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ALEXANDER'S CAPACITY FOR INTERSECTIONS OF ELLIPSOIDS IN $\mathbb{C}^N$

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Abstract. Alexander's projective capacity for some intersections of ellipsoids in $\mathbb{C}^N$ is computed.

1. Introduction. Let S be the unit sphere in $\mathbb{C}^N$. Let σ denote the Lebesgue surface area measure on S. Let

$$s_N := \int_S d\sigma.$$

Let $H_n = H_n(\mathbb{C}^N)$ denote the set of all homogeneous polynomials of degree n (with complex coefficients) in N complex variables.

Let K be a compact subset of $\mathbb{C}^N$. Let

$$\|f\|_K := \sup\{|f(z)| : z \in K\},$$

where $f : K \longrightarrow \mathbb{C}$ is a continuous function.

DEFINITION 1.1. (see [**1**], [**5**]). *Alexander's projective capacity* $\gamma(K)$ is

$$\gamma(K) := \lim_{n\to\infty}(\gamma_n(K))^{\frac{1}{n}} = \inf_n(\gamma_n(K))^{\frac{1}{n}},$$

where

$$\gamma_n(K) := \inf\{\|Q\|_K\},$$

the infimum being taken over all homogeneous polynomials $Q \in H_n$, normalized so that

$$\frac{1}{s_N}\int_S \log\left(|Q(z)|^{\frac{1}{n}}\right) d\sigma(z) = \kappa_N,$$

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where
$$\kappa_N := \frac{1}{s_N}\int_S \log|z_N|d\sigma(z).$$

It is known that
$$\kappa_N = -\frac{1}{2}\sum_{j=1}^{N-1}\frac{1}{j}.$$

Let $L_j > P_j > 0$ $(j = 1, \ldots, N)$.

Let $d_{ij} > 0$ $(i, j = 1, \ldots, N)$. For $i = 1, \ldots, N$ let
$$g_i(z) := \left(\sum_{j=1}^{N} d_{ij}|z_j|^2\right)^{\frac{1}{2}},$$
where $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$.

Let $E_i := \{z \in \mathbb{C}^N : g_i(z) \le 1\}$. Let $g(z) := \max\{g_i(z) : i = 1, \ldots, N\}$. Let $E := \{z \in \mathbb{C}^N : g(z) \le 1\}$. Obviously,
$$E = E_1 \cap \cdots \cap E_N. \tag{1.1}$$

In this paper we take $d_{ii} := L_i - P_i$ for $i = 1, \ldots, N$ and $d_{ij} := L_j$ for $i, j = 1, \ldots, N$ $(i \ne j)$. Then we compute Alexander's projective capacity of the set E.

2. Preliminaries. Let K be a compact subset of $\mathbb{C}^N$.

Definition 2.1. (see **[4]**, **[5]**). The *Siciak homogeneous extremal function* Ψ_K is defined as follows:
$$\Psi(z) = \Psi_K(z) := \lim_{n\to\infty}\left(\Psi_n(z)\right)^{\frac{1}{n}}, \qquad z \in \mathbb{C}^N,$$
where
$$\Psi_n(z) := \sup\{|Q(z)|\},$$
the supremum being taken over all $Q \in H_n$, normalized so that $\|Q\|_K = 1$.

The function $g(z)$ is a norm in $\mathbb{C}^N$ and therefore we have:

Proposition 2.2. (see **[4]**). *If the set E is given by* (1.1) *then*
$$\Psi_E(z) = g(z), \qquad z \in \mathbb{C}^N.$$

Definition 2.3. (see **[5]**, p. 53). The constant $\tau(K)$ is given by the formula:
$$\tau(K) := \exp\left(-\frac{1}{s_N}\int_S \log\Psi_K(z)\,d\sigma(z)\right).$$

Theorem 2.4. (see **[2]**). *If K is a compact subset of $\mathbb{C}^N$ then*
$$\gamma(K) = \exp(\kappa_N)\tau(K).$$

