UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLII
2004

SYMPLECTIC FIELD THEORY APPROACH TO STUDYING ERGODIC MEASURES RELATED WITH NONAUTONOMOUS HAMILTONIAN SYSTEMS

by Anatoliy K. Prykarpatsky

Abstract. An approach to studying ergodic properties of time-dependent periodic Hamiltonian flows on symplectic metric manifolds is developed. Such flows have applications in mechanics and mathematical physics. Based both on J. Mather's results [**9**] about homology of probability invariant measures minimizing some Lagrangian functionals and on the symplectic field theory devised by A. Floer and others [**3**]–[**8**],[**12, 15**] for investigating symplectic actions and Lagrangian submanifold intersections, an analog of Mather's β–function is constructed subject to a Hamiltonian flow reduced invariantly upon some compact neighborhood of a Lagrangian submanifold. Some results on stable and unstable manifolds to hyperbolic periodic orbits having applications in the theory of adiabatic invariants of slowly perturbed integrable Hamiltonian systems are stated within the Gromov-Salamon-Zehnder [**3, 5, 12**] elliptic techniques in symplectic geometry.

Introduction. The past years have given rise to several exciting developments in the field of symplectic geometry and dynamical systems [**3**]–[**12**], which introduced new mathematical tools and concepts suitable for solving numerous problems wich were earlier intractable. When studying periodic solutions to non-autonomous Hamiltonian systems, Salamon & Zehnder [**3**] developed a proper Morse theory for infinite dimensional loop manifolds based on previous results on symplectic geometry of Lagrangian submanifolds of Floer [**4, 6**]. Investigating at the same time ergodic measures related with Lagrangian dynamical systems on tangent spaces to configuration manifolds, Mather [**9**]

1991 *Mathematics Subject Classification.* 37A05, 37B35, 37C40, 37C60, 37J10, 37J40, 37J45.

Key words and phrases. Ergodic measures, holonomy groups, dynamical systems, quasi-complex structures, symplectic field theory.

devised a new approach to studying the correspondingly related invariant probabilistic measures based on a so called β–function. The latter made it possible to describe effectively the so called homology of these invariant probabilistic measures minimizing the corresponding Lagrangian action functional.

As one can easily see, Mather's approach does not allow any direct application to the problem of describing the ergodic measures naturally related to a given periodic non-autonomous Hamiltonian system on a closed symplectic space. Thereby, to overcome constraints to this task in the present work we suggest some new way to imbedding the non-autonomous Hamiltonian case into the Mather β–function theory picture, making use of the mentioned above Salamon & Zehneder and Floer [**3, 4, 6**] loop space homology structures. Further, the Gromov elliptic techniques in symplectic geometry make it possible to construct the invariant submanifolds of our Hamiltonian system, naturally related to the corresponding compact Lagrangian submanifolds, and a β–function analog on them.

1. Symplectic and analytic problem setting. Let $(M^{2n},\omega^{(2)})$ be a closed symplectic manifold of dimension $2n$ with a symplectic structure $\omega^{(2)} \in \Lambda(M^{2n})$ being weakly exact, that is $\omega^{(2)}(\pi_2(M^{2n})) = 0$. Every smooth enough, time-dependent 2π–periodic function $H : M^{2n} \times \mathbb{S}^1 \to \mathbb{R}$ gives rise to the non-autonomous vector field $X_H : M^{2n} \times \mathbb{S}^1 \to T(M^{2n})$ defined by the equality

$$i_{X_H}\omega^{(2)} = -dH, \tag{1}$$

where, as usual, (see [**1, 2**]), the operation "i_{X_H}" denotes the intrinsic derivation of the Grassmann algebra $\Lambda(M^{2n})$ along the vector field X_H. The corresponding flow on $M^{2n} \times \mathbb{S}^1$ takes the form:

$$du/ds = X_H(u;t), \quad dt/ds = 1, \tag{2}$$

where $u : \mathbb{R} \to M^{2n}$ is an orbit, $t \in \mathbb{R}/2\pi\mathbb{Z} \simeq \mathbb{S}^1$ and $s \in \mathbb{R}$ is an evolution parameter. We shall assume that solutions to (2) are complete and determine a one-parametric ψ–flow of diffeomorphisms $\psi^s : M^{2n} \times \mathbb{S}^1 \to M^{2n} \times \mathbb{S}^1$ for all $s \in \mathbb{R}$ which, due to (1), are evidently symplectic, that is $\psi_{t_0}^{s*}\omega^{(2)} = \omega^{(2)}$ where $\psi_{t_0}^s := \psi^s|_{M^{2n}}$ at any fixed $t_0 \in \mathbb{R}/2\pi\mathbb{Z} \simeq \mathbb{S}^1$. Now take an $(n+1)$–dimensional submanifold $\mathcal{L}^{n+1} \subset M^{2n} \times \mathbb{R}$ such that for any closed contractible curve γ with $\gamma \subset \mathcal{L}^{n+1}$ the following integral equality

$$\oint_\gamma (\alpha^{(1)} - H(s)ds) = 0 \tag{3}$$

holds, where $\alpha^{(1)} \in \Lambda^1(M^{2n})$ is such a 1–form on M^{2n} which satisfies the condition $\int_{D^2}(\omega^{(2)} - d\alpha^{(1)}) = 0$ for any compact two–dimensional disk $D^2 \subset M^{2n}$ due to the weak exactness of the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$

and existing globally on $\mathcal{L}^{n+1}$ due to Floer results [**4, 6**]. Assume now also that for the flow of symplectomorphisms $\psi^s_{t_0} : M^{2n} \to M^{2n}$, $s \in \mathbb{R}$, the condition

