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TOTAL SEPARATION AND ASYMPTOTIC STABILITY

by Barnabas M. Garay and József Garay

Abstract. The main goal of this paper is to weaken the invariance assumptions in Liapunov's direct method. Various geometric conditions resembling those provided by the collection of level surfaces of Liapunov functions are investigated. They are used to derive necessary resp. sufficient conditions for the asymptotic stability of equilibrium points of dynamical systems in Banach spaces. The main result — which does not hold true in the infinite–dimensional setting — is a consequence of the Zubov–Ura–Kimura Theorem.

1. The concept of total separation. Let $(X, \|\cdot\|)$ be a Banach space and let $\Phi : \mathbf{R} \times X \to X$ be a dynamical system on X. The origin of X is denoted by 0_X. Throughout this paper, we assume that 0_X is an equilibrium point of Φ. With respect to the dynamical system Φ and its equilibrium point 0_X, we say that a family of triplets $\{(\mathcal{O}^x, \mathcal{S}^x, \mathcal{I}^x)\}_{x \in X \setminus \{0_X\}}$ is a *total separation in* X if

(i) for each $x \in X \setminus \{0_X\}$, $\mathcal{O}^x$, $\mathcal{S}^x$ and $\mathcal{I}^x$ are nonempty, pairwise disjoint subsets of X, $\mathcal{O}^x$ and $\mathcal{I}^x$ are open, $\mathcal{S}^x$ is closed,
(ii) for each $x \in X \setminus \{0_X\}$, $\mathcal{O}^x \cup \mathcal{S}^x \cup \mathcal{I}^x = X$ and $0_X \in \mathcal{I}^x$, and
(iii) for each $x \in X \setminus \{0_X\}$, $\Phi((-\infty, 0), x) \subset \mathcal{O}^x$, $x \in \mathcal{S}^x$ and $\Phi((0, \infty), x) \subset \mathcal{I}^x$.

A total separation is called *dynamically strong* if $\mathcal{I}^x$ is positively invariant for each $x \in X \setminus \{0_X\}$. A total separation is called *weakly nested* if $\mathcal{S}^x = \mathcal{S}^y$ or $\mathcal{S}^x \cap \mathcal{S}^y = \emptyset$ whenever $x, y \in X \setminus \{0_X\}$. If, in addition, $\mathcal{I}^x \neq \mathcal{I}^y$ implies that EITHER $\mathcal{I}^x \subset \mathcal{I}^y$ OR $\mathcal{I}^x \supset \mathcal{I}^y$ for each $x, y \in X \setminus \{0_X\}$, then a weakly nested total separation is termed *nested*.

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For each $x, y \in X \setminus \{0_X\}$, weak nestedness of a total separation implies that $\mathcal{S}^x \cap \Phi(\mathbf{R}, y)$ is either the empty set or a single point and thus $\mathcal{S}^x$ is the common boundary of $\mathcal{O}^x$ and $\mathcal{I}^x$. Note that weakly nested total separations are dynamically strong. As it is demonstrated below, the converse assertion does not hold true.

Example 1. Let $X = \mathbf{R}^n$, equipped with the standard scalar product $\langle \cdot, \cdot \rangle$. Consider the linear dynamical system $\Lambda : \mathbf{R} \times X \to X$, $(t, x) \to e^{-t}x$. For each $x \in X \setminus \{0_X\}$ and parameter $p \in [0, 1)$, set $\mathcal{S}_p^x = X \setminus (\mathcal{O}_p^x \cup \mathcal{I}_p^x)$ where

$$\mathcal{O}_p^x = \{y \in X \mid \frac{\langle y - x, x \rangle}{\|y - x\| \cdot \|x\|} < -p\} \quad , \quad \mathcal{I}_p^x = \{y \in X \mid \frac{\langle y - x, x \rangle}{\|y - x\| \cdot \|x\|} > p\}.$$

It is obvious that the family of triplets $\{(\mathcal{O}_p^x, \mathcal{S}_p^x, \mathcal{I}_p^x)\}_{x \in X \setminus \{0_X\}}$, $p \in [0, 1)$ is a dynamically strong, but not weakly nested total separation (with respect to the dynamical system Λ and its equilibrium point 0_X) in $X = \mathbf{R}^n$.

The total separations in Examples 2 and 3 below are weakly nested but not nested. (Actually, the total separation in Example 2 is 'nowhere nested'. On the other hand, the total separation in Example 3 is 'locally nested' at 0_X. For a precise formulation, see Theorem 2.C below.)

Example 2. Let $X = \mathbf{R}$ and consider the linear dynamical system $\Lambda \colon \mathbf{R} \times X \to X$, $(t, x) \to e^{-t}x$. For $x > 0$, set $\mathcal{S}^x = \{-x^{-1}, x\}$, $\mathcal{I}^x = (-x^{-1}, x)$, and $\mathcal{O}^x = X \setminus (\mathcal{S}^x \cup \mathcal{I}^x)$. Similarly, for $x < 0$, set $\mathcal{S}^x = \{x, -x^{-1}\}$, $\mathcal{I}^x = (x, -x^{-1})$, and $\mathcal{O}^x = X \setminus (\mathcal{S}^x \cup \mathcal{I}^x)$. It is clear that the family of triplets $\{(\mathcal{O}^x, \mathcal{S}^x, \mathcal{I}^x)\}_{x \in X \setminus \{0_X\}}$ is a weakly nested total separation. Note that $\mathcal{I}^x \subset \mathcal{I}^y$ if and only if $\mathcal{I}^x = \mathcal{I}^y$ whenever $0_X \neq x, y \in X = \mathbf{R}$.

