UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXXIX
2001

HOLOMORPHIC FUNCTIONS WITH SINGULARITIES ON ALGEBRAIC SETS

by Józef Siciak

Abstract. The aim of the paper is to prove the following Theorem:

Let P be a non-zero polynomial of two complex variables. Put $A := \{(z_1, z_2); P(z_1, z_2) = 0\}$, $A^1_{z_2} := \{z_1; P(z_1, z_2) = 0\}$, $A^2_{z_1} := \{z_2; P(z_1, z_2) = 0\}$. Let E_1, E_2 be two closed subsets of $\mathbb{C}$ with positive logarithmic capacities. Put $X := (E_1 \times \mathbb{C}) \cup (\mathbb{C} \times E_2)$. Let $f : X \setminus A \ni (z_1, z_2) \mapsto f(z_1, z_2) \in \mathbb{C}$ be a function separately holomorphic on $X \setminus A$, i.e. $f(z_1, \cdot) \in \mathcal{O}(\mathbb{C} \setminus A^2_{z_1})$ for every $z_1 \in E_1$, and $f(\cdot, z_2) \in \mathcal{O}(\mathbb{C} \setminus A^1_{z_2})$ for every $z_2 \in E_2$.

Then there exists a unique function $\tilde{f} \in \mathcal{O}(\mathbb{C}^2 \setminus A)$ with $\tilde{f} = f$ on $X \setminus A$. Theorem remains true for all $n \geq 2$.

If $E_1 = E_2 = \mathbb{R}$ and $P(z_1, z_2) = z_1 - z_2$, we get the result due to O. Öktem [**5**].

1. Introduction. The aim of this paper is to prove the following theorem.

Theorem 1.1. *Given $n \geq 2$, let E_j $(j = 1, \dots, n)$ be a closed subset of the complex plane $\mathbb{C}$ of the positive logarithmic capacity. Put*

(*) $X := (\mathbb{C} \times E_2 \times \dots \times E_n) \cup (E_1 \times \mathbb{C} \times E_3 \times \dots \times E_n) \cup \dots \cup (E_1 \times \dots \times E_{n-1} \times \mathbb{C})$.

Let P be a non-zero polynomial of n complex variables. Put

(**) $A := \{z \in \mathbb{C}^n; P(z) = 0\}, \quad A^j_{z_1,\dots,z_{j-1},z_{j+1},\dots,z_n} := \{z_j \in \mathbb{C}; z \in A\}$

for $(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n) \in \mathbb{C}^{n-1}$, $\quad j = 1, \dots, n$. Let $f : X \setminus A \mapsto \mathbb{C}$ be a function separately holomorphic on $X \setminus A$ in the sense that

$$f(z_1, \dots, z_{j-1}, \cdot, z_{j+1}, \dots, z_n) \in \mathcal{O}(\mathbb{C} \setminus A^j_{z_1,\dots,z_{j-1},z_{j+1},\dots,z_n}),$$

if $z_k \in E_k (k \neq j)$, $\quad j = 1, \dots, n$.

1991 *Mathematics Subject Classification.* 32A10,32A99, 32D15, 32D10.

Key words and phrases. Algebraic set, separately holomomorphic function with singularities on algebraic set, analytic continuation, envelopes of holomorphy, plurisubharmonic functions, pluripotential theory.

Research supported by KBN grant 2 PO3A 04514.

Then $f \in \mathcal{O}(\mathbb{C}^n \setminus A)$*, i.e. there exists a unique function* $\tilde{f} \in \mathcal{O}(\mathbb{C}^n \setminus A)$ *with* $\tilde{f} = f$ *on* $X \setminus A$*.*

If $n = 2$, $E_1 = E_2 = \mathbb{R}$ and $P(z_1, z_2) = z_1 - z_2$, we get the result due to O. Öktem [**5**]. Properties of separately holomorphic functions of the above type were used by O. Öktem ([**5, 6**]) to characterize the range of the exponential Radon transform (which in turn is of interest for mathematical tomography). Theorem 1.1 shows that the *Main conjecture* of paper [**6**] is true at least for a class of special cases interesting for applications in mathematical tomography[1].

Let D_1 and D_2 be two domains in $\mathbb{C}^n$ with $D_1 \subset D_2$. In the sequel we shall say that a function f defined and holomorphic on D_1 is holomorphic on D_2, if there exists a unique function $\tilde{f}$ holomorphic on D_2 such that $\tilde{f} = f$ on D_1.

