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IDEAL AS AN INTERSECTION OF ZERO–DIMENSIONAL IDEALS AND THE NOETHER EXPONENT

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Abstract. The main goal of this paper is to present a method of expressing a given ideal I in the polynomial ring $\mathbb{K}[X_1, \dots, X_n]$ as an intersection of zero-dimensional ideals. As an application, we get an elementary proof for some cases of the Kollár estimation of the Noether exponent of a polynomial ideal presented in [**6**], [**7**]. Moreover, an outline of the effective algorithm is given.

1. Introduction. Let $\mathbb{K}$ be an algebraically closed field. It is well known that any radical ideal I can be expressed as $I = \bigcap_{P \in V(I)} \mathfrak{m}_P$, where $\mathfrak{m}_P$ is the maximal ideal corresponding to P. A natural question arises whether such an intersection is possible for an arbitrary ideal I, i.e. if we can attach an $\mathfrak{m}_P$-primary ideal $\mathfrak{A}_P$ to each $P \in V(I)$ such that $I = \bigcap_{P \in V(I)} \mathfrak{A}_P$.

In this paper a positive answer to this question is given. Using primary decomposition, we reduce the problem to the case where I is primary. This case can be done easily and effectively, so finding the family $\{\mathfrak{A}_P\}$ is as difficult as a primary decomposition is. The proof of the primary case is based on the theory of Gröbner bases.

As an application we present a simple proof of Kollár's Noether exponent estimate for ideals in the polynomial ring of one and two variables and for ideals without embedded primary components.

2. Notation. Let $\mathbb{K}$ be an algebraically closed field. For a given ideal I, $V(I)$ denotes its zero set in $\mathbb{K}^n$.

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3. Zero-dimensional case. Let $I = \mathrm{Id}(f_1, \cdots, f_m)$ be an ideal in the polynomial ring $\mathbb{K}[X] = \mathbb{K}[X_1, \cdots, X_n]$ generated by $\{f_1, \ldots, f_m\} \subset \mathbb{K}[X]$. For a given point $P \in \mathbb{K}^n$ we define

$$\mathcal{M}_P := \{\alpha \in \mathbb{N}^n : \ (X - P)^\alpha \in I\mathcal{O}_P\}, \tag{1}$$

$$\mathcal{D}_P := \mathbb{N}^n \setminus \mathcal{M}_P. \tag{2}$$

Observe that

$$\alpha \in \mathcal{M}_P \Longrightarrow \alpha + \mathbb{N}^n \subset \mathcal{M}_P \Longrightarrow \mathcal{M}_P = \bigcup_{\alpha \in \mathcal{M}_P} (\alpha + \mathbb{N}^n).$$

One can prove that there exists a unique finite set $\alpha^{(1)}, \cdots, \alpha^{(s)} \in \mathcal{M}_P$ such that

$$\alpha^{(i)} \notin (\alpha^{(j)} + \mathbb{N}^n), \qquad \text{for } i \neq j, \tag{3}$$

$$\mathcal{M}_P = \bigcup_{j=1}^{s} (\alpha^{(j)} + \mathbb{N}^n). \tag{4}$$

Lemma 1. *I is zero-dimensional if and only if $\#\mathcal{D}_P(I) < +\infty$ for every $P \in V(I)$.*

Proof. It suffices to prove that P is an isolated point of $V(I)$ if and only if $\#\mathcal{D}_P(I) < +\infty$. Take a $P \in V(I)$. We may assume that $P = 0$. Observe that any of the conditions implies that for every $j = 1, \ldots, n$, there exists a nonzero polynomial $f_j \in I$ of the form $f_j = x_j^{k_j} g_j(x)$ with $g_j(0) \neq 0$ and $k_j \geq 1$. On the other hand, this fact implies that both P is an isolated point of $V(I)$ and $(0, \cdots, k_j, \cdots, 0) \in \mathcal{M}_0(I)$ for $j = 1, \cdots, n$, which proves that $\mathcal{D}_0(I)$ is finite. □

Definition 2. For a given isolated point $P \in V(I)$, $d_P = d_P(I) := 1 + \max\{|\alpha| : \ \alpha \in \mathcal{D}_P(I)\}$ is defined to be the d-multiplicity of I at P.

Remark 3. $d_P(I) = \min\{k \in \mathbb{N} : \ \mathfrak{m}^k \subset I\}$, where $\mathfrak{m} = \mathrm{Id}(X - P)$ is the ideal corresponding to the point P.

