N}{v} - \overset{s}{v}\|_{H_0^1} = 0$, *where* $\{ \overset{s,N}{v}\} = \mathcal{Y} \cap \mathfrak{R}^{s,N}(\nu, f_s)$.

PROOF. In a way similar to that in the proof of Lemma 4.3, we infer that it is sufficient to check that the mapping (4.19) satisfies the assumptions of Theorem 1.10.

Since $\overset{s}{v} \in \mathfrak{R}^s(\nu, f_s)$, $F(\infty, \overset{s}{v}) = 0$, by (4.6). According to Remark 2.6 (c), the mapping $F(\cdot, u) : \overline{\mathbb{N}} \to H_0^1$ is continuous for any $u \in H_0^1$. Moreover, for any $(N, u) \in \overline{\mathbb{N}} \times H_0^1$:

$$d^{II}_{(N,u)}F = \begin{cases} d_u\mathcal{K}_\nu^{s,N} & \text{for } N < \infty, \\ d_u\mathcal{K}_\nu^s & \text{for } N = \infty. \end{cases} \tag{4.20}$$

Replacing P_V with P_V^s in (4.10), we infer that

$$\mathcal{K}_\nu^s \text{ is a Fredholm mapping of index } 0. \tag{4.21}$$

Since $d_{\overset{s}{v}}\mathcal{K}_\nu^s \in EpiH_0^1$, by (1.1),

$$d_{\overset{s}{v}}\mathcal{K}_\nu^s \in AutH_0^1. \tag{4.22}$$

It remains to check that the differential

$$d^{II}F : \overline{\mathbb{N}} \times H_0^1 \to EndH_0^1$$

is continuous at $(\infty, \overset{s}{v})$. Let us denote: $P_V^{s,\infty} := P_V^s$ and $\mathcal{R}_\infty^{-1} := \mathcal{R}^{-1}$. Let $(\overset{k}{N}, \overset{k}{u}) \to (\infty, \overset{s}{v})$ in $(\overline{\mathbb{N}} \times H_0^1)$ as $k \to \infty$. According to (4.20),

$$d^{II}_{(\overset{k}{N},\overset{k}{u})}F = d_{\overset{k}{u}}\mathcal{K}_\nu^{s,\overset{k}{N}} = \nu\mathrm{id} + P_V^{s,\overset{k}{N}} \circ \mathcal{R}^{-1}_{\overset{k}{N}} \circ d_{\overset{k}{u}}\mathcal{A}^{P_V^{s,\overset{k}{N}}}. \tag{4.23}$$

From Corollary 3.8 and from the continuity of the operator P_V^s, it follows that the differential $d_{\overset{s}{v}}\mathcal{A}^{P_V^s}$ is completely continuous. Moreover, by (2.14), Remark 2.6 (c) and Remark A.4 , we infer that

$$P_V^{s,\overset{k}{N}} \circ \mathcal{R}^{-1}_{\overset{k}{N}} \to P_V^{s,\infty} \circ \mathcal{R}_\infty^{-1} \quad \text{pointwise in} \quad (H_0^1)' \quad \text{as} \quad k \to \infty.$$

Therefore, by virtue of Corollary A.6, it is sufficient to prove that

$$d_{\overset{k}{u}}\mathcal{A}^{P_V^{s,\overset{k}{N}}} \to d_{\overset{s}{v}}\mathcal{A}^{P_V^s} \quad \text{in} \quad \mathcal{L}\big(H_0^1, (H_0^1)'\big) \quad \text{as} \quad k \to \infty. \tag{4.24}$$