We shall need the following three known theorems.

THEOREM 1.2. *Let* F_j *be a nonpolar relatively closed subset of a domain* D_j *on the complex* z_j*-plane,* $j = 1, \ldots, n$*. Let* $f : X \mapsto \mathbb{C}$ *be a function of* n *complex variables separately holomorphic on the set* $X := D_1 \times F_2 \times \cdots \times F_n \cup \ldots \cup F_1 \times \cdots \times F_{n-1} \times D_n$*.*

Then the function f *is holomorphic on a neighborhood of the set*

$$D_1 \times (F_2)_{reg} \times \cdots \times (F_n)_{reg} \quad \cup \quad \ldots \quad \cup \quad (F_1)_{reg} \times \cdots \times (F_{n-1})_{reg} \times D_n,$$

where $(F_j)_{reg}$ *is the set of points* a *of* F_j *such that* F_j *is locally regular (in the sense of the logarithmic potential theory) at* a*.*

THEOREM 1.3. *Let* $D \subset \mathbb{C}^m$ *(resp.* $G \subset \mathbb{C}^n$*) be a domain with a pluripolar boundary. Let* E *(resp.* F*) be a non-pluripolar relatively closed subset of* D *(resp.* G*).*

Then every function $f : X \mapsto \mathbb{C}$ *separately holomorphic on the set* $X := E \times G \cup D \times F$ *is holomorphic on* $D \times G$*.*

Theorems 1.2 and 1.3 are direct consequences of (e.g.) the main result of [**4**].

THEOREM 1.4. [**1**] *Let* A *be an analytic subset (of pure codimension 1) of the envelope of holomorphy* $\hat{D}$ *of a domain* $D \subset \mathbb{C}^n$*.*

Then $\hat{D} \setminus A$ *is the envelope of holomorphy of* $D \setminus A$*.*

2. Proof of Theorem 1.1.

We shall show that our theorem follows from the following Lemma.

LEMMA 2.1. *There exists a function* g *holomorphic on the domain* $\mathbb{C}^n \setminus A$ *such that* $g = f$ *on* $F_1 \times \cdots \times F_n$*, where* $F_1 \times \cdots \times F_n \subset \mathbb{C}^n \setminus A$ *and* F_j *is a non-polar subset of* E_j *(*$j = 1, \ldots, n$*).*

[1] M. Janicki and P. Pflug [**2**] have shown that for $n = 2$ the Main Conjecture is true with no additional assumptions.

In order to prove Theorem 1.1 it is sufficient to show that $g = f$ on $X \setminus A$.

First we shall consider the case of $n = 2$. Fix $(a_1, a_2) \in X \setminus A$. We need to show that $g(a_1, a_2) = f(a_1, a_2)$. Without loss of generality we may assume that $a_1 \in E_1$.

For a fixed $z_2 \in F_2$ the functions $f(\cdot, z_2)$ and $g(\cdot, z_2)$ are holomorphic in the domain $\mathbb{C} \setminus A^1_{z_2}$ and identical on the nonpolar subset F_1. Therefore

$$f(z_1, z_2) = g(z_1, z_2), \quad z_1 \in \mathbb{C} \setminus A^1_{z_2}, \quad z_2 \in F_2.$$

Let G_2 be a non-polar subset of F_2 such that $P(a_1, z_2) \neq 0$ for all $z_2 \in G_2$. Then $a_1 \in \mathbb{C} \setminus A_{z_2}$ for all $z_2 \in G_2$. Hence $f(a_1, z_2) = g(a_1, z_2)$ for all $z_2 \in G_2$. The functions $f(a_1, \cdot)$ and $g(a_1, \cdot)$ are holomorphic on the domain $\mathbb{C} \setminus A^2_{a_1}$ and identical on the nonpolar subset G_2 of the domain. Therefore $f(a_1, z_2) = g(a_1, z_2)$ for all $z_2 \in \mathbb{C} \setminus A^2_{a_1}$. In particular, $f(a_1, a_2) = g(a_1, a_2)$ because $a_2 \in \mathbb{C} \setminus A^2_{a_1}$.

