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SEQUENCES OF FIXED POINT INDICES OF ITERATIONS IN DIMENSION 2

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Abstract. Let $\mathrm{ind}(f,0)$ be the local fixed point index at 0. We show that every sequence of integers which satisfies Dold relations can be realized as $\{\mathrm{ind}(f^n,0)\}_{n=1}^{\infty}$, where f is a continuous self-map of a 2–dimensional disk D^2.

1. Introduction. Let f be a continuous self-map of a compact ANR X. There are some restrictions on a sequence of fixed point indices $\{\mathrm{ind}(f^n,X)\}_{n=1}^{\infty}$: it must satisfy congruences established by A. Dold (cf. [**6**]), called Dold relations. Additional assumptions on f or X may give stronger bounds on the shape of $\{\mathrm{ind}(f^n,X)\}_{n=1}^{\infty}$. For example, if $f:\mathbb{R}^k\to\mathbb{R}^k$ is a C^1–map and x_0 is a fixed point of f, then the sequence $\{\mathrm{ind}(f^n,x_0)\}_{n=1}^{\infty}$ is periodic (cf. [**8**], [**5**]). Babenko and Bogatyi proved that every sequence of integers that satisfies Dold relations can be obtained as $\{\mathrm{ind}(f^n,0)\}_{n=1}^{\infty}$, where $f:D^3\to D^3$ is a homeomorphism and D^3 is a unit disk in $\mathbb{R}^3$ (cf. [**1**]). The realization by a homeomorphism is impossible if we replace D^3 by a 2–dimensional unit disk D^2, which is a consequence of the fact that in this case $\{\mathrm{ind}(f^n,0)\}_{n=1}^{\infty}$ takes no more then three values (cf. [**2**], [**7**], [**3**]). The question whether the realization is possible for a continuous self-map of D^2 was asked by Babenko and Bogatyi in [**1**]. This note gives the positive answer to that problem by extending the construction from [**1**], which was performed for some particular classes of sequences.

2. Dold relations and Dold coefficients. Let $f:U\to\mathbb{R}^2$, where U is an open subset of $\mathbb{R}^2$, be a continuous map such that for each integer $n>0$,

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the center 0 is an isolated fixed point for f^n. In this case the fixed point index $\mathrm{ind}(f^n, 0)$ is well defined for f^n restricted to a small neighborhood of 0.

Let us recall the definition of the Möbius function:

DEFINITION 2.1. The Möbius function $\mu : \mathbb{N} \to \mathbb{Z}$ is defined by these three properties:

$$\mu(1) = 1,$$
$$\mu(k) = (-1)^r \text{ if } k \text{ is a product of } r \text{ different primes},$$
$$\mu(k) = 0 \text{ otherwise}.$$

DEFINITION 2.2. For each natural n we define an integer $i_n(f, 0)$ by the equality:

$$i_n(f, 0) = \sum_{k|n} \mu(n/k)\mathrm{ind}(f^k, 0).$$

The following congruences were found by A. Dold (cf. [**6**]):

THEOREM 2.3 (Dold relations). *For every $n \in \mathbb{N}$, there is:*

$$i_n(f) \equiv 0 \pmod{n}.$$

We will call the numbers $A_n(f, 0) = \frac{1}{n} i_n(f, 0)$ Dold coefficients. By the Möbius inversion formula (cf. [**4**]) we obtain:

$$\mathrm{ind}(f^n, 0) = \sum_{k|n} kA_k(f, 0).$$

Let us notice that another way of expressing Dold relations is to say that each $A_n(f, 0)$ is an integer. As we have a one-to-one correspondence between elements of $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$ and $\{A_n(f, 0)\}_{n=1}^{\infty}$ we may reformulate the basic question of realization in the terms of Dold coefficients.

Instead of asking whether for a given sequence of integers $\{c_n\}_{n=1}^{\infty}$, with $n | \sum_{k|n} \mu(n/k)c_k$ (Dold relation must be satisfied) there exists a map f such that $c_n = \mathrm{ind}(f^n, 0)$ for each n, we may ask whether for a given arbitrary sequence of integers $\{b_n\}_{n=1}^{\infty}$ there is a map f such that $b_n = A_n(f, 0)$ for each n.