Let $h \in H_0^1$. By (3.25), we obtain

$$
\begin{aligned}
&\left|d_{\overset{k}{u}}\mathcal{A}^{P_V^{s,\overset{k}{N}}}(h) - d_{\overset{s}{v}}\mathcal{A}^{P_V^s}(h)\right|_{(H_0^1)'} \\
&= \left|\mathcal{A}_{\overset{k}{u},P_V^{s,\overset{k}{N}}h} + \mathcal{A}_{h,P_V^{s,\overset{k}{N}}\overset{k}{u}} - \mathcal{A}_{\overset{s}{v},P_V^s h} - \mathcal{A}_{h,P_V^s \overset{s}{v}}\right|_{(H_0^1)'} \\
&\le \left|\mathcal{A}_{\overset{k}{u},P_V^{s,\overset{k}{N}}h} - \mathcal{A}_{\overset{s}{v},P_V^s h}\right|_{(H_0^1)'} + \left|\mathcal{A}_{h,P_V^{s,\overset{k}{N}}\overset{k}{u}} - \mathcal{A}_{h,P_V^s \overset{s}{v}}\right|_{(H_0^1)'} \\
&\le \frac{3}{2}|\iota|\|\overset{k}{u}\|_{H_0^1}\|(P_V^{s,\overset{k}{N}} - P_V^s)h\|_{L^4} + |\iota|\|\overset{k}{u} - \overset{s}{v}\|_{L^4}|P_V^s|_{EndH_0^1}\|h\|_{H_0^1} \\
&+ \frac{3}{2}|\iota|\|P_V^{s,\overset{k}{N}}(\overset{k}{u}) - P_V^s(\overset{s}{v})\|_{L^4}\|h\|_{H_0^1} \\
&= \frac{3}{2}|\iota|\|\overset{k}{u}\|_{H_0^1}\|\iota\circ(P_V^{s,\overset{k}{N}} - P_V^s)h\|_{L^4} + |\iota|\|\iota\circ(\overset{k}{u} - \overset{s}{v})\|_{L^4}|P_V^s|_{EndH_0^1}\|h\|_{H_0^1} \\
&+ \frac{3}{2}|\iota|\|(\iota\circ P_V^{s,\overset{k}{N}})(\overset{k}{u}) - (\iota\circ P_V^s)(\overset{s}{v})\|_{L^4}\|h\|_{H_0^1}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&|d_{\overset{k}{u}}\mathcal{A}^{P_V^{s,\overset{k}{N}}} - d_{\overset{s}{v}}\mathcal{A}^{P_V^s}|_{\mathcal{L}\left(H_0^1,(H_0^1)'\right)} \\
&\le \frac{3}{2}|\iota|\|\overset{k}{u}\|_{H_0^1}|\iota\circ(P_V^{s,\overset{k}{N}} - P_V^s)|_{\mathcal{L}(H_0^1,L^4)} + |\iota|\|\iota\circ(\overset{k}{u} - \overset{s}{v})\|_{L^4} \\
&+ \frac{3}{2}|\iota||(\iota\circ P_V^{s,\overset{k}{N}})(\overset{k}{u}) - (\iota\circ P_V^s)(\overset{s}{v})|_{\mathcal{L}(H_0^1,L^4)}.
\end{aligned} \tag{4.25}
$$

Since $P_V^{s,\overset{k}{N}} \to P_V^s$ pointwise as $k \to \infty$ and according to (3.28), the imbedding $\iota : H_0^1 \hookrightarrow L^4$ is completely continuous and since L^4 is continuously imbedded in L^2, we apply Corollary A.9 and obtain

$$
\iota\circ P_V^{s,\overset{k}{N}} \to \iota\circ P_V^s \text{ in the space } \mathcal{L}(H_0^1, L^4) \text{ as } k \to \infty. \tag{4.26}
$$

Passing to the limit as $k \to \infty$ in (4.25), by (4.26) and Remark A.3, gives (4.24). □

The main result concerning the sequence $\left(\mathfrak{R}^{s,N}(\nu, f_s)\right)_{N\in\mathbb{N}}$ is expressed in the following theorem

THEOREM 4.7. *If $(\nu, f) \in \mathcal{G}$, then for almost all $s \in \mathbb{N}$:*

(i) $\lim_{N\to\infty}\mathfrak{R}^{s,N}(\nu, f_s) = \mathfrak{R}^s(\nu, f_s)$ *in the Hausdorff metric;*
(ii) *for almost all $N \in \mathbb{N}$: $\#\mathfrak{R}^{s,N}(\nu, f_s) = \#\mathfrak{R}^s(\nu, f_s) < \infty$.*

PROOF. Let us fix $(\nu, f) \in \mathcal{G}$. Then, for any $v \in \mathfrak{R}(\nu, f)$, $d_v\mathcal{K}_\nu \in AutH_0^1$. Moreover, by Theorem 3.12, $\#\mathfrak{R}(\nu, f) < \infty$.

Step 1^0. We will establish that for almost all $s \in \mathbb{N}$

$$P_V^s\mathcal{R}^{-1}f_s \text{ is a regular value of the mapping } \mathcal{K}_\nu^s. \tag{4.27}$$

Indeed, from (4.6) and from the proof of Theorem 4.5, it follows that for almost all $s \in \mathbb{N}$:

$$\begin{aligned} &(\mathcal{K}_\nu^s)^{-1}(\{P_V^s\mathcal{R}^{-1}f_s\}) = \mathfrak{R}^s(\nu, f_s) = \{\overset{s}{v}, \quad v \in \mathfrak{R}(\nu, f)\} \\ &\text{and } \overset{s}{v} \to v \text{ in } H_0^1 \text{ as } s \to \infty. \end{aligned} \tag{4.28}$$

(Let us recall that $\overset{s}{v}$ is the unique solution of the equation $(*)_s$ remaining in a suitable neighbourhood of the solution $v \in \mathfrak{R}(\nu, f)$.)

Let us fix $v \in \mathfrak{R}(\nu, f)$. Since the space of authomorphisms of H_0^1 is open in the space of all endomorphisms of H_0^1,

there exists a neighbourhood $\mathcal{W}(v) \in topEndH_0^1$ of the differential $d_v\mathcal{K}_\nu$ such that $\mathcal{W}(v) \subset AutH_0^1$.