Now consider the case of $n > 2$ and assume that Theorem 1.1 is true in $\mathbb{C}^k$ with $2 \leq k \leq n - 1$. Fix $a = (a_1, \dots, a_n) \in X \setminus A$. Without loss of generality we may assume that $a_1 \in E_1$. Put $a = (a_1, a')$ with $a' = (a_2, \dots, a_n)$. Observe that $A^{(2,\dots,n)}_{a_1} := \{z' \in \mathbb{C}^{n-1}; P(a_1, z') = 0\} \neq \mathbb{C}^{n-1}$.

It is clear that $f(z_1, z') = g(z_1, z')$ if $z_1 \in \mathbb{C} \setminus A^1_{z'}$ and $z' \in F_2 \times \dots \times F_n$. Let G_j $(j = 2, \dots, n)$ be a non-polar subset of F_j such that $P(a_1, z') \neq 0$ for all $z' = (z_2, \dots, z_n) \in G_2 \times \dots \times G_n$. Then the function $g(a_1, \cdot)$ is holomorphic in $\mathbb{C}^{n-1} \setminus A^{(2,\dots,n)}_{a_1}$, and

$$f(a_1, z') = g(a_1, z'), \quad z' \in G_2 \times \dots \times G_n \subset E_2 \times \dots \times E_n \setminus A^{(2,\dots,n)}_{a_1}.$$

Put

$$X' := \mathbb{C} \times E_3 \times \dots \times E_n \cup \dots \cup E_2 \times \dots \times E_{n-1} \times \mathbb{C}.$$

Then the function $f(a_1, \cdot)$ is separately analytic on $X' \setminus A^{(2,\dots,n)}_{a_1}$, and the function $g(a_1, \cdot)$ is holomorphic on

$$\mathbb{C}^{n-1} \setminus A^{(2,\dots,n)}_{a_1}.$$

Moreover, $f(a_1, z') = g(a_1, z')$ for all $z' \in G_2 \times \dots \times G_n$. By the induction assumption we have $f(a_1, z') = g(a_1, z')$ for all $z' \in \mathbb{C}^{n-1} \setminus A^{(2,\dots,n)}_{a_1}$. It is clear that $a' \in \mathbb{C}^{n-1} \setminus A^{(2,\dots,n)}_{a_1}$. Therefore $f(a) = g(a)$. The proof is concluded.

3. Proof of Lemma 2.1. For each k with $1 \leq k \leq n$ the polynomial P can be written in the form

$$(\star) \qquad P(z) = \sum_{j=0}^{d_k} p_{kj}(z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n) z_k^j,$$

where $d_k \geq 0$ and $p_{kd_k} \neq 0 (k = 1, \dots, n)$. It is clear that $d_k = 0$ iff P does not depend on z_k. If $P = const \neq 0$ then $A = \emptyset$.

Put

$$A^k \quad := \quad \{z \in \mathbb{C}^n; p_{kd_k}(z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n) = 0\}, \quad k = 1, \dots, n.$$

Then the set

$$B \quad := \quad A \cup A^1 \cup \cdots \cup A^n$$

is pluripolar. We know that the set $(E_j)_{reg}$ is not polar. Therefore the cartesian product $(E_1)_{reg} \times \cdots \times (E_n)_{reg}$ is not pluripolar, and hence

$$(E_1)_{reg} \times \cdots \times (E_n)_{reg} \setminus B \neq \emptyset.$$

Fix

$$z^o = (z_1^o, \dots, z_n^o) \in (E_1)_{reg} \times \cdots \times (E_n)_{reg} \setminus (A \cup A^1 \cup \cdots \cup A^n).$$

Then there exists $r_o > 0$ such that

$$(\star\star) \qquad (\bar{B}(z_1^o, 2r_o) \times \cdots \times \bar{B}(z_n^o, 2r_o)) \cap (A \cup A^1 \cup \cdots \cup A^n) = \emptyset,$$

where $B(z_j^o, 2r_o) := \{z_j \in \mathbb{C}; |z_j - z_j^o| < 2r_o\}$. In particular, $p_{kd_k}(z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n) \neq 0$ on $\bar{B}(z_1^o, 2r_o) \times \cdots \times \bar{B}(z_{k-1}^o, 2r_o) \times B(z_{k+1}^o, 2r_o) \times \cdots \times \bar{B}(z_n^o, 2r_o)$.

We shall show that Lemma 2.1 follows from the following Main Lemma.