Example 3. Let $X = \mathbf{R}^2$ and consider the linear dynamical system $\Lambda \colon \mathbf{R} \times X \to X$, $(t, x) \to e^{-t}x$. For brevity, we write $\Gamma = \{x = (x_1, x_2) \in X : |x_1 x_2| = 1\}$. It is easy to construct a weakly nested total separation (with respect to the dynamical system Λ and its equilibrium point 0_X) in $X = \mathbf{R}^2$ with the property that $\Gamma = \mathcal{S}^{(1,1)}$. In fact, the $\mathcal{S}^x$'s 'inside' Γ are simple closed curves around 0_X, $\Gamma = \mathcal{S}^{(1,1)}$, the $\mathcal{S}^x$'s 'outside' Γ look like double hyperbolas and consist of four unbounded arcs. With a little more care, these double hyperbolas can be chosen in such a way that, given any two points x, y 'outside' Γ, $\mathcal{I}^x \subset \mathcal{I}^y$ if and only if $\mathcal{I}^x = \mathcal{I}^y$.

As shown by Theorems 1 and 4 below, the concept of nested total separation describes the level surface structure of a Liapunov function from the viewpoint of inclusion properties. The concept of total separation itself is obtained by weakening the invariance requirements to, in our opinion, an unusually large extent (see hypothesis (iii)) but still implying asymptotic stability in Theorem 2.A, the main result of this paper. The proof of Theorem 2.A is based on the

Zubov–Ura–Kimura Theorem. (*see e.g. Corollary 6.1.2 in* [**4**]) *Let* (W, d) *be a locally compact metric space and let* Θ *be a dynamical system on* W*. Finally, let* $\emptyset \neq M$ *be a compact isolated* Θ*–invariant set in* W*. Suppose that* M *is not asymptotically stable. Then* $\emptyset \neq \alpha(x) \subset M$ *for some* $x \notin M$*.*

A modified Bebutov shift in Egawa [**7**] demonstrates that the Zubov–Ura–Kimura Theorem does not hold true without the local compactness condition. Actually, the Zubov–Ura–Kimura Theorem is false in every infinite–dimensional Banach space [**8**]. Essentially the same method shows in Theorem 3 below that also Theorem 2.A is false in every infinite–dimensional Banach space.

We use standard notation and terminology. In particular, $\mathcal{B}(x, \varepsilon)$ stays for the open ball $\{y \in X : \|y - x\| < \varepsilon\}$ of radius ε centered at $x \in X$. The alpha–limit set of the trajectory through x is denoted by $\alpha(x)$. For basic references in stability theory and topological dynamics, see [**1**], [**4**], [**12**], [**14**].

2. Nested separations via Liapunov functions. We begin with the simple result that global asymptotic stability implies the existence of a nested separation. The definition of asymptotic stability is recalled for convenience. The equilibrium point 0_X is *asymptotically stable* if, given an arbitrary $\varepsilon > 0$, there exists a $\delta > 0$ with $\Phi(\mathbf{R}^+, \mathcal{B}(0_X, \delta)) \subset \mathcal{B}(0_X, \varepsilon)$ and its region of attraction $\mathcal{A} = \{y \in X \mid \Phi(t, y) \to 0_X \text{ as } t \to \infty\}$ is a (necessarily open) neighborhood of 0_X in X. Asymptotic stability is *global* if $\mathcal{A} = X$. The definition of (global) asymptotic stability for a nonempty compact invariant set follows a similar pattern and is omitted.

Theorem 1. *Assume that* 0_X *is globally asymptotically stable. Then there is a nested total separation in* X*.*

Proof. This follows immediately from a well–known result in converse Liapunov theory. By Theorem 2.7.14 in [**4**], there exists a continuous function $V : X \to [0, \infty)$ with the properties that $V^{-1}(0) = \{0_X\}$ and, given an arbitrary $x \in X \setminus \{0_X\}$, $t \to V(\Phi(t, x))$ defines a decreasing homeomorphism of $\mathbf{R}$ onto $(0, \infty)$. For each $x \in X \setminus \{0_X\}$, set

$$\mathcal{O}^x = V^{-1}((V(x), \infty)) \quad , \quad \mathcal{S}^x = V^{-1}(V(x)) \quad , \quad \mathcal{I}^x = V^{-1}((0, V(x))).$$

It is clear that the family of triplets $\{(\mathcal{O}^x, \mathcal{S}^x, \mathcal{I}^x)\}_{x \in X \setminus \{0_X\}}$ is a nested total separation in X. Geometrically, $\mathcal{S}^x$ is the level surface of V through $x \in X \setminus \{0_X\}$, and $\mathcal{O}^x$ and $\mathcal{I}^x$ stay for the corresponding outer and inner regions, respectively. $\square$

Theorem 1 holds true if X is replaced by an arbitrary metric space and, of course, 0_X is replaced by a globally asymptotically stable point (or, more generally, by a nonempty compact invariant subset) of this metric space. No alterations in the proof are needed.

It is natural to ask if the $\mathcal{S}^x$'s of a nested total separation can be represented as level surfaces of a suitable Liapunov function. A partial answer is given in Theorem 4 below.

3. Asymptotic stability via total separations. Throughout this subsection, we assume that X is finite dimensional and establish some dynamical consequences implied by the existence of total separations with increasingly finer properties.

THEOREM 2.A. *Let X be finite–dimensional and let $\{(\mathcal{O}^x, \mathcal{S}^x, \mathcal{I}^x)\}_{x\in X\setminus\{0_X\}}$ be a total separation (with respect to a dynamical system Φ and its equilibrium point 0_X) in X. Then 0_X is asymptotically stable.*

PROOF. We first point out that, given an arbitrary $x \in X \setminus \{0_X\}$, the alpha–limit set of x is either empty or unbounded. To the contrary, assume that $\emptyset \neq \alpha(z)$ is bounded for some $z \in X \setminus \{0_X\}$. By a standard application of Zorn Lemma, $\alpha(z)$ contains a nonempty compact minimal invariant set, say M_z. Pick a point $w \in M_z$. We distinguish two cases according to whether $w = 0_X$ or not. If $w = 0_X$, then $0_X \in \mathrm{cl}(\Phi(-\infty, 0), z)) \subset \mathrm{cl}(\mathcal{O}^z) \subset X\setminus\mathcal{I}^z$, a contradiction. If $w \neq 0_X$, then w is a nonequilibrium point and $w \in \alpha(w)$. Consider a time sequence $t_n \to -\infty$ for which $\Phi(t_n, w) \to w$. Since $\Phi(-(t_n - 1), w) \to \Phi(1, w)$ and $\Phi(-(t_n-1), w) \in \mathcal{O}^w$ for n large enough, it follows that $\Phi(1, w) \in \mathrm{cl}(\mathcal{O}^w) \subset X \setminus \mathcal{I}^w$. But $\Phi(1, w) \in \mathcal{I}^w$, a contradiction.