$$\{(\psi^s_{t_0}\mathcal{L}^n_{t_0}, t_0 + s) : s \in \mathbb{R}\} \subset \mathcal{L}^{n+1} \tag{4}$$

holds for some compact Lagrangian submanifold $\mathcal{L}^n_{t_0} \subset M^{2n}$ upon which $\omega^{(2)}|_{\mathcal{L}^n_{t_0}} = 0$. Condition (4) in particular means [**2**] that the following expression

$$\alpha^{(1)} - H(s)ds = d\mathcal{A}(s), \tag{5}$$

$s \in \mathbb{R}$, holds in some vicinity of the Lagrangian submanifold $\mathcal{L}^n_{t_0} \subset M^{2n}$, where a mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ is the so called [**1, 2**] generating function for the defined above continuous set of diffeomorphisms $\psi^s_{t_0} \in$Diff(M^{2n}), $s \in \mathbb{R}$. Expression (5) makes it possible to define naturally the following Poincare-Cartan type functional on a set of almost everywhere differentiable curves $\gamma : [0, \tau] \to M^{\ 2n} \times \mathbb{S}^1$

$$\mathcal{A}^{(\tau)}_{t_0}(\gamma) := \frac{1}{\tau}\int_\gamma (\alpha^{(1)} - H(s)ds), \tag{6}$$

with end points $\{\ \gamma(\tau) = \psi^\tau(\gamma(0))\ \}$, *supp* $\gamma \subset \mathcal{U}(\mathcal{L}^n_{t_0}) \times \mathbb{S}^1$ for all $\tau \in \mathbb{R}$ and $\mathcal{U}(\mathcal{L}^n_{t_0})$ is some compact neighborhood of the Lagrangian submanifold $\mathcal{L}^n_{t_0} \subset M^{2n}$ satisfying the condition $\psi^s_{t_0}\mathcal{U}(\mathcal{L}^n_{t_0}) \subset \mathcal{U}(\mathcal{L}^n_{t_0})$ for all $s \in \mathbb{R}$.

Let us denote by $\Sigma_{t_0}(H)$ the subset of curves γ with support in $\mathcal{U}(\mathcal{L}^n_{t_0}) \times \mathbb{S}^1$ and fixed end-points as before minimizing the functional (6). If the infimum is realized, one easily shows that any such curve $\gamma \in \Sigma_{t_0}(H)$ solves the system (2). For the above set of curves $\Sigma_{t_0}(H)$ to be specified more suitably, choose, following Floer's ideas [**3**]–[**8**],[**12**], an almost complex structure $J : M^{2n} \to End(T(M^{2n}))$ on the symplectic manifold M^{2n}, where by definition $J^2 = -I$, compatible with the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$. Then the expression

$$< \xi, \eta >:= \omega^{(2)}(\xi, J\eta), \tag{7}$$

where $\xi, \eta \in T(M^{2n})$, naturally defines a Riemannian metric on M^{2n}. Subject to the metric (7) our Hamiltonian vector field $X_H : M^{2n} \times \mathbb{S}^1 \to T(M^{2n})$ is now represented as $X_H = J\nabla H$, where $\nabla : \mathcal{D}(M^{2n}) \to T(M^{2n})$ denotes the usual gradient mapping with respect to this metric.

Consider now the space $\Omega := \Omega(M^{2n} \times \mathbb{S}^1)$ of all continuous curves in $M^{2n} \times \mathbb{S}^1$ with fixed end-points. Then one can similarly define the gradient mapping *grad* $\mathcal{A}^{(\tau)}_{t_0}(\gamma) : \Omega \to T(\Omega)$ as follows:

$$(grad\ \mathcal{A}^{(\tau)}_{t_0}(\gamma), \xi) := \frac{1}{\tau}\int_0^\tau ds < J(\gamma_{t_0})\dot{\gamma}_{t_0}(s) + \nabla H(\gamma_{t_0}; s + t_0), \xi >, \tag{8}$$

where $\gamma = \{(\gamma_{t_0}(s); t_0 + s (\mathrm{mod}\, 2\pi)) : s \in [0, \tau]\} \in \Omega$ as before, and $\xi \in T(\Omega)$. Since all critical curves $\gamma \in \Sigma_{t_0}(H)$ minimizing the functional (6) solve (2), this fact suggests a way to construct an invariant subset $\Omega_H \subset \Omega$ such that $\Omega_H := \Omega(\mathcal{U}(\mathcal{L}^n_{t_0}) \times \mathbb{S}^1)$. Namely, define a curve $\gamma \in \Omega_H(\gamma^{(-)}) \subset \Omega_H$ as satisfying [**3**] the following gradient flow in $\mathcal{U}(\mathcal{L}^n_{t_0}) \times \mathbb{S}^1$:

$$\partial\gamma_{t_0}/\partial z = -grad\ \mathcal{A}^{(\tau)}_{t_0}(\gamma), \qquad \partial t/\partial z = 0 \tag{9}$$

for all $z \in \mathbb{R}$ and any $\tau \in \mathbb{R}$ under the asymptotic conditions

$$\lim_{z\to-\infty} \gamma_{t_0}(s; z) = \gamma^{(-)}_{t_0}(s), \quad \lim_{z\to\infty} \gamma_{t_0}(s; z) = \gamma_{t_0}(s) \tag{10}$$