Let

$$D = D^{N-1} := \left\{ (r_1, \ldots, r_N) \in \mathbb{R}^N : r_1 \geq 0, \ldots, r_N \geq 0, \sum_{j=1}^{N} r_j^2 = 1 \right\},$$

$$\Sigma = \Sigma^{N-1} := \left\{ (\theta_1, \ldots, \theta_N) \in \mathbb{R}^N : \theta_1 \geq 0, \ldots, \theta_N \geq 0, \sum_{j=1}^{N} \theta_j = 1 \right\}.$$

Lemma 2.5. (see [**3**], Lemma 3.3). *If $f : D \longrightarrow \mathbb{R}$ is a continuous function then*

$$\frac{1}{s_N} \int_S f\left(|z_1|, \ldots, |z_N|\right) \, d\sigma(z) = \frac{1}{\operatorname{vol}(\Sigma)} \int_\Sigma f\left(\sqrt{\theta_1}, \ldots, \sqrt{\theta_N}\right) \, d\omega(\theta),$$

where ω is the Lebesgue surface area measure on the hyperplane

$$\left\{ (\theta_1, \ldots, \theta_N) \in \mathbb{R}^N : \sum_{j=1}^{N} \theta_j = 1 \right\}$$

and $\operatorname{vol}(\Sigma) := \int_\Sigma d\omega(\theta)$.

Lemma 2.6. (see [**3**], Lemma 3.6). *Let $a_j > 0$ for $j = 1, \ldots, N$. Let*

$$F(a_1, \ldots, a_N) := \frac{1}{2\pi i} \int_C \frac{z^{N-1} \operatorname{Log} z \, dz}{(z - a_1) \ldots (z - a_N)},$$

where C is any contour in the right half-plane enclosing all the points $a_1, \ldots, a_N$, and $\operatorname{Log} z$ is the principal branch of the logarithm. Then

$$\frac{1}{\operatorname{vol}\left(\Sigma^{N-1}\right)} \int_{\Sigma^{N-1}} \log \left(\sum_{j=1}^{N} a_j \theta_j \right) \, d\omega(\theta) = - \sum_{j=1}^{N-1} \frac{1}{j} + F(a_1, \ldots, a_N).$$

3. Main result. For $j = 1, \ldots, N$ let $L_j > P_j > 0$ and let $b_j := 1/P_j$, $c_N := \sum_{i=1}^{N} b_i$, $t_j := b_j / c_N$, $w := -1 + \sum_{i=1}^{N} L_i b_i$, $T := w / c_N$,

$$R_j := F(L_1, \ldots, L_{j-1}, T, L_{j+1}, \ldots, L_N).$$

Theorem 3.1. *If the set E is given by* (1.1) *then*

$$\log \gamma(E) = -\frac{1}{2} \sum_{j=1}^{N} t_j R_j.$$

PROOF. For $\theta \in \mathbb{R}^N$ let $u(\theta) := \sum_{j=1}^N L_j\theta_j$ and $v_i(\theta) := u(\theta) - P_i\theta_i$. We first observe that

$$\log\gamma(E) = \frac{1}{2}\left(-\sum_{j=1}^{N-1}\frac{1}{j} - \frac{1}{\mathrm{vol}(\Sigma)}\int_\Sigma \max(\log v_1(\theta),\dots,\log v_N(\theta))\,d\omega(\theta)\right). \tag{3.1}$$

Indeed, combining Theorem 2.4 with Proposition 2.2 and Lemma 2.5 gives

$$\begin{aligned}
\log\gamma(E) &= \kappa_N + \log\tau(E) = \kappa_N - \frac{1}{s_N}\int_S \log\Psi_E(z)\,d\sigma(z)\\
&= \kappa_N - \frac{1}{s_N}\int_S \log g(z)\,d\sigma(z)\\
&= -\frac{1}{2}\sum_{j=1}^{N-1}\frac{1}{j} - \frac{1}{2}\frac{1}{\mathrm{vol}(\Sigma)}\int_\Sigma \max(\log v_1(\theta),\dots,\log v_N(\theta))\,d\omega(\theta).
\end{aligned}$$