Main Lemma 3.1. *Given δ with $0 < \delta < \min\{1, r_0\}$, put*

$$\Omega_k := B(z_1^o, \delta) \times \cdots \times B(z_{k-1}^o, \delta) \times \mathbb{C} \times B(z_{k+1}^o, \delta) \times \cdots \times B(z_n^o, \delta) \quad 1 \leq k \leq n.$$

If δ is sufficiently small then for each $k = 1, \dots, n$ there exists a function f_k holomorphic on $\Omega_k \setminus A$ such that $f_k(z) = f(z)$ on the set $F_1 \times \cdots \times F_n$, where

$$F_j := E_j \cap B(z_j^o, \delta), \quad j = 1, \dots, n.$$

In order to prove Lemma 2.1 let us observe that by $(\star\star)$ $f_j = f_k = f$ on the non-pluripolar subset $F_1 \times \cdots \times F_n$ of the domain $(\Omega_j \cap \Omega_k) \setminus A$. Therefore the function

$$f_o := f_1 \cup \cdots \cup f_n$$

is well defined and holomorphic on $\Omega \setminus A$ with $\Omega := \Omega_1 \cup \cdots \cup \Omega_n$. Moreover $f_o = f$ on $F_1 \times \cdots \times F_n$. The set Ω is a Reinhardt domain with centre z^o whose envelope of holomorphy is $\mathbb{C}^n$. Therefore by the Grauert-Remmert Theorem 1.4 there exists a function g holomorphic on $\mathbb{C}^n \setminus A$ such that $g = f_o$ on $\Omega \setminus A$; in particular $g = f$ on $F_1 \times \cdots \times F_n$. The proof of Lemma 2.1 is finished.

4. Proof of the Main Lemma. Fix integer k with $1 \leq k \leq n$. We shall consider two cases.

Case 1^o. The polynomial P depends on z_k, i.e. $d_k \geq 1$.

Without loss of generality we may assume that $k = 1$. Let $\{a_1, \dots, a_s\} := \{z_1 \in \mathbb{C}; P(z_1, z_2^o, \dots, z_n^o) = 0\}$ be the zero set of the polynomial $P(\cdot, z_2^o, \dots, z_n^o)$. By (**) the number m given by

$$2m := \min\{|p_{1d_1}(z')|; |z_j - z_j^o| \leq r_o (j = 2, \dots, n)\}$$

is positive.

Let $R_o > \max\{1, r_o\}$ be so large that $B(a_j, 2) \subset B(0, R_o) \quad (j = 1, \ldots, s)$ and

(4.0) $\quad |P(z)| \geq m|z_1|^{d_1} \quad \text{for all} \quad |z_1| \geq R_o, \quad |z_j - z_j^o| \leq r_o (j = 2, \ldots, n).$

Fix ϵ with $0 < \epsilon < 1$ so small that

$$\bar{B}(z_1^o, r_o) \cap \big(\cup_{j=1}^s \bar{B}(a_j, \epsilon)\big) = \emptyset, \quad \bar{B}(a_j, \epsilon) \cap \bar{B}(a_l, \epsilon) = \emptyset \quad (j \neq l)$$

Without loss of generality we may assume that r_o is so small that $P(z) \neq 0$ for all z with $|z_1 - a_j| \geq \frac{\epsilon}{4} \quad (j = 1, \ldots, s), \quad |z_j - z_j^o| \leq r_o \quad (j = 2, \ldots, n)$. Now given $R > R_o$ there exists δ such that $0 < 2\delta < r_o$ and f is bounded and holomorphic on the set

$$\{z \in \mathbb{C}^n; \epsilon < |z_1 - a_j| < \tfrac{3}{2}R \quad (j = 1, \ldots, s), \quad |z_l - z_l^o| < \delta (l = 2, \ldots, n)\}.$$