If we show that

$$d_{\overset{s}{v}}\mathcal{K}_\nu^s \to d_v\mathcal{K}_\nu \text{ in the space } EndH_0^1 \text{ as } s \to \infty, \tag{4.29}$$

then for sufficiently large $s \in \mathbb{N}$: $d_{\overset{s}{v}}\mathcal{K}_\nu^s \in \mathcal{W}(v) \subset AutH_0^1$. Then, by the arbitrariness of v and by (4.28), statement (4.27) is true.
Let us recall

$$\begin{aligned} d_{\overset{s}{v}}\mathcal{K}_\nu^s &= \nu\mathrm{id} + P_V^s \circ \mathcal{R}^{-1} \circ d_{\overset{s}{v}}\mathcal{A}^{P_V^s}, \\ d_v\mathcal{K}_\nu &= \nu\mathrm{id} + P_V \circ \mathcal{R}^{-1} \circ d_v\mathcal{A}^{P_V}. \end{aligned}$$

Since, by Remark 2.6 (b), $P_V^s \to P_V$ in $EndH_0^1$ as $s \to \infty$, it is sufficient to show that

$$d_{\overset{s}{v}}\mathcal{A}^{P_V^s} \to d_v\mathcal{A}^{P_V} \text{ in } \mathcal{L}\big(H_0^1, (H_0^1)'\big) \text{ as } s \to \infty. \tag{4.30}$$

By virtue of (4.12), we infer that (4.30) holds (and, at the same time, (4.29)).

Since the set $\mathfrak{R}(\nu, f)$ is finite, by (4.28) and (4.29), there exists $s_0 \in \mathbb{N}$ such that for $s > s_0$:

$$\mathfrak{R}^s(\nu, f_s) = \{\overset{s}{v}, \quad v \in \mathfrak{R}(\nu, f)\} \tag{4.31}$$

and

$$d_{\overset{s}{v}}\mathcal{K}_\nu^s \in AutH_0^1. \tag{4.32}$$

Step 2^0 Let $s > s_0$. Let $\overset{s}{v} \in \mathfrak{R}^s(\nu, f_s)$. From Lemma 4.6, it follows that there exist $N(\overset{s}{v}) \in \mathbb{N}$ and a neighbourhood $\mathcal{Y}(\overset{s}{v}) \in topH_0^1$ of $\overset{s}{v}$ such that

(i) $\#\big(\mathcal{Y}(\overset{s}{v}) \cap \mathfrak{R}^{s,N}(\nu, f_s)\big) = 1$ for $N > N(\overset{s}{v})$,

(ii) $\lim_{N\to\infty} \| \overset{s,N}{v} - \overset{s}{v}\|_{H_0^1} = 0$, where $\{ \overset{s,N}{v} \} = \mathcal{Y}(\overset{s}{v}) \cap \mathfrak{R}^{s,N}(\nu, f_s)$.

We put $N_* := \max\{N(\overset{s}{v}), \overset{s}{v} \in \mathfrak{R}^s(\nu, f_s)\}$. According to (4.31), $\#\mathfrak{R}^s(\nu, f_s) < \infty$. Therefore, there exists $\delta > 0$ such that the balls $\overline{K}(\overset{s}{v}, \delta) := \{u \in H_0^1 : \|u - \overset{s}{v}\|_{H_0^1} < \delta\}$, ($\overset{s}{v} \in \mathfrak{R}^s(\nu, f_s)$) are pairwise disjoint for any $\overset{s}{v} \in \mathfrak{R}^s(\nu, f_s)$: $\overline{K}(\overset{s}{v}, \delta) \subset \mathcal{Y}(\overset{s}{v})$.

According to (ii)

$$\lim_{N\to\infty} \max\{\| \overset{s,N}{v} - \overset{s}{v}\|_{H_0^1}, \quad \overset{s}{v} \in \mathfrak{R}^s(\nu, f_s)\} = 0. \tag{4.33}$$

Therefore, there exists $N(\delta) \in \mathbb{N}, N(\delta) > N_*$ such that for $N > N(\delta)$:

$$\max\{\| \overset{s,N}{v} - \overset{s}{v}\|_{H_0^1}, \quad \overset{s}{v} \in \mathfrak{R}^s(\nu, f_s)\} < \delta.$$

Let $N > N(\delta)$. Then, the function

$$\mathfrak{R}^s(\nu, f_s) \ni \overset{s}{v} \mapsto \overset{s,N}{v} \in \mathfrak{R}^{s,N}(\nu, f_s)$$

is an injection. In particular, $\#\mathfrak{R}^s(\nu, f_s) \le \#\mathfrak{R}^{s,N}(\nu, f_s)$.

Suppose that the set

$$\mathcal{N} := \{N \in \mathbb{N} : N > N(\delta), \#\mathfrak{R}^s(\nu, f_s) < \#\mathfrak{R}^{s,N}(\nu, f_s)\}$$

is infinite. Let $N \in \mathcal{N}$. The set $\{ \overset{s,N}{v}, \overset{s}{v} \in \mathfrak{R}^s(\nu, f_s)\}$ has exactly $\#\mathfrak{R}^s(\nu, f_s)$ elements. Therefore, it is not the whole $\mathfrak{R}^{s,N}(\nu, f_s)$. Let us choose $\overset{s,N}{a} \in \mathfrak{R}^{s,N}(\nu, f_s) \backslash \{ \overset{s,N}{v}, \overset{s}{v} \in \mathfrak{R}^s(\nu, f_s)\}$. It is clear that $\overset{s,N}{a} \notin \bigcup_{\overset{s}{v}\in\mathfrak{R}^s(\nu,f_s)} \mathcal{Y}(\overset{s}{v})$. By virtue of (3.31) and of Remark 2.6 (c), the sequence $(\overset{s,N}{a})_{N\in\mathcal{N}}$ is bounded. Thus, the Banach-Alaoglu theorem yields that

there exist $\overset{s}{a} \in H_0^1$ and an infinite subset $\mathcal{N}_1 \subset \mathcal{N}$

such that $\overset{s,N}{a} \to \overset{s}{a}$ weakly in H_0^1 as $\mathcal{N}_1 \ni N \to \infty$.