with the corresponding curves $\gamma^{(-)}_{t_0}$, $\gamma_{t_0} : \mathbb{R} \to M^{2n}$ satisfying system (2), and moreover, with the curve $\gamma^{(-)}_{t_0} : \mathbb{R} \to M^{2n}$ being taken to be hyperbolic [**1, 2**] with $supp\ \gamma^{(-)}_{t_0} \subset \mathcal{L}^n_{t_0}$. Now we can construct a so called [**1**] unstable manifold $W^u(\gamma^{(-)}_{t_0})$ to this hyperbolic curve $\gamma^{(-)}_{t_0}$ defined for all $\tau \in \mathbb{R}$, varying smoothly a $supp\ \gamma^{(-)}_{t_0} \subset \mathcal{L}^n_{t_0}$. Thus, due to the above construction, the functional manifold $W^u(\gamma^{(-)}_{t_0})$, if subject to weak Whitney topology it is compact, can be imbedded as a point compact submanifold into M^{2n}, thereby interpreting supports of all curves solving (9) and (10) where $supp\ \gamma_{t_0} \subset \mathcal{L}^n_{t_0}$, as a compact neighborhood $\mathcal{L}^{(-)}_{t_0}(H) \subset \mathcal{U}(\mathcal{L}^n_{t_0})$ of the compact Lagrangian submanifold $\mathcal{L}^n_{t_0} \subset M^{2n}$ looked for above.

The same construction can evidently be made in the case of conditions (10) replaced with either

$$\lim_{z\to+\infty} \gamma_{t_0}(s; z) = \gamma^{(+)}_{t_0}(s), \quad \lim_{z\to-\infty} \gamma_{t_0}(s; z) = \gamma_{t_0}(s), \tag{10a}$$

or

$$\lim_{z\to-\infty} \gamma_{t_0}(s; z) = \gamma^{(-)}_{t_0}(s), \quad \lim_{z\to\infty} \gamma_{t_0}(s; z) = \gamma^{(+)}_{t_0}(s), \tag{10b}$$

where $\gamma^{(-)}_{t_0} : \mathbb{R} \to M^{2n}$ and $\gamma^{(+)}_{t_0} : \mathbb{R} \to M^{2n}$ are some strictly different hyperbolic curves on M^{2n} with $supp\ \gamma^{(\pm)}_{t_0} \subset \mathcal{L}^n_{t_0}$ and solving (2). Based on (10a) one similarly constructs the stable manifold $W^s(\gamma^{(+)}_{t_0}(s))$ to a hyperbolic curve $\gamma^{(+)}_{t_0}$ and further the corresponding compact neighborhood $\mathcal{L}^{(+)}_{t_0}(H) \subset \mathcal{U}(\mathcal{L}^n_{t_0})$ of the compact Lagrangian submanifold $\mathcal{L}^n_{t_0} \subset M^{2n}$ which is of crucial importance when studying intersection properties of stable $W^s(\gamma^{(+)}_{t_0})$ and unstable $W^u(\gamma^{(-)}_{t_0})$ manifolds. Similarly based on (10b), one constructs the neighborhood $\mathcal{L}_{t_0}(H) \subset \mathcal{U}(\mathcal{L}^n_{t_0})$ of the compact Lagrangian submanifold $\mathcal{L}^n_{t_0} \subset M^{2n}$ being of interest when investigating so called adiabatic perturbations of integrable autonomous Hamiltonian flows on the symplectic manifold M^{2n}.

Now we make use of some statements [**3, 5, 12**] about the properties of the set Ω_H constructed above. For a generic choice of the Hamiltonian function $H : M^{2n} \times \mathbb{S}^1 \to \mathbb{R}$, the functional space of curves Ω_H is proved to be finite–dimensional and compact which instantly gives rise to hereditary finite–dimensionality of the compact neighborhood $\mathcal{L}_{t_0}^{(-)}(H)$ with the compact manifold structure. To see this, linearize equation (9) in the direction of a vector field $\xi \in T(\Omega_H)$. This leads to the linearized first-order differential operator :

$$F_{t_0}(u)\xi := \nabla_z\xi + J(u)\nabla_s\xi + \nabla_\xi J(u)\partial u/\partial s + \nabla_\xi \nabla H(u; t_0 + s), \tag{11}$$

where an element $u \in \Omega_H|_{M^{2n}}$ is bounded and satisfies the following equation stemming from (1.9):

$$\partial u/\partial z + J(u)\partial u/\partial s + \nabla H(u; s + t_0) = 0 \tag{12}$$