LEMMA 2.4. (cf. [**1**]) *Let $f : X \to X$, $g : Y \to Y$ be continuous maps of absolute neighbourhood retracts X, Y with isolated fixed points p and q, respectively. Then for the fixed point (p,q) of the map: $f \vee g : X \vee Y \to X \vee Y$, where $X \vee Y$ is a fan (bouquet) of spaces X and Y, there is:*

$$\mathrm{ind}(f \vee g, (p, q)) = \mathrm{ind}(f, p) + \mathrm{ind}(g, q) - 1,$$
$$A_1(f \vee g, (p, q)) = A_1(f, p) + A_1(g, q) - 1,$$
$$A_n(f \vee g, (p, q)) = A_n(f, p) + A_n(g, q),$$

for $n > 1$.

3. The realization of an arbitrary sequence of integers. Let A_d be an arbitrary integer, $d > 1$. Consider two cases: (1) $A_d > 0$. Let T_d be a fan (see an example of T_3 in Picture 1) composed of d isometric isosceles triangles with one common vertex at 0. Define a map $f_d : T_d \to T_d$ as the composition $f_d = g_d\theta_d$, where θ_d is a preserving orientation isometry which maps each triangle in counter-clockwise order on the nearest one ($\theta_d^d = id$), g_d is equal to g on each triangle of the fan, where g is time-one map of the flow given in Picture 3a ($\mathrm{ind}(g, 0) = 2$).

By Lemma 2.4 and the definition of fixed point index, we obtain:

$$\mathrm{ind}(f_d^n, 0) = \begin{cases} 1 & \text{if } d \nmid n, \\ 1 + d & \text{if } d | n. \end{cases}$$

It easy to check, using Definition 2.2, that in terms of Dold coefficients this is equivalent to:

$$A_n(f_d, 0) = \begin{cases} 1 & \text{if } n = 1 \text{ or } n = d, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.1. *We may look at the above construction as at a kind of inductive procedure. We have 0, the fixed point itself; it has index equal to 1. Then, every time we add a single triangle from the family T_d with $\mathrm{ind}(g, 0) = 2$, we must, by Lemma 2.4, add the number (2-1) to index of f_d^n, where $d|n$.*

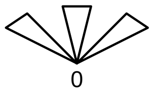
Now we take a fan $T_{d,A_d} = \bigvee_{d=1}^{A_d} T_d$ which consists of A_d copies of T_d, with $f_{d,A_d} = \bigvee_{d=1}^{A_d} f_d : T_{d,A_d} \to T_{d,A_d}$. Then, again by Lemma 2.4, there is:

$$A_n(f_{d,A_d}, 0) = \begin{cases} 1 & \text{if } n = 1, \\ A_d & \text{if } n = d, \\ 0 & \text{otherwise.} \end{cases}$$

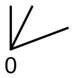
(2) $A_d \le 0$. We repeat the construction from [**1**], in which segments are used instead of triangles. Let I_d be a fan composed of d segments of the same length with 0 as their only common point (see an example of I_3 in Picture 2). Define $f_d : I_d \to I_d$ as the composition $f_d = h_d\theta_d$, where θ_d isometrically maps each segment on the nearest one in the counter-clockwise order, h_d is equal to h on each segment, where h is time-one map of the flow given in Picture 4a ($\mathrm{ind}(h, 0) = 0$). Then, by Lemma 2.4:

$$\mathrm{ind}(f_d^n, 0) = \begin{cases} 1 & \text{if } d \nmid n, \\ 1 - d & \text{if } d | n. \end{cases}$$

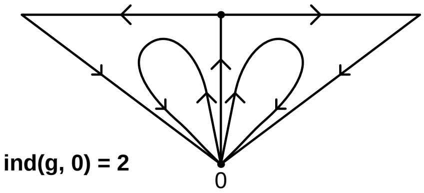
Pic. 1



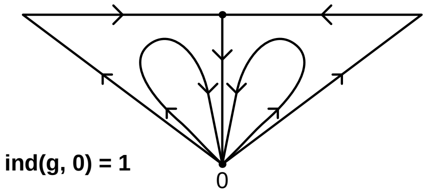
Pic. 2



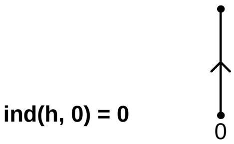
Pic. 3a



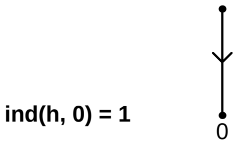
Pic. 3b



Pic. 4a



Pic. 4b



$$A_n(f_d,0)=\begin{cases}1 & \text{if } n=1,\\ -1 & \text{if } n=d,\\ 0 & \text{otherwise.}\end{cases}$$