As a by–product, we obtain that $\{0_X\}$ as an invariant set is isolated (i.e. $\{0_X\}$ is maximal invariant set within a neighborhood of itself) and $\{0_X\} = \alpha(x)$ for no $x \in X \setminus \{0_X\}$. Thus all conditions of the Zubov–Ura–Kimura Theorem are satisfied. Consequently, 0_X is asymptotically stable. □

THEOREM 2.B. *Assume, in addition, that the total separation is dynamically strong. Then $\alpha(x) = \emptyset$ for each $x \in X \setminus \{0_X\}$.*

PROOF. To the contrary, assume that $\alpha(p) \neq \emptyset$ for some $p \in X \setminus \{0_X\}$. Pick a $q \in \alpha(p)$ and consider a time sequence $t_n \to -\infty$ for which $\Phi(t_n, p) \to q$ as $n \to \infty$. We may assume that $t_{n+1} < t_n - 2$ for each n. Note that $\Phi(t_n - 1, p) \to \Phi(-1, q)$ and $\Phi(t_n + 1, p) \to \Phi(1, q)$. Since $\Phi(-1, q) \in \mathcal{O}^q$ and $\Phi(1, q) \in \mathcal{I}^q$, it follows for n large enough, say $n \geq N_o$, that $\Phi(t_n - 1, p) \in \mathcal{O}^q$, $\Phi(t_n + 1, p) \in \mathcal{I}^q$ and $\Phi(\tau_n, p) \in \mathcal{S}^q$ for some (not necessarily unique) $\tau_n \in (t_n - 1, t_n + 1)$. By the construction, $t_{N_o} - 1 > t_{N_o+1} + 1$. Now we make use of the additional assumption. The positive invariance of $\mathcal{I}^q$ implies that $\Phi(t_{N_o} - 1, p) \in \mathcal{I}^q \subset X \setminus \mathcal{O}^q$, a contradiction. □

Theorem 2.C. *Assume, in addition, that the total separation is weakly nested. Then there exists a* $z_0 \in \mathcal{A} \setminus \{0_X\}$ *such that* $cl(\mathcal{I}^{z_0}) = \mathcal{S}^{z_0} \cup \mathcal{I}^{z_0}$ *is compact and connected in* $\mathcal{A}$*. Moreover, if* $\dim(X) \geq 2$*, then the family of triplets* $\{(\mathcal{O}^x, \mathcal{S}^x, \mathcal{I}^x)\}_{x \in X \setminus \{0_X\}}$ *is nested at* 0_X *in the sense that (for any* z_0 *having the properties as above)* $\mathcal{S}^x \cup \mathcal{I}^x \subset \mathcal{I}^{z_0}$ *whenever* $x \in \mathcal{I}^{z_0} \setminus \{0_X\}$ *and* $\mathcal{S}^y \cup \mathcal{I}^y \subset \mathcal{I}^x$ *whenever* $y \in \mathcal{I}^x \setminus \{0_X\}$*.*

Proof. We already know from Theorem 2.A that 0_X is asymptotically stable. Let $\mathcal{A}$ denote the region of attraction of 0_X. Applying Theorem 2.7.14 of [**4**] again, there exists a continuous function $W : \mathcal{A} \to [0, \infty)$ such that $W^{-1}(0) = 0_X$ and, given an arbitrary $x \in \mathcal{A} \setminus \{0_X\}$, $t \to W(\Phi(t, x))$ defines a decreasing homeomorphism of $\mathbf{R}$ onto $(0, \infty)$. Set $S = W^{-1}(1)$. In view of Corollary 2.11.38 of [**4**], an easy consequence of Theorem 2.7.14 via local compactness, S is a compact global section for $\Phi|_{\mathcal{A} \setminus \{0_X\}}$. In other words, S is a compact subset of $\mathcal{A} \setminus \{0_X\}$ and, given an arbitrary $x \in \mathcal{A} \setminus \{0_X\}$, there exists the unique $\tau(x) \in \mathbf{R}$ such that $\Phi(\tau(x), x) \in S$ and the function $\tau : \mathcal{A} \setminus \{0_X\} \to \mathbf{R}$ is continuous. Also projection $\Pi : \mathcal{A} \setminus \{0_X\} \to S$, $x \to \Phi(\tau(x), x)$ is continuous.

The first assertion of Theorem 2.C is trivial if $\dim(X) = 1$. (As it is demonstrated by Example 2 above, the second assertion is false if $\dim(X) = 1$.)

From now on, assume that $\dim(X) \geq 2$. Note that both $\mathcal{A} \setminus \{0_X\}$ and, a fortiori, its projective image $\Pi(\mathcal{A} \setminus \{0_X\})$ are connected and arcwise connected.

Claim 1. *For* $z \in \mathcal{A} \setminus \{0_X\}$*, let* $c(\mathcal{S}^z \cap \mathcal{A}; z)$ *denote the connected component of* $\mathcal{S}^z \cap \mathcal{A}$ *containing* z*. Then* $\Pi(c(\mathcal{S}^z \cap \mathcal{A}; z))$ *is an arcwise connected open subset of* S*.*

Proof of Claim 1. Fix a $z \in \mathcal{A} \setminus \{0_X\}$ and choose $\varepsilon > 0$ in such a way that

$$\mathcal{B}(\Phi(-1, z), \varepsilon) \subset \mathcal{O}^z \cap \mathcal{A} \quad \text{and} \quad \mathcal{B}(\Phi(1, z), \varepsilon) \subset \mathcal{I}^z \cap \mathcal{A}.$$