and ∇_z, ∇_s and ∇_ξ denote here the corresponding covariant derivatives with respect to metric (7) on M^{2n}. If $u \in \Omega_H$ satisfies (12), a curve γ_{t_0} in M^{2n} has *supp* $\gamma_{t_0} \subset \mathcal{L}_{t_0}^n$ and a curve $\gamma_{t_0}^{(-)}$ in $\mathcal{L}_{t_0}^n$ is hyperbolic and nondegenerate [**3**], then the operator $F_{t_0}(u) : T(\Omega_H) \to T(\Omega_H)$ defined by (12) is a Fredholm operator [**12**] between appropriate Sobolev spaces. The corresponding pair (H, J) with $J : M^{2n} \to End(T(M^{2n}))$ satisfying (7) is called regular [**3**] if every hyperbolic solution to (2) is nondegenerate [**1, 3**] and the operator $F_{t_0}(u)$ is onto for $u \in \Omega_H$. In general one can prove that the space $(\mathcal{H}, \mathcal{J})_{reg} \subset (\mathcal{H}, \mathcal{J})$ of regular pairs $(H, J) \in (\mathcal{H}, \mathcal{J})$ is dense with respect to the C^∞–topology. Thus, for the regular pairs, it follows from an implicit function theorem [**1**] that the space $\Omega_H(\gamma_{t_0}^{(-)})$ is indeed, for any curve γ_{t_0} with *supp* $\gamma_{t_0} \subset \mathcal{L}_{t_0}^n$, a finite–dimensional compact functional submanifold whose local dimension near $u \in \Omega_H(\gamma_{t_0}^{(-)})$ is exactly the Fredholm index of the operator $F_{t_0}(u)$. As a simple inference from the finite–dimensionality and compactness of the set $\Omega_H(\gamma_{t_0}^{(-)})$, there follows compactness one gets that the corresponding point set $\mathcal{L}_{t_0}^{(-)}(H)$ is a finite–dimensional and compact submanifold smoothly imbedded into M^{2n}. The same is evidently true for the point manifolds $\mathcal{L}_{t_0}^{(+)}(H)$ and $\mathcal{L}_{t_0}(H)$ supplying us with compact neighborhoods of the compact Lagrangian submanifold $\mathcal{L}_{t_0}^n \subset M^{2n}$. Let us specify the structure of the manifold $\mathcal{L}_{t_0}^{(-)}(H)$ more exactly making use of the Floer type analytical results [**3, 8, 12**] about the space of solutions to problem (9)–(10). There follows that for any two curves $\gamma^{(-)}, \gamma : [0, \tau] \to \mathcal{L}_{t_0}^n \times \mathbb{S}^1$ satisfying system (2), the following functional

$$\Phi_{t_0}^{(\tau)}(u) := \frac{1}{\tau}\int_0^\tau ds \int_{\mathbb{R}} dz(|\partial u/\partial z|^2 + |\partial u/\partial s - X_H(u; s + t_0)|^2), \tag{13}$$

if bounded, satisfies the characteristic equality

$$\Phi_{t_0}^{(\tau)}(u) = \mathcal{A}_{t_0}^{(\tau)}(\gamma^{(-)}) - \mathcal{A}_{t_0}^{(\tau)}(\gamma) \tag{14}$$

for any $\tau \in \mathbb{R}$. Thereby, in the case of a nonvanishing right hand side of (14), the functional space $\Omega_H(\gamma^{(-)})$ will not be *a priori* nontrivial. Similarly, for any bounded $u \in \mathcal{L}_{t_0}^{(+)}(H)$, one finds that

$$\Phi_{t_0}^{(\tau)}(u) = \mathcal{A}_{t_0}^{(\tau)}(\gamma) - \mathcal{A}_{t_0}^{(\tau)}(\gamma^{(+)}), \tag{14a}$$

where the corresponding curve $\gamma_{t_0}^{(+)} : [0,\tau] \to M^{2n}$ satisfies system (2), is hyperbolic having *supp* $\gamma_{t_0}^{(+)} \subset \mathcal{L}_{t_0}^n$, and the curve $\gamma_{t_0} : [0,\tau] \to M^{2n}$ also satisfies system (2), having *supp* $\gamma_{t_0} \subset \mathcal{L}_{t_0}^n$, and finally, for $u \in \mathcal{L}_{t_0}(H)$

$$\Phi_{t_0}^{(\tau)}(u) = \mathcal{A}_{t_0}^{(\tau)}(\gamma^{(-)}) - \mathcal{A}_{t_0}^{(\tau)}(\gamma^{(+)}), \tag{14b}$$

where $\gamma^{(\pm)} : [0,\tau] \to M^{2n} \times \mathbb{S}^1$, $\tau \in \mathbb{R}$, are taken to be strictly different, hyperbolic and with varying supports *supp* $\gamma^{(\pm)} \subset \mathcal{L}_{t_0}^n$. The case of $\gamma_{t_0}^{(+)} = \gamma_{t_0}^{(-)}$ needs some modification of the construction presented above; we shall not dwell on this here. Thus we have constructed the corresponding neighborhoods $\mathcal{L}_{t_0}^{(\pm)}(H)$ and $\mathcal{L}_{t_0}(H)$ of the compact Lagrangian submanifold $\mathcal{L}_{t_0}^n \subset M^{2n}$ consisting of all bounded solutions to the corresponding equations (9), (10) and (10a,b). Now based on this fact and analytical expressions (14) and (14a,b) the following important lemma may be proved.

Lemma 1.1. *All neighborhoods $\mathcal{L}_{t_0}^{(\pm)}(H)$ and $\mathcal{L}_{t_0}(H)$ constructed via the scheme presented above are compact and invariant with respect to the Hamiltonian flow of diffeomorphisms $\psi^s \in$Diff$(M^{2n}) \times \mathbb{S}^1$, $s \in \mathbb{R}$.*

Let us now consider the case of the neighborhood $\mathcal{L}_{t_0}(H) \subset M^{2n}$. The preceding characterization of the space of curves Ω_H leads us, using Mather's approach [**9**], to another description important in applications of the compact neighborhood $\mathcal{L}_{t_0}(H)$, by means of the space of normalized probability measures $\mathcal{M}_{t_0}(H) := \mathcal{M}(T(\mathcal{L}_{t_0}(H)) \times \mathbb{S})$ with compact support and invariant with respect to our Hamiltonian ψ–flow of diffeomorphisms $\psi^s \in$Diff$(M^{2n}) \times \mathbb{S}^1$, $s \in \mathbb{R}$, naturally extended on $T(\mathcal{L}_{t_0}(H)) \times \mathbb{S}$. Due to Lemma 1.1 the Hamiltonian ψ–flow can be reduced invariantly upon the compact submanifold $\mathcal{L}_{t_0}(H) \times \mathbb{S} \subset M^{2n} \times \mathbb{S}$. For the properties of this reduced ψ–flow upon $\mathcal{L}_{t_0}(H) \times \mathbb{S}$ to be studied in more detail, let us assume that our extended Hamiltonian ψ_*–flow on $T(\mathcal{L}_{t_0}(H)) \times \mathbb{S}$ is ergodic, that is the $\lim_{\tau\to\infty} \mathcal{A}_{t_0}^{(\tau)}(\gamma)$ doesn't depend on initial points $(u_0, \dot{u}_0; t_0) \in T(\mathcal{L}_{t_0}(H)) \times \mathbb{S}$.