By Theorem 2.9 (b), $\overset{s}{a} \in \mathfrak{R}^s(\nu, f_s)$ and $\overset{s,N}{a} \to \overset{s}{a}$ in H_0^1 as $\mathcal{N}_1 \ni N \to \infty$. We get the contradiction

$$\bigcup_{\overset{s}{v}\in\mathfrak{R}^s(\nu,f_s)} \mathcal{Y}(\overset{s}{v}) \not\ni \overset{s}{a} \in \mathfrak{R}^s(\nu, f_s) \subset \bigcup_{\overset{s}{v}\in\mathfrak{R}^s(\nu,f_s)} \mathcal{Y}(\overset{s}{v}).$$

Therefore, $\#\mathcal{N} < \infty$ and for $N > \sup \mathcal{N}$:

$$\#\mathfrak{R}^{s,N}(\nu, f_s) = \#\mathfrak{R}^s(\nu, f_s).$$

Let us now prove statement (i). Let $\varepsilon > 0$. Then, by (4.33), there exists $\overline{N} > \sup \mathcal{N}$ such that $N > \overline{N}$:

$$\max\{\| \overset{s,N}{v} - \overset{s}{v}\|_{H_0^1}, \quad \overset{s}{v} \in \mathfrak{R}^s(\nu, f_s)\} \leq \varepsilon. \tag{4.34}$$

Let $N > \overline{N}$. Then, from (4.34), it follows that

$$\mathfrak{R}^s(\nu, f_s) \subset \mathfrak{R}^{s,N}(\nu, f_s) + \overline{K}(\varepsilon). \tag{4.35}$$

We will show that

$$\mathfrak{R}^{s,N}(\nu, f_s) \subset \mathfrak{R}^s(\nu, f_s) + \overline{K}(\varepsilon). \tag{4.36}$$

Indeed, let $u \in \mathfrak{R}^{s,N}(\nu, f_s) = \{ \overset{s,N}{v}, \overset{s}{v} \in \mathfrak{R}^s(\nu, f_s)\}$. Then $u = \overset{s,N}{v}$ for some $\overset{s}{v} \in \mathfrak{R}^s(\nu, f_s)$. Therefore $\|u - \overset{s}{v}\|_{H_0^1} = \| \overset{s,N}{v} - \overset{s}{v}\|_{H_0^1} \leq \varepsilon$, and hence $u \in \{\overset{s}{v}\} + \overline{K}(\varepsilon) \subset \mathfrak{R}^s(\nu, f_s) + \overline{K}(\varepsilon)$.

The inclusions (4.35), (4.36) mean that the Hausdorff distance between $\mathfrak{R}^{s,N}(\nu, f_s)$ and $\mathfrak{R}^s(\nu, f_s)$ is not greater than ε for $N > \overline{N}$. □

Theorem 4.5 asserts that each solution of the Navier-Stokes problem is approximated by the whole sequence $(\overset{s}{v})_{s\in\mathbb{N}}$ whenever the data (ν, f) belong to $\mathcal{G}$. In turn, if we take sufficiently large $s \in \mathbb{N}$, then each $\overset{s}{v}$ is the limit of the whole sequence $(\overset{s,N}{v})_{N\in\mathbb{N}}$ for the same data $(\nu, f) \in \mathcal{G}$, which is a consequence of Theorem 4.7(i). Moreover, in this way the problem of uniqueness of the Navier-Stokes problem has been reduced to the problem in the finite-dimensional space for sufficiently large $s, N \in \mathbb{N}$.

Appendix

Some auxiliary results from functional analysis.

Remark A.1. *If $x_k \to x$ weakly in a Hilbert space H, $y_k \to y$ in H, then $(x_k|y_k)_H \to (x|y)_H$ as $k \to \infty$.*

Proof. In the equality

$$(x_k|y_k)_H - (x|y)_H = (x_k|y_k - y)_H + (x_k - x|y)_H \qquad \text{for } k \in \mathbb{N},$$

we see that $(x_k - x|y)_H \to 0$ as $k \to \infty$, since $x_k \to x$ weakly in H. Since $|(x_k|y_k - y)_H| \leq |x_k|_H |y_k - y|_H$ and a weakly convergent sequence is bounded, $(x_k|y_k - y)_H \to 0$ as $k \to \infty$. □

REMARK A.2. *Let H, H_1 be Hilbert spaces. Assume that $A \in \mathcal{L}(H, H_1)$ and (A_k) is a sequence of operators from $\mathcal{L}(H, H_1)$ such that $A_k^* \to A^*$ pointwise as $k \to \infty$. Let $x_k \to x$ weakly in H as $k \to \infty$. Then $A_k(x_k) \to A(x)$ weakly in H_1 as $k \to \infty$.*