By continuity, there exist positive constants $\eta < \varepsilon$ and δ such that

$$\Phi(-\tau(z) - 1, \mathcal{B}(\Pi(z), \delta)) \subset \mathcal{B}(\Phi(-1, z), \eta)$$

and

$$\Phi(2, \mathcal{B}(\Phi(-1, z), \eta)) \subset \mathcal{B}(\Phi(1, z), \varepsilon),$$
$$\Phi(-\tau(z) + 1, \mathcal{B}(\Pi(z), \delta)) \subset \mathcal{B}(\Phi(1, z), \varepsilon).$$

Consider a point $q \in S \cap \mathcal{B}(\Pi(z), \delta)$. By the construction, there exists the unique $\sigma_z = \sigma_z(q) \in (-\tau(z) - 1, -\tau(z) + 1)$ for which $\Phi(\sigma_z(q), q) \in \mathcal{S}^z \cap \mathcal{A}$ and $\Pi(\Phi(\sigma_z(q), q)) = q$. It follows that $\Pi(\mathcal{S}^z \cap \mathcal{A})$ is open in S. As a direct consequence of weak nestedness, Π is a bijection between $\mathcal{S}^z \cap \mathcal{A}$ and $\Pi(\mathcal{S}^z \cap \mathcal{A})$. For $s \in \Pi(\mathcal{S}^z \cap \mathcal{A})$, there exists a unique $\sigma_z(s) \in \mathbf{R}$ with $\Phi(\sigma_z(s), s) \in \mathcal{S}^z \cap \mathcal{A}$ and $\tau(\Phi(\sigma_z(s), s)) = -\sigma_z(s)$. A standard compactness argument shows that the mapping $\sigma_z : \Pi(\mathcal{S}^z \cap \mathcal{A}) \to \mathbf{R}$ is continuous. Hence Π is a homeomorphism between the components of $\mathcal{S}^z \cap \mathcal{A}$ and those of $\Pi(\mathcal{S}^z \cap \mathcal{A})$. Together with the components of $\Pi(\mathcal{S}^z \cap \mathcal{A})$, also $\Pi(c(\mathcal{S}^z \cap \mathcal{A}; z))$ is open in S.

In order to prove arcwise connectedness of $\Pi(c(\mathcal{S}^z \cap \mathcal{A}; z))$, we return to a $q \in S \cap \mathcal{B}(\Pi(z), \delta)$. Consider the straight line segment γ connecting the points $\Phi(-1, z) = \Phi(-\tau(z) - 1, \Pi(z))$ and $\Phi(-\tau(z) - 1, q)$ in $\mathcal{B}(\Phi(-1, z), \eta) \subset \mathcal{O}^z \cap \mathcal{A}$. By the construction $\Pi(\gamma)$ is an arc connecting $\Pi(z)$ and q in $\Pi(\mathcal{S}^z \cap \mathcal{A})$. The desired arcwise connectedness follows immediately. $\square$

Claim 2. *For some $t > 0$ and $z \in \mathcal{A} \setminus \{0_X\}$, let $w = \Phi(t, z)$. Then $\Pi(c(\mathcal{S}^w \cap \mathcal{A}; w)) \supset \Pi(c(\mathcal{S}^z \cap \mathcal{A}; z))$. Moreover, given an arbitrary $r \in c(\mathcal{S}^z \cap \mathcal{A}; z)$, there exists a $\sigma > 0$ such that $\Phi(\sigma, r) \in c(\mathcal{S}^w \cap \mathcal{A}; w)$.*

Proof of Claim 2. To the contrary, assume that the first assertion is false. Pick a point $q \in \Pi(c(\mathcal{S}^z \cap \mathcal{A}; z)) \setminus \Pi(c(\mathcal{S}^w \cap \mathcal{A}; w))$. Since $\Pi(c(\mathcal{S}^z \cap \mathcal{A}; z))$ is arcwise connected, there exists a continuous function $\Gamma : [0, 1] \to \Pi(c(\mathcal{S}^z \cap \mathcal{A}; z)) \subset S$ such that $\Gamma(0) = \Pi(z) = \Pi(w)$, $\Gamma(\mu) \in \Pi(c(\mathcal{S}^w \cap \mathcal{A}; w))$ for each $\mu \in [0, 1)$ but $p = \Gamma(1) \notin \Pi(c(\mathcal{S}^w \cap \mathcal{A}; w))$ for some $p \in S$.

For brevity, set $L = \liminf_{\mu \to 1} \sigma_w(\Gamma(\mu))$ where (with z replaced by w) the continuous function $\sigma_w : \Pi(\mathcal{S}^w \cap \mathcal{A}) \to \mathbf{R}$ is defined as in the proof of Claim 1. We next point out that $L = -\infty$. By a simple compactness argument, attraction is uniform and thus $\Phi([T^*, \infty), S) \subset \mathcal{I}^w$ for some $T^* > 0$. This makes $L = \infty$ impossible. Suppose that L is finite. By the construction, $\Phi(L - 1, \Gamma(\mu)) \in \mathcal{O}^w$ for each $\mu \in [0, 1)$ and thus $\Phi(L - 1, \Gamma(1)) \in \mathrm{cl}(\mathcal{O}^w) = X \setminus \mathcal{I}^w$. On the other hand, $\Phi(\mathbf{R}, \Gamma(1)) = \Phi(\mathbf{R}, p) \subset \mathcal{I}^w$. In particular, $\Phi(L - 1, \Gamma(1)) \in \mathcal{I}^w$, a contradiction.

The remaining task is easy. Suppose that the second assertion is false. In other words, suppose that $\sigma_z(\Pi(r)) \geq \sigma_w(\Pi(r))$. Let $\Delta : [0, 1] \to \Pi(c(\mathcal{S}^z \cap \mathcal{A}; z)) \subset \Pi(c(\mathcal{S}^w \cap \mathcal{A}; w)) \subset S$ be a continuous function with $\Delta(0) = \Pi(z) = \Pi(w)$ and $\Delta(1) = \Pi(r)$. Note that $\sigma_z(\Delta(0)) < \sigma_w(\Delta(0))$. In view of the Bolzano Theorem, there exists a $\mu_0 \in (0, 1]$ with $\sigma_z(\Delta(\mu_0)) = \sigma_w(\Delta(\mu_0))$. Thus $\mathcal{S}^z = \mathcal{S}^{\sigma_z(\Delta(\mu_0))} = \mathcal{S}^{\sigma_w(\Delta(\mu_0))} = \mathcal{S}^w$, a contradiction. $\square$

Now we are in a position to finish the proof of Theorem 2.C.