Recall now that a basic result [**13**] of functional analysis (the Riesz representation theorem) states that the set of Borel probability measures on a compact metric space X is a subset of the dual space $C(X)^*$ of the Banach space $C(X)$ of continuous functions on X. It is obviously a convex set and it is well known [**13**] to be metrizable and compact with respect to the weak topology on $C(X)^*$ defined by $C(X)$, also called the weak $(*)$–topology. The restriction of this topology to the set of Borel measures is frequently called the vague topology on measures [**9**]. Since the space $\mathcal{P}_{t_0} := T(\mathcal{L}_{t_0}(H)) \times \mathbb{S}$ is metrizable and can be as well compactified, it follows that the set of Borel probability measures on $\mathcal{P}_{t_0}$ is a metrizable, compact and convex subset of the dual space to the Banach space of continuous functions on $\mathcal{P}_{t_0}$. The corresponding set $\mathcal{M}_{t_0}(H)$ is then evidently a compact, convex subset of this set. The well known result of Kryloff and Bogoliuboff [**14**] states that any ψ–flow on a compact metric space X has an invariant probability measure. This result one can suitably adapt [**9**] to our metric compactified space $\mathcal{P}_{t_0} := T(\mathcal{L}_{t_0}(H)) \times \mathbb{S}$ as follows. Take a trajectory $\gamma \in \Omega_H$ of the extended ψ_*–flow on $\mathcal{P}_{t_0}$with *supp* $\gamma \subset \mathcal{L}_{t_0}(H) \times \mathbb{S}$ defined on a time interval $[0, \tau) \subset \mathbb{R}$ and let a measure μ_τ on $T(\mathcal{L}_{t_0}(H)) \times \mathbb{S}$ be evenly distributed along the orbit γ. Then evidently $||\psi_*^s \mu_\tau - \mu_\tau|| \leq 2s/\tau$ for all $s \in [0, \tau)$. Denote by μ a point of accumulation of the set $\{\mu_\tau : \tau \in \mathbb{R}_+\}$ as $\tau \to \infty$ with respect to the vague topology before mentioned. For any continuous function $f \in C(\mathcal{P}_{t_0})$, any $s \in [0, \tau)$ and any $\tau_0, \varepsilon > 0$, there exists $\tau > \tau_0$ such that $|\int_{\mathcal{P}_{t_0}} f \circ \psi_*^{\bar{s}} d\mu - \int_{\mathcal{P}_{t_0}} f \circ \psi_*^{\bar{s}} d\mu_\tau| < \varepsilon$ for all $\bar{s} \in \{0, s\}$. Then it follows from the above estimations

$$
\begin{aligned}
|\int_{\mathcal{P}_{t_0}} f \circ \psi_*^s d\mu - \int_{\mathcal{P}_{t_0}} f d\mu| \ &\leq |\int_{\mathcal{P}_{t_0}} f \circ \psi_*^s d\mu - \int_{\mathcal{P}_{t_0}} f \circ \psi_*^s d\mu_\tau| \\
&+ |\int_{\mathcal{P}_{t_0}} f \circ \psi_*^s d\mu_\tau - \int_{\mathcal{P}_{t_0}} f d\mu_\tau| + |\int_{\mathcal{P}_{t_0}} f d\mu_\tau - \int_{\mathcal{P}_{t_0}} f d\mu| \\
&\leq 2\varepsilon + ||f|| \ ||\psi_*^s \mu_\tau - \mu_\tau|| \ \leq 2\varepsilon + 2s||f||/\tau,
\end{aligned}
$$

that is $|\int_{\mathcal{P}_{t_0}} f \circ \psi_*^s d\mu - \int_{\mathcal{P}_{t_0}} f d\mu| = 0$ since $\varepsilon > 0$ can be taken arbitrarily small and $\tau_0 > 0$ arbitrarily large. Thereby one sees that the constructed measure $\mu \in \mathcal{M}_{t_0}(H)$ is normalized and invariant with respect to the extended Hamiltonian ψ_*–flow on $\mathcal{P}_{t_0}$.

Thus, in the case of the ψ_*–flow ergodic on $T(\mathcal{L}_{t_0}(H)) \times \mathbb{S}$, the above mentioned limit exists

$$
\lim_{\tau \to \infty} \mathcal{A}_{t_0}^{(\tau)}(\gamma) = \int_{\mathcal{P}_{t_0}} (\alpha^{(1)} - H) d\mu \tag{15}
$$

with a 1–form $\alpha^{(1)} \in \Lambda^1(M^{2n})$ being considered above as a function $\alpha^{(1)} : \mathcal{P}_{t_0} \to \mathbb{R}$, since the submanifold $\mathcal{L}_{t_0}(H)$ is by construction compact and invariantly imbedded into M^{2n}, due to Lemma 1.1. So, it is natural to study properties of the functional