PROOF. By virtue of the Riesz theorem, it is sufficient to show that for any $y \in H_1$:

$$\big(A_k(x_k)|y\big)_{H_1} \to \big(A(x)|y\big)_{H_1} \quad \text{as } k \to \infty.$$

Using the properties of the adjoint operators, we obtain

$$\begin{aligned}\big(A_k(x_k)|y\big)_{H_1} - \big(A(x)|y\big)_{H_1} &= \big(x_k|A_k^*(y)\big)_H - \big(x|A^*(y)\big)_H \\ &= \big(x_k|(A_k^* - A^*)(y)\big)_H + \big(x_k - x|A^*(y)\big)_H \qquad \text{for } k \in \mathbb{N}.\end{aligned}$$

We infer that $\big(x_k - x|A^*(y)\big)_H \to 0$ as $k \to \infty$. From the inequality $\big|\big(x_k|(A_k^* - A^*)(y)\big)_H\big| \le |x_k|_H |(A_k^* - A^*)(y)|_H$, it follows that $\big(x_k|(A_k^* - A^*)(y)\big)_H \to 0$, since (x_k) is bounded as a weakly convergent sequence and $|(A_k^* - A^*)(y)|_H \to 0$. □

REMARK A.3. *Assume that X, Y are Banach spaces. Let (A_k) be a sequence of operators from $\mathcal{L}(X, Y)$ such that $A_k \to A$ pointwise. Let $x_k \to x$ in X. Then $A_k(x_k) \to A(x)$ in Y.*

PROOF. The assertion is a consequence of the Banach-Steinhaus theorem and of the estimate

$$|A_k(x_k) - A(x)|_Y \le |A_k|_{\mathcal{L}(X,Y)} |x_k - x|_X + |A_k(x) - A(x)|_Y \qquad \text{for } k \in \mathbb{N}.$$

□

REMARK A.4. *Assume that X, Y, Z are Banach spaces. Let (A_k) be a sequence of operators from $\mathcal{L}(X, Y)$ such that $A_k \to A$ pointwise and (L_k) – a sequence of operators from $\mathcal{L}(Y, Z)$ such that $L_k \to L$ poitwise. Then $L_k \circ A_k \to L \circ A$ pointwise.*

PROOF. Let us fix $x \in X$. The statement is a consequence of the Banach-Steinhaus theorem and of the estimate

$$\begin{aligned}&|(L_k \circ A_k)(x) - (L \circ A)(x)|_Z \\ &\le |(L_k \circ A_k)(x) - (L_k \circ A)(x)|_Z + |(L_k \circ A)(x) - (L \circ A)(x)|_Z \\ &\le |L_k|_{\mathcal{L}(Y,Z)} |A_k(x) - A(x)|_Y + |L_k(A(x)) - L(A(x))|_Z, \qquad k \in \mathbb{N}\end{aligned}$$

□

LEMMA A.5. *Let X, Y, Z be Banach spaces. Assume that $A : X \to Y$ is a linear completely continuous operator and (L_k) is a sequence of operators from $\mathcal{L}(Y, Z)$ such that $L_k \to L$ poitwise. Then $L_k \circ A \to L \circ A$ in $\mathcal{L}(X, Z)$.*

PROOF. By virtue of the Banach-Steinhaus theorem, there exists $C > 0$ such that $\sup_{k\in\mathbb{N}} |L_k|_{\mathcal{L}(Y,Z)} \leq C$. Then $|L|_{\mathcal{L}(Y,Z)} \leq C$.

Let $\varepsilon > 0$. Since A is completely continuous, the image $A\big(\overline{K}_X(0,1)\big)$ of the closed unite ball is a relatively compact subset of Z. From the Hausdorff theorem (see Th. 17.2 in [KL]), it follows that there exist $y_1, \ldots, y_s \in Y$ such that

$$A\big(\overline{K}_X(0,1)\big) \subset \bigcup_{i=1}^{s} \overline{K}_Y(y_i, \frac{\varepsilon}{2C}). \tag{A.1}$$

Let $x \in \overline{K}_X(0,1)$. According to (A.1), there exists $j \in \{1, \ldots, s\}$ such that $|A(x) - y_j|_Y \leq \frac{\varepsilon}{2C}$. By virtue of the estimate

$$\begin{aligned}
&|(L_k \circ A)(x) - (L \circ A)(x)|_Z \leq |L_k(A(x)) - L_k(y_j)|_Z + |L_k(y_j) - L(y_j)|_Z \\
&+ |L(y_j) - L(A(x))|_Z \leq |L_k|_{\mathcal{L}(Y,Z)}|A(x) - y_j|_Y + |L_k(y_j) - L(y_j)|_Z \\
&+ |L|_{\mathcal{L}(Y,Z)}|y_j - A(x)|_Y \leq \varepsilon + \sum_{i=1}^{s} |L_k(y_i) - L(y_i)|_Z \quad \text{for } k \in \mathbb{N},
\end{aligned}$$

we infer that

$$|(L_k \circ A) - (L \circ A)|_{\mathcal{L}(X,Z)} \leq \varepsilon + \sum_{i=1}^{s} |L_k(y_i) - L(y_i)|_Z \quad \text{for } k \in \mathbb{N}.$$