Proposition 4. *The following conditions hold*

1. *$P \notin V(I) \Longleftrightarrow \mathcal{M}_P = \mathbb{N}^n$.*
2. *If P is an isolated point of $V(I)$ and Q_P is the primary component of I with associated prime $I(P)$, then*

$$\mathcal{M}_P(I) = \mathcal{M}_P(Q_P) \quad \text{and} \quad \mathcal{D}_P(I) = \mathcal{D}_P(Q_P).$$

3. *P is an isolated point of $V(I)$ if and only if $\#\mathcal{D}_P(I) < +\infty$.*

PROOF. To prove (1) it is enough to observe that if $P \notin V(I)$ then $1 \in I\mathcal{O}_P$.

Without loss of generality we may assume that $P = 0$. Since 0 is an isolated point of $V(I)$, Q_0 does not depend on a primary decomposition of I (see e.g. [**1**], Thm. 8.56). Thus $I = Q_0 \cap J_0$, where J_0 contains the rest of primary components. Since $Q_0\mathcal{O}_0 = I\mathcal{O}_0$, $\mathcal{M}_0(Q_0) = \mathcal{M}_0(I)$. To prove the opposite inclusion, take $\beta \in \mathcal{M}_0(I)$. Then there exists $f = x^\beta(1 + \sum_{\alpha>0} a_\alpha X^\alpha) \in I$. Therefore, $f \in Q_0$ and finally $x^\beta \in Q_0\mathcal{O}_0$.

Because of (2) we may assume that I is zero-dimensional. Applying Lemma 1 finishes the proof. □

Effective construction of $\mathcal{D}_P$. Again, it is enough to consider the case $P = 0$. Let $J_\alpha = I : \mathrm{Id}(X^\alpha)$. Observe that

$$\text{(5)} \quad \begin{aligned} \mathcal{M}_0(I) &= \{\alpha \in \mathbb{N}^n : \exists g \in \mathbb{K}[X] : g(0) \neq 0, X^\alpha g \in I\} \\ &= \{\alpha \in \mathbb{N}^n : \exists g \in J_\alpha, g(0) \neq 0\}. \end{aligned}$$

Now it suffices to compute the Gröbner basis $G_\alpha = \{g_\alpha^{(1)}, \cdots, g_\alpha^{(s_\alpha)}\}$ of J_α (see e.g. [**3**] or [**1**]) for "all" α (because of (1) and Lemma 1 the computation ends after a finite number of steps) and check whether there exists $j \in \{1, \ldots, s_\alpha\}$ such that $g_\alpha^{(j)}(0) \neq 0$.

THEOREM 5. *If I is a zero-dimensional ideal, then* $d(I) := \max_{P\in V(I)} d_P(I)$ *is the Noether exponent* $N(I) = \min\{k \in \mathbb{N} : (\mathrm{rad}(I))^k \subset I\}$.

PROOF. The proof that $d(I) \geq N(I)$ follows directly from the fact that for every $P \in V(I)$ we have $\mathfrak{m}^{d_P(I)} \subset I\mathcal{O}_P$. To prove the opposite, one can assume that $0 \in V(I)$, $d_0(I) = d(I)$. Take an $\alpha \in \mathcal{D}_0$ such that $|\alpha| = d_0(I) - 1$, and $g \in \mathbb{K}[X]$ such that $g(0) \neq 0$ and $g(P) = 0$ for all $P \in V(I) \setminus \{0\}$. Observe that $X_j g(X) \in \mathrm{rad}(I)$, $j = 1, \ldots, n$, but $(\mathrm{rad}(I))^{d_0(I)-1} \ni X^\alpha (g(X))^{|\alpha|} \notin I$. □

LEMMA 6. *Let I be a zero-dimensional ideal. Then $d(I) \leq \dim \mathbb{K}[X]/I$.*

PROOF. Let $V(I) = \{P_1, \ldots, P_s\}$ and let $Q_1, \ldots, Q_s$ be the primary decomposition of I such that $V(Q_i) = P_i$. Since $I \subset Q_i$, $d_{P_i}(I) = d_{P_i}(Q_i)$ (by Proposition 4) and $\dim \mathbb{K}[X]/Q_i \leq \dim \mathbb{K}[X]/I$, it suffices to prove the case $s = 1$ and $P := P_1 = 0$.