For $i = 1,\dots,N$ define

$$\begin{aligned}
\Lambda_i &:= \left\{\theta \in \Sigma^{N-1} : P_i\theta_i \le P_j\theta_j, j = 1,\dots,N\right\},\\
m_i &:= \frac{1}{\mathrm{vol}(\Sigma)}\int_{\Lambda_i} \max(\log v_1(\theta),\dots,\log v_N(\theta))\,d\omega(\theta)\\
&= \frac{1}{\mathrm{vol}(\Sigma)}\int_{\Lambda_i} \log v_i(\theta)\,d\omega(\theta).
\end{aligned}$$

Obviously,

$$\frac{1}{\mathrm{vol}(\Sigma)}\int_\Sigma \max(\log v_1(\theta),\dots,\log v_N(\theta))\,d\omega(\theta) = \sum_{i=1}^N m_i. \tag{3.2}$$

We next show that

$$m_i = t_i\left(-\sum_{j=1}^{N-1}\frac{1}{j} + R_i\right). \tag{3.3}$$

Without loss of generality we can assume that $i = 1$. Clearly,

$$m_1 = \frac{1}{\mathrm{vol}(\Sigma)}\int_\Lambda \log v_1(\theta)\,d\omega(\theta), \tag{3.4}$$

where $\Lambda := \Lambda_1$.

An analysis similar to that in the proof of Theorem 3.2 ([**3**], p. 257) shows that

$$
\begin{aligned}
&\frac{1}{\operatorname{vol}(\Sigma)} \int_\Lambda \log v_1(\theta)\, d\omega(\theta) \\
&= t_1 \frac{1}{\operatorname{vol}(\Sigma)} \int_\Sigma \log \left(T\eta_1 + \sum_{j=2}^{N} L_j \eta_j \right) d\omega(\eta).
\end{aligned}
\tag{3.5}
$$

Indeed, we change the variables:

$$
\begin{aligned}
\theta_1 &= t_1 \eta_1, \\
\theta_j &= t_j \eta_1 + \eta_j
\end{aligned}
$$

for $j = 2, \ldots, N$. Let us observe that the simplex Λ has the vertices: $(t_1, t_2, \ldots, t_N)$, $(0, 1, 0, \ldots, 0)$, $(0, 0, 1, \ldots, 0), \ldots, (0, 0, 0, \ldots, 1)$.

It is easy to see that $\eta_1, \ldots, \eta_N$ are the barycentric coordinates on Λ.

Applying (3.4), (3.5) and Lemma 2.6 we get

$$
\begin{aligned}
m_1 &= t_1 \left(- \sum_{j=1}^{N-1} \frac{1}{j} + F(T, L_2, L_3, \ldots, L_N) \right) \\
&= t_1 \left(- \sum_{j=1}^{N-1} \frac{1}{j} + R_1 \right).
\end{aligned}
$$

Now (3.3) is proved for $i = 1$. In the same manner we can show that (3.3) is true for $i = 2, \ldots, N$.

Combining (3.1) with (3.2) and (3.3) we get

$$
\begin{aligned}
\log \gamma(E) &= \frac{1}{2} \left(- \sum_{j=1}^{N-1} \frac{1}{j} - \sum_{i=1}^{N} m_i \right) \\
&= \frac{1}{2} \left(- \sum_{j=1}^{N-1} \frac{1}{j} + \sum_{j=1}^{N-1} \frac{1}{j} - \sum_{i=1}^{N} t_i R_i \right) \\
&= -\frac{1}{2} \sum_{i=1}^{N} t_i R_i,
\end{aligned}
$$

which completes the proof.

□

Remark 3.2. Now suppose that $P_j = 0$ for at least one j . Then the set E is an ellipsoid:

$$E = \left\{(z_1, \dots, z_N) \in \mathbb{C}^N : L_1|z_1|^2 + \dots + L_N|z_N|^2 \le 1\right\}.$$

Therefore in this case (see [**3**], Theorem 3.1):

$$\log \gamma(E) = -\frac{1}{2} F(L_1, \dots, L_N).$$

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