By compactness, there is a finite collection of points $\mathcal{Z} = \{z_0, z_1, \ldots, z_K\}$ in $\mathcal{A} \setminus \{0_X\}$ such that $\{\Pi(c(\mathcal{S}^{z_k} \cap \mathcal{A}; z_k)) \mid k = 0, 1, \ldots, K\}$ is an open cover of S. There is no loss of generality in assuming that $\mathcal{S}^{z_k} \cap \mathcal{S}^{z_\ell} = \emptyset$ for $k \neq \ell$. For $z_k, z_\ell \in \mathcal{Z}$ chosen arbitrarily, we say that z_k is *not greater than* z_ℓ and write $z_k \Uparrow z_\ell$ if $\Pi(c(\mathcal{S}^{z_k} \cap \mathcal{A}; z_k)) \cap \Pi(c(\mathcal{S}^{z_\ell} \cap \mathcal{A}; z_\ell)) \neq \emptyset$ and, for some $r \in \Pi(c(\mathcal{S}^{z_k} \cap \mathcal{A}; z_k)) \cap \Pi(c(\mathcal{S}^{z_\ell} \cap \mathcal{A}; z_\ell))$, $\sigma_{z_k}(r) \leq \sigma_{z_\ell}(r)$. Note that $z_k \Uparrow z_k$ for each k. As a direct consequence of Claim 2, $z_k \Uparrow z_\ell$ and $z_\ell \Uparrow z_k$ imply that $z_k = z_\ell$, $k = \ell$. Similarly, $z_k \Uparrow z_\ell$ and $z_\ell \Uparrow z_m$ imply that $z_k \Uparrow z_m$. Thus $\Uparrow$ defines a partial ordering on $\mathcal{Z}$. For brevity, we say that $z \in \mathcal{Z}$ is a *maximal element* if $z_k \Uparrow z$ for each $k = 0, 1, \ldots, K$. We claim that there exists a maximal element. Suppose not. By a repeated application of Claim 2, it follows then that there is a subset of $\mathcal{Z}$, say $\{z_1, z_2, \ldots, z_M\}$, $M \geq 2$ such that $\Pi(c(\mathcal{S}^{z_k} \cap \mathcal{A}; z_k)) \cap \Pi(c(\mathcal{S}^{z_\ell} \cap \mathcal{A}; z_\ell)) = \emptyset$ for $k \neq \ell, k \in \{1, 2, \ldots, M\}$ and $\{\Pi(c(\mathcal{S}^{z_m} \cap \mathcal{A}; z_m)) | m = 1, 2, \ldots, M\}$ is an open cover of S. This contradicts the connectedness of S. The maximal element, say $z = z_0$, is unique and satisfies $\Pi(c(\mathcal{S}^{z_0} \cap \mathcal{A}; z_0)) = S$. Further, $s \to \Phi(\sigma_{z_0}(s), s)$ defines a homeomorphism of S onto $c(\mathcal{S}^{z_0} \cap \mathcal{A}; z_0) = \mathcal{S}^{z_0} \cap \mathcal{A} = \mathcal{S}^{z_0}$. This ends the proof of the first assertion of Theorem 2.C.

The second assertion follows by a double application of Claim 2. □

In general, asymptotic stability of 0_X ensured by Theorem 2 is only local.

EXAMPLE 4. Let $X = \mathbf{R}^2$. For brevity, we write $L = \{x = (x_1, x_2) \in X | x_2 = -2\}$ and $\Gamma = \{x \in X : |x_1| < \pi/2 \text{ and } x_2 = -2 + 1/\cos(x_1)\}$. It is easy to construct a C^∞ dynamical system Φ_0 on X with the following properties: 1. The origin 0_X is asymptotically stable and its region of attraction is the vertical strip $\mathcal{A}_0 = \{x \in X : |x_1| < \pi/2\}$; 2. Trajectories outside $\mathcal{A}_0$ are upward vertical straight lines; 3. With the exception of 0_X and of the single 'entirely downward' trajectory $\gamma_0 = \{(0, x_2) \in X | x_2 > 0\}$, all other trajectories in $\mathcal{A}_0$ intersect curve Γ at exactly one point; 4. Trajectory segments below the horizontal line L are upward straight line segments. Having done this, it is not hard to construct a nested total separation (with respect to the dynamical system Φ_0 and its equilibrium point 0_X) in $X = \mathbf{R}^2$. In fact, the $\mathcal{S}^x$'s below L are horizontal lines, $L = \mathcal{S}^{(0,-2)}$, the $\mathcal{S}^x$'s between L and Γ look like parabolas, $\Gamma = \mathcal{S}^{(0,-1)}$, the $\mathcal{S}^x$'s above Γ are simple closed curves around 0_X.

Theorem 2.A holds true if the pair $(X, 0_X)$ is replaced by $(\mathcal{X}, x_0)$ where $\mathcal{X}$ is a locally compact metric space and $x_0 \in \mathcal{X}$. If, in addition, $\mathcal{X} \setminus \{x_0\}$ is connected and locally arcwise connected, then also Theorem 2.C remains valid. No essential alterations in the proofs are needed.