Indeed, f is separately holomorphic on the set

$$(\natural) \qquad D_1 \times F_2 \times \cdots \times F_n \cup \cdots \cup F_1 \times \cdots \times F_{n-1} \times D_n$$

with $F_1 := E_1 \cap \bar{B}(z_1^o, r_o)$, $F_j := E_j \cap B(z_j^o, r_o) \quad (j = 2, \ldots, n)$, $D_1 := \mathbb{C} \setminus \big(\bar{B}(a_1, \frac{\epsilon}{4}) \cup \cdots \cup \bar{B}(a_s, \frac{\epsilon}{4})\big)$, $D_j := B(z_j^o, r_o) \quad (j = 2, \ldots, n)$. For each j the set F_j is locally regular at z_j^o. Hence by Theorem 1.2 there exists δ such that $O < 2\delta < r_o$ and f is holomorphic on the domain

$$(\dagger) \quad \{z \in \mathbb{C}^n; \tfrac{\epsilon}{2} < |z_1 - a_j| < 2R (j = 1, \ldots, s), \quad |z_\ell - z_\ell^o| < 2\delta \, (\ell = 2, \ldots, n)\}.$$

Observe that the function

$$W(\omega, z) := \frac{P(\omega, z') - P(z_1, z')}{\omega - z_1} \equiv \sum_{l=1}^{d_1} p_{1l}(z')[\omega^{l-1} + \omega^{l-2} z_1 + \cdots + z_1^{l-1}]$$

is a polynomial of $n+1$ variables $\omega, z_1, \ldots, z_n$.

It is clear that for every $j \in \mathbb{Z}$ the function

$$(\&) \qquad \Phi_j(\omega, z) := W(\omega, z) \frac{f(\omega, z')}{P(\omega, z')^{j+1}}$$

is holomorphic on the set $\{(\omega, z) \in \mathbb{C}^{n+1}; \frac{\epsilon}{2} < |\omega - a_j| < 2R \, (j = 1, \ldots, s), \; z_1 \in \mathbb{C}, \; z' \in B(z_2^o, 2\delta) \times \cdots \times B(z_n^o, 2\delta)\}$.

Therefore the function

$$(4.1) \qquad c_{1j}(z) := \tfrac{1}{2\pi i} \int_{C(0,R)} \Phi_j(\omega, z) d\omega$$

is holomorphic on the set $\mathbb{C} \times B(z_2^o, 2\delta) \times \cdots \times B(z_n^o, 2\delta)$; here $C(0, R)$ denotes the positively oriented circle of centre 0 and radius R. Moreover, by (4.0) for every compact subset K of $\mathbb{C}$ there exists a positive constant $M = M(K, R)$ such that

$$(4.2) \qquad |c_{1j}(z)| \leq M^{|j|}$$

for all $j \in \mathbb{Z}$ and $z \in K \times B(z_2^o, \delta) \times \cdots \times B(z_n^o, \delta)$.

For a fixed $z' \in F_2 \times \cdots \times F_n$ with $F_j := E_j \cap B(z_j^o, \delta)$ the function $\Phi_j(\cdot, \cdot, z')$ is holomorphic on $\{\omega \in \mathbb{C}; P(\omega, z') \neq 0\} \times \mathbb{C}$. Hence, by the Cauchy residue theorem,

$$c_{1j}(z) = \frac{1}{2\pi i} \int_{\partial D_+(z',\rho)} \Phi_j(\omega, z) d\omega, \quad z \in \mathbb{C} \times (F_2 \times \cdots \times F_n), \tag{4.3}$$

where ρ is any positive real number and

$$D_+(z', \rho) := \{z_1 \in \mathbb{C}; |P(z_1, z')| < \rho\}.$$

In the formula (4.3) the integration is taken over the positively oriented boundary of the open set $D_+(z', \rho)$ (the interior of the lemniscate on the z_1-plane).

We claim that the required function f_1 may be given by the formula (a *generalized Laurent series*)

$$f_1(z) := \sum_{-\infty}^{\infty} c_{1j}(z) P(z)^j, \quad z \in \Omega_1 \setminus A,$$

where c_{1j} is defined by (4.1). It remains to show that the series is convergent locally uniformly in $\Omega_1 \setminus A$, and $f_1 = f$ on $F_1 \times \cdots \times F_n$.