Passing in the above inequality to the limes superior, we obtain

$$\overline{\lim_{k\to\infty}} |(L_k \circ A) - (L \circ A)|_{\mathcal{L}(X,Z)} \leq \varepsilon,$$

and since ε is arbitrary, $\lim_{k\to\infty} |(L_k \circ A) - (L \circ A)|_{\mathcal{L}(X,Z)} = 0$. □

COROLLARY A.6. *Let X, Y, Z be Banach spaces. Assume that (A_k) is a sequence of operators from $\mathcal{L}(X,Y)$ such that $A_k \to A$ in $\mathcal{L}(X,Y)$ and A is completely continuous. Let (L_k) be a sequence of operators from $\mathcal{L}(Y,Z)$ such that $L_k \to L$ pointwise. Then $L_k \circ A_k \to L \circ A$ in $\mathcal{L}(X,Z)$.*

PROOF. Since

$$\begin{aligned}
&|L_k \circ A_k - L \circ A|_{\mathcal{L}(X,Z)} \\
&\leq |L_k|_{\mathcal{L}(Y,Z)}|A_k - A|_{\mathcal{L}(X,Y)} + |(L_k \circ A) - (L \circ A)|_{\mathcal{L}(X,Z)}, \quad \text{for } k \in \mathbb{N},
\end{aligned}$$

the statement is a consequence of the Banach-Steinhaus theorem and of Lemma A.5. □

LEMMA A.7. *Assume that H_1 is a separable Hilbert space. Let (A_k) be a sequence of selfadjoint endomorphisms of a Hilbert space H such that $A_k \to A$ pointwise. Let $K : H \to H_1$ be a linear completely continuous operator. Then $K \circ A_k \to K \circ A$ in $\mathcal{L}(H, H_1)$.*

PROOF.

Step 1^0. Assume that $H_1 := \mathbb{R}$. Then K is a linear functional on H; denote $K =: \xi \in H'$. By the Riesz theorem, $\xi = (\cdot|a)_H$ for some $a \in H$. Let $x \in \overline{K}_H(0,1)$. By the selfadjointness of the operators A_k, we obtain

$$\begin{aligned}
&|(\xi \circ A_k)(x) - (\xi \circ A)(x)| = |\xi(A_k(x)) - \xi(A(x))| \\
&= |(A_k(x)|a)_H - (A(x)|a)_H| = |(x|A_k(a))_H - (x|A(a))_H| \\
&= |(x|A_k(a) - A(a))_H| \leq |A_k(a) - A(a)|_H, \quad k \in \mathbb{N}.
\end{aligned}$$

Therefore

$$|(\xi \circ A_k) - (\xi \circ A)|_{H'} \leq |A_k(a) - A(a)|_H \to 0.$$

Step 2^0. Now, we assume that $K(H)$ is a finite-dimensional subspace of H_1. Let $b_1, \ldots, b_n$ form a basis of $K(H)$, where $n = dimK(H)$ and let $b_1^*, \ldots, b_n^*$ be the dual basis. Then

$$K \circ A_k = \sum_{i=1}^{n} \Lambda_i \circ (\xi_i \circ A_k), \qquad K \circ A = \sum_{i=1}^{n} \Lambda_i \circ (\xi_i \circ A), \tag{A.2}$$

where

$$\begin{aligned}
&\xi_i : H \ni x \mapsto (b_i^* \circ K)(x) \in \mathbb{R}, \\
&\Lambda_i : \mathbb{R} \ni r \mapsto r \cdot b_i \in H_1, \qquad i = 1, \ldots, n.
\end{aligned}$$

Indeed, let $x \in H$. Then

$$\begin{aligned}
&\Big(\sum_{i=1}^{n} \Lambda_i \circ (\xi_i \circ A_k)\Big)(x) = \sum_{i=1}^{n} \Lambda_i\big(\xi_i(A_k(x))\big) = \sum_{i=1}^{n} \Lambda_i\big(b_i^*((K \circ A_k)(x))\big) \\
&= \sum_{i=1}^{n} b_i^*\big((K \circ A_k)(x)\big) \cdot b_i = (K \circ A_k)(x).
\end{aligned}$$

Since $A_k \to A$ pointwise and K is continuous, step 1^0 yields that

$$\xi_i \circ A_k \to \xi_i \circ A \text{ in } H'$$

and hence, by the continuity of the operators Λ_i

$$\Lambda_i \circ (\xi_i \circ A_k) \to \Lambda_i \circ (\xi_i \circ A) \quad \text{in } \mathcal{L}(H, H_1) \tag{A.3}$$

for any $i \in \{1, \ldots, n\}$. Then, the assertion follows from (A.2) and (A.3).