With a slight abuse of terminology, the global section S we used in proving Theorem 2.C is called a Liapunov sphere in $\mathcal{A} \setminus \{0_X\}$. It is immediate that any two Liapunov spheres (for $\Phi|_{\mathcal{A} \setminus \{0_X\}}$) in $\mathcal{A} \setminus \{0_X\}$ are homeomorphic. If $\dim(X) \geq 4$, Liapunov spheres of abstract dynamical systems (for which 0_X is an asymptotically stable equilibrium) need not be topological manifolds [**6**]. For $\dim(X) \geq 1$, Brown Theorem [**5**] (quoted as Theorem 2.8.10 in [**4**]) states that the region of attraction itself is homeomorphic to X. On the other hand, Liapunov spheres in infinite-dimensional Banach spaces are homeomorphic to the unit sphere $\{x \in X \mid \|x\| = 1\}$ and regions of attractions of asymptotically stable equilibria themselves are homeomorphic to X [**9**]. (Recall that, by a famous result in infinite-dimensional topology [**3**], $\{x \in X : \|x\| = 1\}$ and X are homeomorphic.)

4. A negative result in infinite dimension. On the other hand, as it is demonstrated by Theorem 3 below, Theorem 2 does not remain valid in infinite-dimensional Banach spaces.

THEOREM 3. *Let X be an infinite-dimensional Banach space. Then there is a dynamical system Ψ and a nested total separation $\{(\mathcal{O}^x, \mathcal{S}^x, \mathcal{I}^x)\}_{x\in X\setminus\{0_X\}}$ (with respect to the dynamical system $\Psi : \mathbf{R}\times X \to X$ and its equilibrium point 0_X) in X such that $\cap\{\mathcal{I}^x \mid x \in X \setminus \{0_X\}\} = \{0_X\}$ but 0_X is unstable.*

PROOF. Fix an 'upward' unit vector $e_0 \in X$. In view of the Banach–Hahn Theorem, X can be represented as $X = \{x = y + \lambda e_0 \mid y \in Y \text{ and } \lambda \in \mathbf{R}\}$ where Y is a suitably chosen 'horizontal' codimension one subspace. There is no loss of generality in assuming that $\|x\| = \|y\| + |\lambda|$ whenever $x = y + \lambda e_0$ with $y \in Y$ and $\lambda \in \mathbf{R}$. Note that Y is an infinite-dimensional Banach space with origin 0_Y and $X = Y \times \mathbf{R}$.

Consider now the system of ordinary differential equations

$$\dot{y} = 0_Y \quad \& \quad \dot{\lambda} = \|x\| \qquad \text{on} \quad Y \times \mathbf{R} = X.$$

Since the right–hand side is Lipschitz continuous and linearly bounded, the solutions of $\dot{y} = 0_Y$ & $\dot{\lambda} = \|x\|$ define a dynamical system $\Theta : \mathbf{R} \times X \to X$. Geometrically, Θ is an 'upward vertical' flow. Note that 0_X is an equilibrium point of Θ. For brevity, we write $\gamma_- = \{\nu e_0 \mid \nu < 0\}$ and $\gamma_+ = \{\nu e_0 \mid \nu > 0\}$. All trajectories outside $\gamma_- \cup \{0_X\} \cup \gamma_+$ are 'upward vertical' straight lines. The trajectories through $-e_0$ and e_0 are γ_- and γ_+, respectively.

For $c > 0$, the formula $h_c(y) = -c + c^{-1}\|y\|$ defines a continuous function $h_c : Y \to \mathbf{R}$. A direct computation shows that, given an arbitrary $y + \lambda e_0 = x \in X \setminus (\{0_X\} \cup \gamma_+)$, there is the unique $c(x) > 0$ for which $h_{c(x)}(y) = \lambda$. For $x \in X \setminus \{0_X\}$, define

$$o^x = \{z = w + \mu e_0 \mid w \in Y, \mu \in \mathbf{R} \text{ and } h_{c(x)}(w) < \mu\},$$

$$s^x = \{z = w + \mu e_0 \mid w \in Y, \mu \in \mathbf{R} \text{ and } h_{c(x)}(w) = \mu\},$$

$$i^x = \{z = w + \mu e_0 \mid w \in Y, \mu \in \mathbf{R} \text{ and } h_{c(x)}(w) > \mu\},$$

and observe that $\cap\{i^x \mid x \in X \setminus (\{0_X\} \cup \gamma_+)\} = \{0_X\} \cup \gamma_+$.

The construction of Ψ is based on the concept of deleting homeomorphisms of Y, i.e. of homeomorphisms of Y onto $Y \setminus \{0_Y\}$. By the main result in [**3**], there exists a deleting homeomorphism $\mathcal{H} : Y \to Y \setminus \{0_Y\}$ such that $\mathcal{H}(y) = y$ whenever $\|y\| \geq 1$. By putting

$$\mathcal{G}(x) = \begin{cases} \lambda\mathcal{H}(y/\lambda) + \lambda e_0 & \text{if} \quad x = y + \lambda e_0 \quad \text{with} \quad \lambda > 0 \\ x & \text{if} \quad x = y + \lambda e_0 \quad \text{with} \quad \lambda \leq 0, \end{cases}$$

a homeomorphism $\mathcal{G}$ of X onto $X \setminus \gamma_+$ is defined. The desired dynamical system is defined by $\Psi(t, x) = \mathcal{G}^{-1}(\Theta(t, \mathcal{G}(x)))$, $(t, x) \in \mathbf{R} \times X$. Finally, for

$x \in X \setminus \{0_X\}$, define

$$\mathcal{O}^x = \mathcal{G}^{-1}(o^{\mathcal{G}(x)}) \quad , \quad \mathcal{S}^x = \mathcal{G}^{-1}(s^{\mathcal{G}(x)}) \quad , \quad \mathcal{I}^x = \mathcal{G}^{-1}(i^{\mathcal{G}(x)}).$$

By the construction, it is readily checked that $\cap\{\mathcal{I}^x \mid x \in X \setminus \{0_X\}\} = \{0_X\}$ and that the triplet $\{(\mathcal{O}^x, \mathcal{S}^x, \mathcal{I}^x)\}_{x \in X\setminus\{0_X\}}$ is a nested total separation with respect to the dynamical system $\Psi : \mathbf{R} \times X \to X$ and its equilibrium point 0_X. In addition, we show that $\|\Psi(t,x)\| \to \infty$ as $t \to \infty$ for each $x \notin \gamma_- \cup \{0_X\}$, a very strong form of instability of 0_X. □

The existence of deleting homeomorphisms is one of the central results in the topology of infinite-dimensional Banach spaces. It shows immediately that Borsuk's nonretractibility theorem for n–dimensional spheres does not remain valid in the infinite-dimensional setting. Deleting homeomorphisms/diffeomorphisms were used in explaining why results like the n–dimensional Rolle Theorem [**13**], [**1**], or a great part of the theory of ordinary differential equations including Peano's existence theorem fail in infinite-dimensional Banach spaces, see [**11**], [**10**].