Let $l = \dim \mathbb{K}[X]/I$. To end the proof, it is enough to show that for any $a = (a_1, \ldots, a_n) \in \mathbb{K}^n$, $(a_1 X_1 + \cdots + a_n X_n)^l \in I\mathcal{O}_0$. Fix an a and let T be a linear isomorphism such that $T(X_1) = a_1 X_1 + \cdots + a_n X_n$. Let $T^* I \cap \mathbb{K}[X_1] = \mathrm{Id}(f)$. Since I is primary, we may take $f = X_1^k$. Observe that

$$k = \dim \mathbb{K}[X_1]/(X_1^k) \leq \dim \mathbb{K}[X]/T^* I = \dim \mathbb{K}[X]/I = l$$

— this implies that $X_1^l \in T^* I$. □

Lemma 7. *Let $I = \mathrm{Id}(f_1, \dots, f_m)$ be an ideal, where the f_i are non-zero polynomials, and let $1 \le k \le n$. Then there exist linear forms $L_j \in \mathcal{L}(\mathbb{K}^m, \mathbb{K})$, $j = 2, \dots, k$ such that for $I_k = \mathrm{Id}(f_1, L_2 \circ f, \dots, L_k \circ f)$ where f denotes the m-tuple $f_1, \dots, f_m$, the components of $V(I_k)$ not contained in $V(I)$ are at most $n-k$-dimensional.*

Proof. The case $k = 1$ is trivial. Fix a $k \ge 2$ and suppose than the forms $L_2, \dots, L_{k-1}$ are constructed. Let $V_1, \dots, V_s$ be components of $V(I_k)$ not contained in $V(I)$. For each $j = 1, \dots, s$, there exist $P_j \in V_j$ such that the sequence $f_1(P_j), \dots, f_m(P_j)$ has at least one non-zero element. Let $H_j = \{\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{K}^m : \ \alpha_1 f_1(P_j) + \ldots + \alpha_m f_m(P_j) \ne 0\}$, $j = 1, \dots, s$. Since the sets H_j are Zariski-open, the set $H := \cap_{j=1}^s H_j \ne \emptyset$. Take $\alpha \in H$ and let $L_k(Y_1, \dots, Y_m) = \alpha_1 Y_1 + \ldots + \alpha_m Y_m$. Since $V(L_k \circ f)$ intersects each V_j properly, the dimension decreases. □

Corollary 8. *Let $I = \mathrm{Id}(f_1, \dots, f_m)$ be an ideal and suppose the numbers $d_i := \deg f_i$ form a non-increasing sequence. Then there exists an ideal $J \subset I$ such that all components of $V(J)$ not contained in $V(I)$ are zero-dimensional, $J = \mathrm{Id}(g_1, \dots, g_n)$, $\deg g_n = \deg f_m$, and $\deg g_i \le \deg f_i$ for $i = 1, \dots, n-1$, where if $n > m$, set $f_(1) = \cdots = f_{n-m-1} = 0$.*

Proof. If is enough to renumber the generators and apply Lemma 7 followed by Gaussian elimination of the forms L_i. □

Theorem 9. *Let I be as in* Corollary 8. *Then $N(I) \le d_1 d_2 \cdot \cdots \cdot d_{n-1} d_m$.*

Proof. Let J be as in Corollary 8. Applying Lemma 5, Lemma 6, and Bezout's theorem, we get

$$N(I) = d(I) = \dim \mathbb{K}[X]/I \le \dim \mathbb{K}[X]/J \le g_1 g_2 \cdot \cdots \cdot g_n \le f_1 f_2 \cdot \cdots f_{n-1} f_m.$$

□

4. The case of one and two variables. In the ring of polynomials of one variable all ideals are zero-dimensional.

Theorem 10. *The estimate is true for ideals in the ring of polynomials of two variables.*

Proof. Take an ideal $I = \mathrm{Id}(f_1, \dots, f_m)$ as in Corollary 8 and assume that I is one-dimensional. Let $g_1, \dots, g_s$ be irreducible polynomials corresponding to the hypersurfaces contained in $V(I)$. Let $r_1, \dots, r_s$ be such that $g := g_1^{r_1} \cdot \cdots \cdot g_s^{r_s} = \mathrm{GCD}(f_1, \dots, f_m)$. Put $\widetilde{f_i} := f_i/g$, $i = 1, \dots, m$. Observe that $J := \mathrm{Id}(\widetilde{f_1}, \dots, \widetilde{f_m})$ is zero-dimensional.

Put $d = (d_1 - \deg g)(d_m - \deg g) + \max\{r_1, r_2, \dots, r_s\} \le d_1 d_m$ and let $p_1, \dots, p_d \in \mathrm{rad}(I)$. Obviously, $p_1, \dots, p_d \in \mathrm{rad}(J)$ and, consequently, $p_1 \cdot \cdots \cdot p_d \in I$. □

Remark 11. The above technique can be used to "remove" components of codimension one during the effective calculation of the Noether exponent.