Step 3^0. The general case. By virtue of the Banach-Steinhaus theorem, the sequence of norms $(|A_k|_{EndH})$ is bounded. Let $C := \sup_{k \in \mathbb{N}} |A_k|_{EndH}$. Then $|A|_{EndH} \leq C$. Let $\varepsilon > 0$. Since H_1 is a separable Hilbert space, the subspace of all finite-dimensional operators is dense in the space of all completely continuous operators (see Th. 46.5 in [**16**]). Therefore, there exists a finite-dimensional

operator $L : H \to H_1$ such that $|K - L|_{\mathcal{L}(H,H_1)} \leq \frac{\varepsilon}{2C}$. Then

$$|K \circ A_k - K \circ A|_{\mathcal{L}(H,H_1)} \leq |K \circ A_k - L \circ A_k|_{\mathcal{L}(H,H_1)} + |L \circ A_k - L \circ A|_{\mathcal{L}(H,H_1)}$$
$$+ |L \circ A - K \circ A|_{\mathcal{L}(H,H_1)} \leq |K - L|_{\mathcal{L}(H,H_1)}|A_k|_{EndH} + |L \circ A_k - L \circ A|_{\mathcal{L}(H,H_1)}$$
$$+ |L - K|_{\mathcal{L}(H,H_1)}|A|_{EndH} \leq 2C|K - L|_{\mathcal{L}(H,H_1)} + |L \circ A_k - L \circ A|_{\mathcal{L}(H,H_1)}$$
$$\leq \varepsilon + |L \circ A_k - L \circ A|_{\mathcal{L}(H,H_1)} \qquad \text{for } k \in \mathbb{N}.$$

Hence, by step 2^0

$$\overline{\lim_{k\to\infty}} |K \circ A_k - K \circ A|_{\mathcal{L}(H,H_1)} \leq \varepsilon$$

and by the arbitrariness of ε, $\lim_{k\to\infty}|K \circ A_k - K \circ A|_{\mathcal{L}(H,H_1)} = 0$. □

LEMMA A.8. *(Lions) Let E, E_1, E_2 be Banach spaces. Assume that $A : E_1 \to E$ is a completely continuous linear operator and $L : E \to E_2$ is a linear continuous monomorphism. Then for every $\alpha > 0$ there exists $\beta > 0$ such that for every $x \in E_1$*

$$|A(x)|_E \leq \alpha|x|_{E_1} + \beta|(L \circ A)(x)|_{E_2}.$$

For A and L imbeddings, the above lemma is proved in **[23]** Section III, Th.3.3.

Assume that

(A.4) *E is a Banach space such that there exist a separable Hilbert space H_1 and a continuous linear monomorphism $L : E \to H_1$.*

For example, this assumption is fulfilled if a Banach space is continuously imbedded in a separable Hilbert space.

COROLLARY A.9. *Assume* (A.4). *Let $K : H \to E$ be a completely continuous linear operator and let (A_k) be a sequence of endomorphisms of a Hilbert space H such that $A_k \to A$ pointwise. Then $K \circ A_k \to K \circ A$ in $\mathcal{L}(H, E)$.*

PROOF. Since K is completely continuous, $L \circ K : H \to H_1$ is a completely continuous operator between Hilbert spaces. By virtue of Lemma A.7

$$L \circ K \circ A_k \to L \circ K \circ A \quad \text{in } \mathcal{L}(H, H_1). \tag{A.5}$$

According to the Banach-Steinhaus theorem, there exists $C > 0$ such that $\sup_{k\in\mathbb{N}} |A_k|_{EndH} \leq C$ and $|A|_{EndH} \leq C$. Let $\varepsilon > 0$. Let $x \in \overline{K}_H(0, 1)$. By virtue of Lemma A.8, there exists $\beta > 0$ such that

$$|(K \circ A_k)(x) - (K \circ A)(x)|_E$$
$$\leq \frac{\varepsilon}{2C}|A_k(x) - A(x)|_H + \beta|(L \circ K \circ A_k)(x) - (L \circ K \circ A)(x)|_{H_1}$$
$$\leq \varepsilon + \beta|(L \circ K \circ A_k) - (L \circ K \circ A)|_{\mathcal{L}(H,H_1)} \quad \text{for } k \in \mathbb{N}.$$

Taking the supremum over $x \in \overline{K}_H(0,1)$, we obtain

$$|(K \circ A_k) - (K \circ A)|_{\mathcal{L}(H,E)} \leq \varepsilon + \beta |(L \circ K \circ A_k) - (L \circ K \circ A)|_{\mathcal{L}(H,H_1)} \quad \text{for } k \in \mathbb{N}.$$

Passing to the limes superior, by (A.5), gives

$$\overline{\lim_{k\to\infty}} |K \circ A_k - K \circ A|_{\mathcal{L}(H,E)} \leq \varepsilon.$$

Since ε is arbitrary, $\lim_{k\to\infty} |K \circ A_k - K \circ A|_{\mathcal{L}(H,E)} = 0$. □