$$\mathcal{A}_{t_0}(\mu) := \int_{\mathcal{P}_{t_0}} (\alpha^{(1)} - H) d\mu \tag{16}$$

on the space $\mathcal{M}_{t_0}(H)$, where we for brevity omitted the natural pullback of the 1–form $\alpha^{(1)} \in \Lambda^1(M^{2n})$ upon the invariant compact submanifold $\mathcal{L}_{t_0}(H) \subset M^{2n}$. Being interested namely in ergodic properties of ψ_*–orbits on $T(\mathcal{L}_{t_0}(H)) \times \mathbb{S})$, we shall below develop an analog of the J. Mather Lagrangian measure homology technique [**9, 10**] for a more general and complicated case of the reduced Hamiltonian ψ–flow on the invariant compact submanifold $\mathcal{L}_{t_0}(H) \subset M^{2n}$. In particular, we shall construct an analog of the so called Mather β–function [**9**] on the homology group $H_1(\mathcal{L}_{t_0}(H); \mathbb{R})$. The corresponding linear domains of the Mather β–function generate exactly ergodic components of a measure $\mu \in \mathcal{M}_{t_0}(H)$ minimizing functional (16), being of great importance for studying regularity properties of ψ_*–orbits on $T(\mathcal{L}_{t_0}(H)) \times \mathbb{S}$. The results can be further extended to adiabatically perturbed integrable Hamiltonian systems depending on a small parameter $\varepsilon \downarrow 0$ via the continuous dependence $H(t) := \tilde{H}(\varepsilon t)$, where $\tilde{H}(\tau + 2\pi) = \tilde{H}(\tau)$ for all $\tau \in [0, 2\pi]$. It also makes also possible to state the existence of so called adiabatic invariants with compact supports in $\mathcal{L}_{t_0}(H)$ with numerous applications in mathematical physics and mechanics. Some of the results can also be applied to investigating the problem of transversal intersections of the corresponding stable and unstable manifolds to hyperbolic curves or singular points, connected closely with the existence of highly irregular motions in a periodic time–dependent Hamiltonian dynamical system under regard.

2. Invariant measures and Mather's type β–function. Before studying average functional (16) on the measure space $\mathcal{M}_{t_0}(H)$, let us first analyze properties of the functional

$$\oint_\sigma a^{(1)} := \prec a^{(1)}, \sigma \succ \tag{17}$$

on $H^1(\mathcal{L}_{t_0}(H); \mathbb{R})$ at a fixed $\sigma \in H_1(\mathcal{L}_{t_0}(H); \mathbb{R})$. Since the 1–form $a^{(1)} \in H^1(\mathcal{L}_{t_0}(H); \mathbb{R})$ in (17) can be considered as a function $a^{(1)} : \mathcal{P}_{t_0} \to \mathbb{R}$, by virtue of the Riesz theorem [**13**], there exists a Borel measure $\mu : \mathcal{P}_{t_0} \to \mathbb{R}_+$ (still not necessary ψ–invariant), such that

$$\prec a^{(1)}, \sigma \succ = \int_{\mathcal{P}_{t_0}} a^{(1)} d\mu. \tag{18}$$

The following lemma characterizing the right hand side of (18) holds.

Lemma 2.1. *Let a 1–form* $a^{(1)} = d\lambda^{(0)} \in \Lambda^1(\mathcal{L}_{t_0}(H))$ *be exact, that is the cohomology class* $[d\lambda^{(0)}] = 0 \in H^1(\mathcal{L}_{t_0}(H); R)$. *Then for any* $\mu \in \mathcal{M}_{t_0}(H)$

$$\oint_\sigma a^{(1)} = 0. \tag{19}$$

Proof. Indeed, for $a^{(1)} = d\lambda^{(0)}$, where $\lambda^{(0)} : \mathcal{L}_{t_0}(H) \to \mathbb{R}$ is an absolutely continuous mapping, the following holds for any $\tau \in \mathbb{R}_+$ due to the Fubini theorem:

$$\begin{aligned} |\int_{\mathcal{P}_{t_0}} d\lambda^{(0)} d\mu.| &= |\frac{1}{\tau}\int_0^\tau ds \int_{\mathcal{P}_{t_0}} d\lambda^{(0)}(\psi_*^s d\mu)| \\ &= |\frac{1}{\tau}\int_{\mathcal{P}_{t_0}} d\mu \int_0^\tau dsd(\lambda^{(0)} \circ \psi_*^s)/ds| \\ &= |\frac{1}{\tau}\int_{\mathcal{P}_{t_0}} d\mu[\lambda^{(0)} \circ \psi_*^\tau - \lambda^{(0)} \circ \psi_*^0]| \ \leq 2||\lambda^{(0)}||/\tau, \end{aligned} \tag{20}$$

as $\tau \to \infty$, the latter inequality gives rise to required equality (19), which proves the lemma. □

Thus, the right hand side of (18) defines a well defined functional

$$H^1(\mathcal{L}_{t_0}(H); \mathbb{R}) \ni a^{(1)} \to \int_{\mathcal{P}_{t_0}} a^{(1)} d\mu \in \mathbb{R} \tag{21}$$

on the cohomology space $H^1(\mathcal{L}_{t_0}(H); \mathbb{R})$. All the above can be formulated as the following theorem.