5. Higher-dimensional case. The main goal of this part is to present a given ideal I as an intersection of primary ideals and then to use induction. The proof does not work for ideals with embedded primary components.

Proposition 12. *Let $I = \mathrm{Id}(f_1, \ldots, f_m)$ be a primary ideal and let $k \in \mathbb{N}$ such that for every $y = (y_1, \ldots, y_k) \in \mathbb{K}^k$ the ideal $I_y = \mathrm{Id}(f_1(y, Z), \ldots, f_m(y, Z))$ in the ring $\mathbb{K}[Z] = \mathbb{K}[Z_1, \ldots, Z_{n-k}]$ is proper and zero-dimensional. Let $f \in \mathbb{K}[X]$. Then the following conditions are equivalent, writing $X = Y \cup Z$ in an obvious notation:*

1. *$f \in I$,*
2. *$f \in \bigcap_{y \in \mathbb{K}^k} (I + \mathrm{Id}(Y - y))$,*
3. *$\forall y \in \mathbb{K}^k : \ f_y = f(y, Z) \in I_y$,*
4. *there exists a nonempty Zariski-open set $U \subset \mathbb{K}^k$ such that $\forall y \in U$: $f_y \in I_y$.*

Observe that the theorem is not true without the assumption that I is primary. For example, take $I = \mathrm{Id}(YZ, Z^2) = \mathrm{Id}(Z) \cap \mathrm{Id}(Y, Z^2)$, $k = 1$ and $f = Z$. Then f satisfies condition (4) with $U = \mathbb{K} \setminus \{0\}$, but $f \notin I$.

Proof. The implications *1*$\Longrightarrow$*2*$\Longrightarrow$*3*$\Longrightarrow$*4* are trivial. To prove *4*$\Longrightarrow$*1* suppose that $G = (g_1, \ldots, g_s)$, $g_i \in \mathbb{K}[Y][Z]$ is the comprehensive Gröbner basis ([**10**], see also [**1**]) of I for parameters $y \in U$. Observe that for y from a Zariski-open set $U' \subset U \subset \mathbb{K}^k$ the division of $f(y, Z)$ by $(g_1(y, Z), \ldots, g_s(y, Z))$ is conducted the same way (i.e. before each step the multidegree of all the polynomials involved do not depend on y). Since $\forall y \in \mathbb{K}^k$, $f_y \in I_y$, the remainders of the divisions are 0. Let $q_1, \ldots, q_s \in \mathbb{K}(Y)[Z]$ be such that $f(y, Z) = \sum_{i=1}^{s} q_i(y, Z) g_i(y, Z)$ for $y \in U'$. Multiplying the equation by the common denominator $s(Y)$ of coefficients of all q_i we get $s(Y) f(Y, Z) = \sum_{i=1}^{s} r_i(Y, Z) g_i(Y, Z)$, where $r_i \in \mathbb{K}[Y][Z]$. This implies that $s(Y) f(Y, Z) \in I$. Since I is primary and $I \cap \mathbb{K}[Y] = \{0\}$, we get $f \in I$. □

Theorem 13. *The estimate is true for ideals without embedded primary components.*

Proof. We apply induction on the number of variables. The cases $n = 1$ and $n = 2$ are already solved.

Take $n \geq 3$ and an $I = \mathrm{Id}(f_1, \ldots, f_m)$ as in Corollary 8 and let $d := d_1 d_2 \cdot \cdots \cdot d_{n-1} d_m$. Let $Q_1 \cap \cdots \cap Q_s$ be a primary decomposition of I. Observe that for a generic linear isomorphism T, each of the components $T^* Q_1, \ldots, T^* Q_s$ of $T^* I$ satisfies the hypotheses of Proposition 12. It suffices to prove that for any $p_1, \ldots, p_d \in \mathrm{rad}(T^* I)$, $p_1 \cdot \cdots \cdot p_d \in T^* Q_i$ for any $i = 1, \ldots, s$.

Let Q be a primary component of T^*I. If Q is zero-dimensional then it is an isolated component and $(\operatorname{rad} Q)^d \subset Q$ since the multiplicity of Q does not exceed d.

Assume now that $k := \dim Q > 0$. Let $U \subset \mathbb{K}^k$ be such that, for $y \in U$, T^*I_y has no embedded primary components. Fix $y \in U$. Since $(p_1)_y, \ldots, (p_d)_y \in \operatorname{rad}(T^*I_y)$ and T^*I_y has no embedded primary components, we get $(p_1 \cdot \cdots \cdot p_d)_y \in T^*I_y \subset Q_y$ by the inductive hypothesis. Applying Proposition 12 ends the proof.