References

1. Adams R., *Sobolev spaces,* Academic Press, 1975.
2. Bożek B., Holly K., Olajossy A., *Stationary Navier-Stokes equations: on qualitative and numerical aspects of the Galerkin method,* Opuscula Mathematica, Zeszyty Naukowe AGH, Bull. 11, No. 1397, 1991, 1–61.
3. Dugundji J., Granas A., *Fixed point theory,* PWN, Warszawa, 1982.
4. Foiaş C., Saut J.C., *Remarques sur les équations de Navier-Stokes stationnaires,* Ann. Scuola Norm. Sup. Pisa, Sèrie IV, **10** (1983), 169–177.
5. Foiaş C., Temam R., *Structure of the set of stationary solutions of the Navier-Stokes equations,* Comm. Pure Appl. Math. Vol. XXX (1977), 149–164.
6. Foiaş C., Temam R., *Remarques sur les équations de Navier-Stokes stationnaires et les phénomènes successifs de bifurcation,* Ann. Scuola Norm. Sup. Pisa, Ser. IV, **5** (1978), 29–63.
7. Fortin M., Peyret R., Temam R., *Résolution numériques des equations de Navier-Stokes pour un fluide incompressible,* J. de Mécanique, 10, No.3 (1971), 357–390.
8. Fortin M., *Calcul numérique écoulements des fluides de Bingham et des fluides newtoniens incompressible par la méthode des éléments finis,* Thése, Universite de Paris, 1972.
9. Galdi G.P., *An introduction to the mathematical theory of the Navier-Stokes equations, Vol. I Linearized steady problems,* Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1994.
10. Galdi G.P., *An introduction to the mathematical theory of the Navier-Stokes equations, Vol. II Nonlinear steady problems,* Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1994
11. Girault V., Raviart P.A., *Finite element methods for Navier-Stokes equations,* Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1986.
12. Gobert J., *Une inegalite fondamentale de la theorie de l'elasticite,* Bull.-Soc.-Roy.-Sci.-Liege **31** (1962), 182–191.
13. Holly K., *Some application of the implicit function theorem to the stationary Navier-Stokes equations,* Ann. Polon. Math. **LIV.2** (1991), 93–109.
14. Holly K., *Non-uniqueness of a Galerkin equation corresponding to the stationary Navier-Stokes system,* Univ. Iagell. Acta Math. **XXXI** (1994), 154–174.
15. Holly K., Motyl E., *Inversion of the divdiv*-operator and three numerical methods in hydrodynamics,* Selected Problems in Mathematics, Cracow University of Technology, 1995, 35–94.
16. Kołodziej W., *Wybrane rozdziały analizy,* Biblioteka Matematyczna, Tom 36, Warszawa, 1982.

17. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow,* Gordon and Breach, New York, English translation, second edition, 1969.
18. Leray J., *Etude de diverses équations intégrales non linéaires et de quelques problémes que pose l'hydrodynamique,* J. Math. Pures Appl. **12** (1933), 1–82.
19. Leray J., *Essai sur les mouvements plans d'un liquide visqueux que limitent des parois,* J. Math. Pures Appl. **13** (1934), 331–418.
20. Leray J., *Sur le mouvement d'un liquide visqueux emplissant l'espace,* Acta Math. **63** (1934), 193–248.
21. Lions J.L., *Quelques méthodes de résolution des problèmes auxlimites non linèaires,* Dunod, Paris, 1969.
22. Lions J.L., Magenes, *Nonhomogeneous boundary value problems and applications,* Springer-Verlag, Berlin, New York, 1972.
23. Marcinkowska H., *Dystrybucje, przestrzenie Sobolewa, równania różniczkowe,* Biblioteka Matematyczna, Tom 75, Warszawa, 1993.
24. Meyers N., Serrin J., *H=W,* Proc. Nat. Acad. Sci. USA **51** (1964), 1055–1056.
25. Mlak W., *Wstęp do teorii przestrzeni Hilberta,* Biblioteka Matematyczna, Tom 35, Warszawa, 1972.
26. Nečas J., *Equations aux dérivées particles,* Presses de l'.Universit'e de Montr'eal, 1965.
27. Pasquarelli F., Quarteroni A., Sacchi-Landriani G., *Spectral approximations of the Stokes problem by divergence-free functions,* Journal of Scientific Computing, Vol. 2, **3** (1987), 195–225.
28. Rudin W., *Functional analysis,* McGraw-Hill Book Company, New York, 1973.
29. Saut J.C., Temam R., *Generic properties of nonlinear boundary value problems,* Comm. Partial Differential Equations, **4** (1979), 293–319.
30. Saut J.C., Temam R., *Generic properties of Navier-Stokes equations: Genericity with respect to the boundary values,* Indiana Univ. Math. J. **29** (1980), 427–446.
31. Smale S., *An infinite-dimensional version of Sard's theorem,* Amer. J. Math. **87** (1965), 861–866.
32. Temam R., *Navier-Stokes equations. Theory and numerical analysis,* North Holland Publishing Company, Amsterdam–New York–Oxford, 1979.
33. Temam R., *Navier-Stokes equations and nonlinear functional analysis,* SIAM, Philadelphia, Pensylvania, 1995.
34. Thomasset F., *Étude d'une méthode d'éléments finis de degré 5. Application aux problémes de plaques et d'écoulement de fluides,* Thése de 3^{éme} cycle, Univesité de Paris-Sud., 1974.
35. Thomasset F., *Implementation of finite element methods for Navier-Stokes equations,* Springer Series in Comp. Physics, 1981.
36. Urban K., *On divergence-free wavelets,* Advances in Computational Mathematics **4** (1995), 51–81.

Received November 24, 1997

Instytut Matematyki
Politechnika Krakowska
Warszawska 24
PL - 31-155 Kraków