5. Liapunov functions via nested separations. Concluding this paper, we present sufficient conditions ensuring that the $\mathcal{S}^x$'s of a nested total separation can be represented as level surfaces of a suitable Liapunov function.

THEOREM 4. *Assume that X is finite dimensional, 0_X is globally asymptotically stable and that the family of triplets $\{(\mathcal{O}^x, \mathcal{S}^x, \mathcal{I}^x)\}_{x\in X\setminus\{0_X\}}$ is a nested total separation (with respect to the dynamical system Φ and its equilibrium point 0_X) in X. Then there exists a continuous function $V : X \to \mathbf{R}^+$ which is strictly decreasing along nontrivial trajectories, $V(0_X) = 0$, and satisfies $V(\tilde{x}) = V(x)$ whenever $\tilde{x} \in \mathcal{S}^x$, $x \in X \setminus \{0_X\}$.*

PROOF. The family of closed sets $\mathcal{F} = \{\mathcal{S}^x \subset X \mid X \setminus \{0_X\}\}$ defines a decomposition of $X \setminus \{0_X\}$. For brevity, we write $\mathcal{S}^x \leq_0 \mathcal{S}^y$ if $\mathcal{I}^x \subset \mathcal{I}^y$ and $\mathcal{S}^x <_0 \mathcal{S}^y$ if $\mathcal{I}^x \cup \mathcal{S}^x \subset \mathcal{I}^y$. It is easily checked that $\mathcal{S}^x <_0 \mathcal{S}^y$ if and only if $x \in \mathcal{I}^y$ and thus $\mathcal{S}^x \leq_0 \mathcal{S}^y$ if and only if $\mathcal{S}^x <_0 \mathcal{S}^y$ or $\mathcal{S}^x = \mathcal{S}^y$. We conclude that $\leq_0$ defines a total ordering on $\mathcal{F}$. Note that $\leq_0$ is a closed relation, i.e. $\mathcal{S}^{x_k} \leq_0 \mathcal{S}^{y_k}$ and $x_k \to x \in X \setminus \{0_X\}$, $y_k \to y \in X \setminus \{0_X\}$ imply that $\mathcal{S}^x \leq_0 \mathcal{S}^y$. In fact, the opposite relation $\mathcal{S}^y <_0 \mathcal{S}^x$ implies that $\mathcal{S}^y <_0 \mathcal{S}^{\Phi(-\delta,y)} <_0 \mathcal{S}^{\Phi(\delta,x)} <_0 \mathcal{S}^x$ for some $\delta > 0$. Hence $y_k \in \mathcal{I}^{\Phi(-\delta,y)}$ and $x_k \in \mathcal{O}^{\Phi(\delta,x)}$ for k large enough, a contradiction.

By letting $\{0_X\} = \mathcal{S}^{0_X} \hat{\leq}_0 \mathcal{S}^x$ for each $x \in X$, the order relation $\leq_0$ extends to $\hat{\mathcal{F}} = \mathcal{F} \cup \{0_X\}$. It is crucial that also the extended order relation $\hat{\leq}_0$ is closed. In fact, the required closedness property is equivalent to pointing out that $\mathcal{I}^{x_k} \subset \mathcal{I}^{y_k}$, $k = 1, 2, \ldots$ and $y_k \to 0_X$ as $k \to \infty$ imply $x_k \to 0_X$. With z_0 as in Theorem 2.C, observe that $\mathcal{I}^{y_k} \subset \mathcal{I}^{z_0}$ for k large enough. Thus $\{x_k\}_{k=1}^{\infty}$

is contained in a compact subset of X and we may assume that $x_k \to x$ for some $x \in X$. Suppose that $x \neq 0_X$. Then $x_k \in \mathcal{O}^{\Phi(1,x)}$ for k large enough. On the other hand, $x_k \in \mathrm{cl}(\mathcal{I}^{y_k}) \subset \mathrm{cl}(\mathcal{I}^{\Phi(1,x)}) = X \setminus \mathcal{O}^{\Phi(1,x)}$ for k large enough, a contradiction.

Set $w_1 = z_0$, $U_1 = \mathcal{I}^{w_1}$. As a direct consequence of Claim 2, $U_1 \setminus \{0_X\} = \cup\{\mathcal{S}^{\Phi(t,w_1)} \mid t \geq 0\}$. By letting

$$V_1(x) = \begin{cases} e^{-t} & \text{if} \quad x \in \mathcal{S}^{\Phi(t,w_1)} \quad \text{for some} \quad t \geq 0 \\ 0 & \text{if} \quad x = 0_X, \end{cases}$$

a function $V_1 : \mathrm{cl}(U_1) \to [0,1]$ is defined. We claim that V_1 is continuous. In fact, by letting $\hat{\Phi}(\infty, w_1) = 0_X$, a continuous extension of $\Phi(\cdot, w_1)$ to the extended line $\hat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ is defined. In order to prove continuity at some $x \in \mathcal{S}^{\hat{\Phi}(t,w_1)} \subset \mathrm{cl}(U_1)$, consider a sequence $x_k \in \mathcal{S}^{\hat{\Phi}(t_k,w_1)} \subset \mathrm{cl}(U_1)$ with $x_k \to x$. It is enough to prove that $t_k \to t$ in $\hat{\mathbf{R}}$. Suppose not. By passing to a subsequence, we may assume that $t_k \to t_*$ for some $t_* \neq t$ in $\hat{\mathbf{R}}$. Since $\hat{\Phi}(t_k, w_1) \to \hat{\Phi}(t_*, w_1)$ as $k \to \infty$ and $x_k \in \mathcal{S}^{x_k} = \mathcal{S}^{\hat{\Phi}(t_k,w_1)}$ for each k, the closedness ot the extended order relation $\hat{\leq}_0$ implies that $x \in \mathcal{S}^x = \mathcal{S}^{\hat{\Phi}(t_*,w_1)}$. Hence $t = t_*$, a contradiction. Note that $\mathrm{cl}(U_1) \setminus U_1 = \mathcal{S}^{w_1}$ and $V_1(w_1) = 1$. By the construction, $V_1(\tilde{x}) = V_1(x)$ whenever $\tilde{x} \in \mathcal{S}^x$, $x \in \mathrm{cl}(U_1)$.