We already know that the functions c_{1j} are holomorphic on $\Omega_1 := \mathbb{C} \times B(z_2^o, \delta) \times \cdots \times B(z_n^o, \delta)$. Passing to the proof of our claim let us observe that, given $z' \in F_2 \times \cdots \times F_n$ and $0 < r < 1$, we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(z',r)} \frac{f(\omega, z')}{\omega - z_1} d\omega, \quad z_1 \in D(z', r) := \{z_1 \in \mathbb{C}; r < |P(z_1, z')| < \frac{1}{r}\}.$$

Hence

$$f(z) = \frac{1}{2\pi i} \int_{\partial D_+(z', \frac{1}{r})} \frac{f(\omega, z')}{\omega - z_1} d\omega - \frac{1}{2\pi i} \int_{\partial D_-(z',r)} \frac{f(\omega, z')}{\omega - z_1} d\omega$$

for all $z_1 \in D(z', r)$, where $D_+(z', \frac{1}{r}) := \{z_1 \in \mathbb{C}; |P(z_1, z')| < \frac{1}{r}\}$, $D_-(z', r) := \{z_1 \in \mathbb{C}; |P(z_1, z')| > r\}$.

Observe that

$$\frac{f(\omega, z')}{\omega - z_1} = \frac{P(\omega, z') - P(z_1, z')}{\omega - z_1} \cdot \frac{f(\omega, z')}{P(\omega, z') - P(z_1, z')} = \sum_{j=0}^{\infty} \Phi_j(\omega, z) P(z)^j$$

for all $\omega \in \mathbb{C}$ with $|P(\omega, z')| = \frac{1}{r}$ and all $z_1 \in D_+(z', \frac{1}{r})$, the series being uniformly convergent with respect to $\omega \in \partial D_+(z', \frac{1}{r})$.

Similarly,

$$\frac{f(\omega, z')}{\omega - z_1} = -\sum_{j=1}^{\infty} \Phi_j(\omega, z) P(z)^{-j}$$

for all $\omega \in \partial D_-(z', r)$ and all $z_1 \in D_-(z', r)$, the series being uniformly convergent with respect to $\omega \in \partial D_-(z', r)$.

By (4.3) it follows that

$$f(z) = \sum_{j=-\infty}^{\infty} c_{1j}(z)P(z)^j, \quad z_1 \in D(z', 0), \quad z' \in F_2 \times \cdots \times F_n. \tag{4.4}$$

Moreover, for every $\rho > 0$, for every $z' \in F_2 \times \cdots \times F_n$, and for every compact subset K of $\mathbb{C}$ there exists $M = M(\rho, z', K) > 0$ such that

$$|c_{1j}(z)| \leq M\rho^{-j}, \quad j \in \mathbb{Z}, \quad z_1 \in K, \quad z' \in F_2 \times \cdots \times F_n.$$

Hence for all $r > 0$, $\quad z_1 \in \mathbb{C}$, $\quad z' \in F_2 \times \cdots \times F_n$ one gets the inequalities

$$|c_{1j}(z_1, z')| \leq M(\frac{1}{r}, z', \{z_1\})r^j, \quad j \geq 0,$$

$$|c_{1j}(z_1, z')| \leq M(r, z', \{z_1\})r^{|j|}, \quad j \leq 1.$$

By the arbitrary nature of $r > 0$ it follows that

$$\limsup_{|j|\to\infty} \frac{1}{|j|} \log |c_{1j}(z)| = -\infty, \quad z_1 \in C, \quad z' \in F_2 \times \cdots \times F_n.$$

By (4.2) the sequence $\{\frac{1}{|j|} \log |c_{1j}|\}$ is locally uniformly upper bounded on Ω_1. Put $u(z) := \limsup \frac{1}{|j|} \log |c_{1j}(z)|$, $z \in \Omega_1$. Then the upper semicontinuous regularization u^* of u is plurisubharmonic in Ω_1, and by the Bedford-Taylor theorem [**3**] on negligible sets the set $\{z \in F_1 \times \cdots \times F_n; -\infty = u(z) = u^*(z)\}$ is non-pluripolar. Therefore $u^* \equiv -\infty$ in Ω_1.

Given a compact subset K of $\Omega_1 \setminus A$, there exists $r = r(K)$ with $0 < r < 1$ such that $r < |P(z)| < \frac{1}{r}$ for all $z \in K$. Fix $k > 0$ so large that $\frac{1}{r}e^{-k} < \frac{1}{2}$. By the Hartogs Lemma there exists $j_o = j_o(k, K)$ such that

$$\frac{1}{|j|} \log |c_{1j}(z)P(z)^j| \leq -k + \log \frac{1}{r}, \quad z \in K, \quad |j| > j_o,$$

i.e.

$$|c_{1j}(z)|P(z)^j| \leq 2^{-|j|}, \quad z \in K, \quad |j| > j_o.$$

It follows that the series $\sum_{j=-\infty}^{\infty} c_{1j}(z)P(z)^j$ is uniformly convergent on every compact subset of $\Omega_1 \setminus A$. Its sum f_1 is holomorphic on $\Omega_1 \setminus A$. By (4.4) $f_1 = f$ on $F_1 \times \cdots \times F_n$. The proof of Case 1^o is completed.