Theorem 2.2. *Let an element* $\sigma \in H_1(\mathcal{L}_{t_0}(H); \mathbb{R})$ *be fixed. Then there exists a* ψ*–invariant probability measure (not unique)* $\mu \in \mathcal{M}_{t_0}(H)$ *such that representation* (18) *holds and, vice versa, for any measure* $\mu \in \mathcal{M}_{t_0}(H)$ *there exists the homology class* $\sigma := \rho_{t_0}(\mu) \in H_1(\mathcal{L}_{t_0}(H); \mathbb{R})$ *such that*

$$\prec a^{(1)}, \rho_{t_0}(\mu) \succ = \int_{\mathcal{P}_{t_0}} a^{(1)} d\mu \tag{22}$$

for all $a^{(1)} \in H^1(\mathcal{L}_{t_0}(H); \mathbb{R})$.

Definition 2.3. ([**10**]) *For any measure* $\mu \in \mathcal{M}_{t_0}(H)$ *the homology class* $\rho_{t_0}(\mu) \in H_1(L_{t_0}(H); \mathbb{R})$ *is called its homology.*

Corollary 2.4. *The homology mapping* $\rho_{t_0} : \mathcal{M}_{t_0}(H) \to H_1(L_{t_0}(H); \mathbb{R})$ *defined within Theorem 2.2 is surjective.*

Sketch of a proof of Theorem 2.2. The fact that for each $\mu \in \mathcal{M}_{t_0}(H)$ there exists the unique homology class $\sigma := \rho_{t_0}(\mu) \in H_1(\mathcal{L}_{t_0}(H); \mathbb{R})$ is based on the well known Poincaré duality theorem [**1**]. The inverse statement is about the surjectivity of the mapping $\rho_{t_0} : \mathcal{M}_{t_0}(H) \to H_1(\mathcal{L}_{t_0}(H); \mathbb{R})$. For it to be stated, consider, following [**8, 9, 10**], a covering space $\bar{\mathcal{L}}_{t_0}(H)$ over $\mathcal{L}_{t_0}(H)$ defined by the condition that $\pi_1(\mathcal{L}_{t_0}(H)) = \ker h_{t_0}$, where $h_{t_0} : \pi_1(\bar{\mathcal{L}}_{t_0}(H)) \to H_1(\mathcal{L}_{t_0}(H); \mathbb{R})$ denotes the Hurewicz homomorphism [**10**]. Since the functional (22) is actually defined on the covering space $\bar{\mathcal{L}}_{t_0}(H)$, it is necessary to lift all curves $\gamma \in \Omega_H$ on $\mathcal{L}_{t_0}(H) \times \mathbb{S}$ to curves $\tilde{\gamma} \in \tilde{\Omega}_H$ on $\overline{\mathcal{L}}_{t_0}(H) \times \mathbb{S}$. In the case of the abelian homotopy group $\pi_1(\mathcal{L}_{t_0}(H))$, the covering space $\bar{\mathcal{L}}_{t_0}(H)$ becomes universal, but in general it is obtained as some universal covering of $\mathcal{L}_{t_0}(H)$ factorized further with respect to the action of the kernel of the corresponding Hurewicz homomorphism $h_{t_0} : \pi_1(\bar{\mathcal{L}}_{t_0}(H)) \to H_1(\mathcal{L}_{t_0}(H); \mathbb{R})$.

Take now any element $\sigma \in H_1(\mathcal{L}_{t_0}(H); \mathbb{R})$ and construct a set of so called Deck transformations $\tau^{-1}\sigma_\tau \in Im\ h_{t_0} \subset H_1(\mathcal{L}_{t_0}(H); \mathbb{R})$, $\tau \in \mathbb{R}_+$, approximating it in such a way that weakly $\lim_{\tau\to\infty} \tau^{-1}\sigma_\tau = \sigma$ holds. Put further $\tilde{x}_\tau := \sigma_\tau \circ \tilde{x}_0 \in \overline{\mathcal{L}}_{t_0}(H) \times \mathbb{S}$, $\tau \in \mathbb{R}_+$, where $\tilde{x}_0 \in \overline{\mathcal{L}}_{t_0}(H) \times \mathbb{S}$ is taken arbitrary and consider such a curve $\tilde{\gamma} : [0, \tau] \to \overline{\mathcal{L}}_{t_0}(H) \times \mathbb{S}$ with end-points $\tilde{\gamma}(0) = \tilde{x}_0$, $\tilde{\gamma}(\tau) = \tilde{x}_\tau$ whose projection on $\mathcal{L}_{t_0}(H) \times \mathbb{S}$ is the curve $\gamma \in \Sigma_{t_0}(H)$ minimizing functional (6). Consider also a set $\{\mu_\tau : \tau \in \mathbb{R}_+\}$ of probability measures on $\mathcal{P}_{t_0}$ evenly distributed along corresponding curves $\gamma \in \Sigma_{t_0}(H)$ for each $\tau \in \mathbb{R}_+$ and denote by μ a point of its accumulation as $\tau \to \infty$. Due to the uniform distribution of measures μ_τ, $\tau \in \mathbb{R}_+$, along curves $\gamma \in \Sigma_{t_0}(H)$ having the end-points agreed with above chosen Deck transformations $\sigma_\tau \in H_1(\mathcal{L}_{t_0}(H); \mathbb{R})$, $\tau \in \mathbb{R}_+$, it immediately follows from the Birkhoff–Khinchin ergodic theorem [**1, 2**] that

$$\int_{\mathcal{P}_{t_0}} a^{(1)} d\mu_\tau = \prec a^{(1)}, \tau^{-1}\sigma_\tau) \succ \tag{23}$$

for any $a^{(1)} \in H^1(\mathcal{L}_{t_0}(H); \mathbb{R})$. Passing now to the limit in (23) as $\tau \to \infty$ and taking into account that weakly $\lim_{\tau\to\infty} \tau^{-1}\sigma_\tau = \sigma$, one gets right away that the equality (22) holds for some measure $\mu \in \mathcal{M}_{t_0}(H)$ such that $\rho_{t_0}(\mu) = \sigma \in H_1(\mathcal{L}_{t_0}(H); \mathbb{R})$, thereby proving the surjectivity of the mapping $\rho_{t_0} : \mathcal{M}_{t_0}(H) \to H_1(\mathcal{L}_{t_0}(H); \mathbb{R})$ and this theorem. □