Set $z_1 = \Phi(-1, w_1)$. Since $\leq_0$ is a total ordering on $\mathcal{F}$ and $\cup\{\mathcal{I}^x \mid x \in X \setminus \{0_X\}\}$ is an open cover of the compact ball $B_k = \{x \in X \ : \ \|x\| \leq k-1\}$, there exists a $z_k \in X \setminus \{0_X\}$ for which $B_k \subset \mathcal{I}^{z_k}$, $k = 2, 3, \ldots$. There is no loss of generality in assuming that $\mathcal{S}^{z_k} <_0 \mathcal{S}^{z_{k+1}}$, $k = 1, 2, \ldots$.

Our strategy is to construct a sequence of points $w_1, w_2, \ldots$ in $X \setminus \{0_X\}$, an accompanying sequence of open subsets $U_1 \subset U_2 \subset \ldots$ of X, and an accompanying sequence of continuous functions $V_k : \mathrm{cl}(U_k) \to [0,k]$ such that $z_k \in U_k = \mathcal{I}^{w_k}$, $V_k(w_k) = k$, $\mathrm{cl}(U_k) \subset U_{k+1}$, $V_{k+1}|_{\mathrm{cl}(U_k)} = V_k$, $k = 1, 2, \ldots$ and, last but not least, $\cup_{k=1}^{\infty} U_k = X$ and function $V : X \to \mathbf{R}^+$ defined by $V(x) = V_k(x)$ for $x \in U_k$ satisfies $V(\tilde{x}) = V(x)$ whenever $\tilde{x} \in \mathcal{S}^x$, $x \in X$.

We proceed by induction on k. The starting case $k = 1$ is already settled. Suppose that w_m, U_m, and V_m are already constructed for some $m \geq 1$ in such a way that $V_m(\tilde{x}) = V_m(x)$ whenever $\tilde{x} \in \mathcal{S}^x$, $x \in \mathrm{cl}(U_m)$. By the construction, $z_\ell \notin \mathrm{cl}(U_m)$ and $B_{m+1} \subset \mathcal{I}^{z_\ell}$ for some $\ell = \ell(m) \in \{m+2, m+3, \ldots\}$. Let $w_{m+1} = z_\ell$ and let $U_{m+1} = \mathcal{I}^{w_{m+1}}$. Observe that $z_{m+1} \in \mathcal{I}^{z_{m+2}} \subset \mathcal{I}^{z_\ell} = U_{m+1}$. There exists a unique $T_{m+1} > 0$ for which $\Phi(T_{m+1}, w_{m+1}) \in \mathcal{S}^{w_m}$. Recall that $V_m(w_m) = m$. By letting

$$V_{m+1}(x) = \begin{cases} m + 1 - t/T_{m+1} & \text{if } x \in \mathcal{S}^{\Phi(t,w_{m+1})} \text{ for some } t \in [0, T_{m+1}] \\ V_m(x) & \text{if } x \in \mathrm{cl}(U_m), \end{cases}$$

a continuous function $V_{m+1} : \mathrm{cl}(U_{m+1}) \to [0, m+1]$ is defined. Continuity is an immediate consequence of the closedness of ordering $\leq_0$. It is also clear that $V_{m+1}(w_{m+1}) = m+1$ and $V_{m+1}(\tilde{x}) = V_{m+1}(x)$ whenever $\tilde{x} \in \mathcal{S}^x$, $x \in \mathrm{cl}(U_{m+1})$. Since $B_{m+1} \subset U_{m+1}$ for $m = 1, 2, \ldots$, the construction ends in a countably infinite number of steps and leads to the desired function V. □

The connectedness assumptions on X in Theorem 4 can be weakened to the same extent as they were weakened in Theorem 2.C to. In particular, Theorem 4 remains valid if X is replaced by a locally compact metric space $\mathcal{X}$ such that $\mathcal{X} \setminus \{x_0\}$ is connected and locally arcwise connected. Here of course $x_0 \in \mathcal{X}$ is assumed to be a globally asymptotically stable equilibrium.

Finally, we return to the sequence $w_1, w_2, \ldots$ constructed in the proof of Theorem 4. Suppose there exists such an index i_* that $\Phi(\mathbf{R}, w_{i_*}) \cap \mathcal{S}^x \neq \emptyset$ for each $x \in X \setminus \{0_X\}$. Then the construction of V can be finished in i_* steps. Moreover, starting with w_{i_*} directly, the number of construction steps reduces to *one*. Unfortunately, as it is demonstrated by a careful choice of the $\mathcal{S}^x$'s 'outside' Γ in Example 3, there exist nested total separations for which

$$\{w \in X \mid \Phi(\mathbf{R}, w) \cap \mathcal{S}^x \neq \emptyset \text{ for each } x \in X \setminus \{0_X\}\} = \emptyset.$$

This explains the role of the sequences $\{w_k\}_{k=1}^{\infty}$, $\{U_k\}_{k=1}^{\infty}$, and $\{V_k\}_{k=1}^{\infty}$ in the proof of Theorem 4.

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University of Technology
Department of Mathematics
Műegyetem rakpart 3–9
H-1521 Budapest, Hungary
e-mail: garay@math.bme.hu

L. Eötvös University
Department of Taxonomy and Ecology
Ludovika tér 2, H-1083 Budapest, Hungary
e-mail: garayj@ludens.elte.hu