Case 2^o. The polynomial P does not depend on z_k.

Without loss of generality we may assume that $k = n$. Now the function f is separately holomorphic on the set (♮) with $D_j := B(z_j^o, r_o)$, $F_j := E_j \cap B(z_j^o, r_o)$ $(j = 1, \ldots, n-1$, $D_n := \mathbb{C}$, $F_n := E_n \cap \bar{B}(z_n^o, r_o)$. Given $R > 0$, by Theorem 1.2 there exists sufficiently small $\delta > 0$ such that f is holomorphic on the domain

$$\{z \in \mathbb{C}^n; |z_j - z_j^o| < 2\delta \quad (j = 1, \ldots, n-1), |z_n| < 2R. \tag{‡}$$

The function

$$\text{(a)} \qquad c_{nj}(z) \equiv c_{nj}(z') := \tfrac{1}{2\pi i} \int_{C(0,R)} \frac{f(z',\omega)}{\omega^{j+1}} d\omega, \quad j \geq 0,$$

with $z' := (z_1, \dots, z_{n-1})$, is holomorphic on the set $B(z_1^o, 2\delta) \times \dots \times B(z_{n-1}^o, 2\delta) \times \mathbb{C}$. Moreover, for every compact subset K of $\mathbb{C}$ there exists a positive constant $M = M(K,R)$ such that

$$\text{(b)} \quad |c_{nj}(z)| \leq M R^{-j}, \quad j \geq 0, \quad z \in \Omega_n := B(z_1^o, \delta) \times \dots B(z_{n-1}^o, \delta) \times K.$$

It is clear that for every $\rho > 0$

$$\text{(c)} \qquad c_{nj}(z) = \tfrac{1}{2\pi i} \int_{C(0,\rho)} \frac{f(z',\omega)}{\omega^{j+1}} d\omega, \quad z \in F_1 \times \dots \times F_{n-1} \times \mathbb{C},$$

where $F_j := E_j \cap B(z_j^o, \delta)$. Moreover,

$$\text{(d)} \qquad f(z) = \sum_{j=0}^{\infty} c_{nj}(z) z_n^j, \quad z \in F_1 \times \dots \times F_{n-1} \times \mathbb{C}.$$

Put $u_j(z) := \frac{1}{j} \log |c_{nj}(z)|$. The sequence $\{u_j\}$ is locally uniformly upper bounded on Ω_n , and $\limsup_{j \to \infty} u_j(z) = -\infty$ for all $z \in F_1 \times \dots \times F_{n-1} \times \mathbb{C}$. Hence by the Hartogs Lemma and by the Bedford-Taylor theorem on negligible sets, the series $\sum_{j=0}^{\infty} c_{nj}(z) z_n^j$ is locally uniformly convergent on Ω_n, and its sum f_n is identical with f on $F_1 \times \dots \times F_n$. The proof of case 2^o is finished, and so is the proof of the Main Lemma.

References

1. Grauert H., Remmert R., *Konvexität in der komplexen Analysis,* Comment. Math.Helv. **31** (1976), 152–160, 161–183.
2. Jarnicki M., Pflug P., *Cross Theorem,* Ann. Polon. Math., (to appear).
3. Klimek M., *Pluripotential Theory,* The Clarendon Press 1991.
4. Nguyen Thanh Van, Zeriahi A., *Une extension du théorème de Hartogs sur les fonctions séparément analytiques,* In *Analyse complexe multivariable (Guadeluppe, 1988),* EditEl, Rende, 1991, 183–194.
5. Öktem Ozan, *Extension of separately analytic functions and applications to range characterization of the exponential Radon transform,* Ann. Polon. Math. **70** (1998), 195–213.
6. ______, *Extension of separately analytic functions and applications to mathematical tomography,* Department of Mathematics, Stockholm University 1999 (Thesis).

Received April 27, 2001

Jagiellonian University
Institute of Mathematics
Reymonta 4
30-059 Kraków
Poland
e-mail: siciak@im.uj.edu.pl