Now we take a fan $I_{d,A_d} = \bigvee_{d=1}^{|A_d|} I_d$ which consists of $|A_d|$ copies of I_d ($I_{d,0}=\{0\}$), with $f_{d,A_d}=\bigvee_{d=1}^{|A_d|} f_d : I_{d,A_d} \to I_{d,A_d}$. Then, again by Lemma 2.4, there is:

$$A_n(f_{d,A_d},0)=\begin{cases}1 & \text{if } n=1,\\ A_d & \text{if } n=d,\\ 0 & \text{otherwise.}\end{cases}$$

For $d=1$, we introduce a slight change: if $A_1>0$, then we define T_{1,A_1} as a fan composed of A_1-1 triangles, if $A_1\le 0$, then we define I_{1,A_1} as composed of $|A_1|+1$ segments, in both cases taking f_{1,A_d} in the same way as for $d\neq 1$.

Remark 3.2. *Notice that if we replace the map g and h by the time-one map of the flow in Pictures 3b and 4b, respectively, then all Dold coefficients* *for $n > 1$ will equal zero, $A_n(f_d, 0) = 0$. It is a consequence of the fact that an addition of another triangle or segment with* $\mathrm{ind}(g, 0) = 1$ *or* $\mathrm{ind}(h, 0) = 1$ *does not change the index of f_d^n. By Lemma 2.4, it will be constant and equal to 1, so the inductive procedure of enlarging the index, described in Remark 3.1, does not work.*

Now assume that an arbitrary sequence of integers $\{A_d\}_{d=1}^{\infty}$ is given. For each A_d we choose an appropriate X_{d,A_d}, where $X_{d,A_d} \in \{T_{d,A_d}, I_{d,A_d}\}$ in such a way that it is placed in the planar sector defined in polar coordinates ϕ, ρ by: $|\phi - \frac{\pi}{2d}| < \frac{\pi}{8d^3}$, $\rho < \frac{1}{d}$ and the height from the corner 0 of each T_d or the length of each I_d is equal to $\frac{1}{2d}$.

Let us take $f_\infty = \bigvee_{d=1}^{\infty} f_{d,A_d}$ which is a self-map of $K_\infty = \bigvee_{d=1}^{\infty} X_{d,A_d}$.

Let us state the following well-known fact:

Lemma 3.3. *Let X be a compact AR, $F : X \to X$, continuous. If p is the only fixed point of F, then* $\mathrm{ind}(F, p) = 1$.

For an arbitrary given n we represent f_∞ as $F_n \vee G_{n+1}$, where $F_n = \bigvee_{d=1}^{n} f_{d,A_d}$ is a self-map of $\bigvee_{d=1}^{n} X_{d,A_d}$, $G_{n+1} = \bigvee_{d=n+1}^{\infty} f_{d,A_d}$ is a self-map of $\bigvee_{d=n+1}^{\infty} X_{d,A_d}$. Notice that $\mathrm{Fix}\,(G_{n+1}^k) = \{0\}$ for each $k \leq n$, so by Lemma 3.3 we get that $\mathrm{ind}(G_{n+1}^k, 0) = 1$ for $k \leq n$. As a consequence we obtain that $A_1(G_{n+1}, 0) = 1$, $A_k(G_{n+1}, 0) = 0$ for each $1 < k \leq n$. By Lemma 2.4 we have: $A_k(f_\infty, 0) = A_k(F_n, 0) + A_k(G_{n+1}, 0)$ for $k > 1$ and $A_1(f_\infty, 0) = A_1(F_n, 0) + A_1(G_{n+1}, 0) - 1$, thus $A_k(f_\infty, 0) = A_k(F_n, 0)$ for $k \leq n$. Applying again Lemma 2.4 we get: $A_k(F_n, 0) = A_k$ for $k \leq n$.

Finally we define $f : D^2 \to D^2$ by $f = f_\infty r$, where $r : D^2 \to K_\infty$ is retraction and get that $A_k(f, 0) = A_k$, for $k = 1, 2, 3, \ldots$. This completes the construction.

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