□

6. Reducing the number of generators.

Lemma 14. *Let I, J and Q be ideals such that $V(I) \cup V(Q) = V(J)$ and $V(I) \cap V(Q) = \emptyset$. Fix $d \in \mathbb{N}$. If $\operatorname{rad}(J)^d \subset J$ then $\operatorname{rad}(I)^d \subset I$.*

Proof. Let $I = Q_1 \cap \ldots \cap Q_k$ be a primary decomposition. For $j = 1, \ldots, k$ there exists a polynomial $h_j \in Q \setminus \operatorname{rad}(Q_j)$. Let $P_j \in V(Q_j)$ be such that $h_j(P_j) \neq 0$. Using the construction in 7 we get a polynomial $h \in Q$ such that $h(P_j) \neq 0$ for $j = 1, \ldots, k$.

Take $p_1, \ldots, p_d \in \operatorname{rad}(I)$. Observe that $hp_1, \ldots, hp_d \in \operatorname{rad}(J)$. It follows that $h^d p_1 \cdot \cdots \cdot p_d \in J$. Assume that $p_1 \cdot \cdots \cdot p_d \notin Q_j$ for some $j \in \{1, \ldots, k\}$. Since Q_j is primary, $h^{dn} \in Q_j$ for some $n \in \mathbb{N}$. It follows that $h \in \operatorname{rad}(Q_j)$ — contradiction. This proves that $\operatorname{rad}(I)^d \subset I$. □

Corollary 15. *To prove the Kollár estimate it is enough to consider the case of n generators.*

Proof. It suffices to apply Corollary 8, take Q corresponding to the components of $V(J)$ not contained in $V(I)$ and then apply Lemma 14. □

7. Closing remarks. (1) Let A be a Noetherian ring. Then we note that:

0_A *is an intersection of ideals which are powers of maximal ideals (and so are zero-dimensional).*

For consider $a \in A \backslash \{0_A\}$ and let $I = 0 : a$. Then $I \subset M$, for some maximal ideal M. By Krull's Intersection Theorem, there exists $n \in \mathbb{N}$ such that $a/1 \notin M_M^n$. Hence $a \notin M^n$ and the result follows.

(2) Let A be an excellent (or indeed J-2) ring. Then by a general version of Zariski's Main Lemma on holomorphic functions (see [**5**], [**4**]), 0_A is an intersection of ideals of the form $\mathfrak{m}^e$, where $\mathfrak{m}$ is a maximal ideal and e is the maximum of the Noether exponents of the primary components in a primary decomposition of 0_A.

(3) We remark that Proposition 12 supplies a proof using comprehensive Gröbner bases of the following striking result, which can be regarded as a 'Nullstellensatz with Normalization':

Let A be an affine ring over the field $\mathbb{K}$ with 0_A a primary ideal. Via Noether Normalization, write $A = \mathbb{K}[Y_1, ..., Y_k, z_1, ..., z_{n-k}]$ where $Y_1, ..., Y_k$ are algebraically independent and A is integral over $\mathbb{K}[Y_1, ..., Y_k]$. Then if $U \subset \mathbb{K}^k$ is a non-empty Zariski-open set, $\bigcap_{y \in U} Id(Y - y).A = 0_A$.

In this connection, we note the following proof of Theorem 13 in the unmixed case that avoids this result (for which it would be of interest to have a 'classical' proof).

Alternative proof of Theorem 13 *in the unmixed case:*

Let I be an ideal in $\mathbb{K}[X_1, ..., X_n]$ with primary decomposition $I = Q_1 \cap ... \cap Q_s$, with $\text{ht } I = \text{ht } Q_i$, $i = 1, ..., s$. Set $P_i = \text{rad}(Q_i)$, $i = 1, ..., s$. Let A denote $\mathbb{K}[X_1, ..., X_n]/I$, and for each i let $\mathfrak{p}_i$ denote P_i/I. Via Noether Normalization, we have an integral extension $B \subset A$, with B a polynomial ring. By the basic properties of integral extensions, $\mathfrak{p}_i \cap B = 0$, $i = 1, ..., s$. Hence, setting $S = B \backslash \{0\}$, S consists of non-zerodivisors in A. Then $S^{-1}A$ is a zero-dimensional affine ring integral over $\mathbb{L}$, the quotient field of B, and the Noether exponent of 0_A is the same as the Noether exponent of $0_{S^{-1}A}$. Moreover, the defining ideal of $S^{-1}A$ arises from I by a linear (even triangular) transformation of variables, so the degrees of the generators get no worse. Hence we have reduced the proof to the zero-dimensional case.

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