Return now to treating average functional (16) subject to the space of all invariant measures $\mathcal{M}_{t_0}(H)$. Namely, consider the following β–function $\beta_{t_0} : H_1(\mathcal{L}_{t_0}(H); \mathbb{R}) \to \mathbb{R}$ defined as

$$\beta_{t_0}(\sigma) := \inf_{\mu} \{\mathcal{A}_{t_0}(\mu) : \rho_{t_0}(\mu) = \sigma \in H_1(\mathcal{L}_{t_0}(H); \mathbb{R})\} \tag{24}$$

It will be further called a Mather type β–function, due to its analogy to the definition before given in [**9, 10**]. The following lemma holds.

LEMMA 2.5. *Let a 1–form $a^{(1)} \in H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$ be taken arbitrary. Then the Mather type β–function*

$$\beta_{t_0}^{(a)}(\sigma) := \inf_{\mu}\{\mathcal{A}_{t_0}^{(a)}(\mu) : \rho_{t_0}(\mu) = \sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})\}, \tag{25}$$

where, by definition

$$\mathcal{A}_{t_0}^{(a)}(\mu) := \int_{\mathcal{P}_{t_0}} (\alpha^{(1)} + a^{(1)} - H)d\mu, \tag{26}$$

satisfies the following equation:

$$\beta_{t_0}^{(a)}(\sigma) = \beta_{t_0}(\sigma) + \prec a^{(1)}, \sigma) \succ . \tag{27}$$

PROOF. The proof easily follows from definition (25) and equality (22). □

Assume now that the infimum in (24) is attained at a measure $\mu(\sigma) \in \mathcal{M}_{t_0}(H)$. Then evidently, $\rho_{t_0}(\mu(\sigma)) = \sigma$ for any homology class $\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$. Denote by $\mathcal{M}_{t_0}^{(\sigma)}(H)$ the set of all measures in $\mathcal{M}_{t_0}(H)$ minimizing the functional (24). In the next chapter we shall proceed to studying its ergodic and homology properties.

3. Ergodic measures and their homologies. Consider the introduced above Mather type β–function $\beta_{t_0}^{(a)} : H_1(\mathcal{L}_{t_0}(H);\mathbb{R}) \to \mathbb{R}$ for any $a^{(1)} \in H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$. It is evidently a convex function on $H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$, that is for any $\lambda_1, \lambda_2 \in [0,1]$, $\lambda_1 + \lambda_2 = 1$, and $\sigma_1, \sigma_2 \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ the following inequality holds:

$$\beta_{t_0}^{(a)}(\lambda_1\sigma_1 + \lambda_2\sigma_2) \leq \lambda_1\beta_{t_0}^{(a)}(\sigma_1) + \lambda_2\beta_{t_0}^{(a)}(\sigma_2). \tag{28}$$

As usually dealing with convex functions, one says that an element $\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ is extremal point [**13**] if $\beta_{t_0}^{(a)}(\lambda_1\sigma_1 + \lambda_2\sigma_2) < \lambda_1\beta_{t_0}^{(a)}(\sigma_1) + \lambda_2\beta_{t_0}^{(a)}(\sigma_2)$ for all $\lambda_1, \lambda_2 \in (0,1)$, $\lambda_1 + \lambda_2 = 1$, and $\sigma = \lambda_1\sigma_1 + \lambda_2\sigma_2$. Accordingly, we shall call a convex set $Z_{t_0}(H) \subset H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ a linear domain of the Mather type function (25) if

$$\beta_{t_0}^{(a)}(\lambda_1\sigma_1 + \lambda_2\sigma_2) = \lambda_1\beta_{t_0}^{(a)}(\sigma_1) + \lambda_2\beta_{t_0}^{(a)}(\sigma_2) \tag{29}$$

for any $\sigma_1, \sigma_2 \in Z_{t_0}(H)$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. It is now easy to see that if $\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ is extremal, then the set $\mathcal{M}_{t_0}^{(\sigma)}(H)$ contains [**15**] ergodic minimizing measure components. Namely, following [**9, 10**] one states that if $Z_{t_0}(H)$ is a linear domain and $\mathcal{P}_{t_0}^{(\sigma)} \subset \mathcal{P}_{t_0}$ is the closure of the union of the supports of measures $\mu(\sigma) \in \mathcal{M}_{t_0}^{(\sigma)}(H)$ with $\sigma \in Z_{t_0}(H)$, then the set $\mathcal{P}_{t_0}^{(\sigma)}$

is compact and the inverse mapping $(p_{t_0}|_{\mathcal{P}^{(\sigma)}_{t_0}})^{-1} : p_{t_0}(\mathcal{P}^{(\sigma)}_{t_0}) \to \mathcal{P}^{(\sigma)}_{t_0}$ is Lipschitzian, where $p_{t_0} : \mathcal{P}_{t_0} \to \mathcal{L}_{t_0}(H) \times \mathbb{S}$ is the standard projection, being injective upon $\mathcal{P}^{(\sigma)}_{t_0}$. Moreover, one can show [**9**] that if a measure $\mu \in \mathcal{M}^{(\sigma)}_{t_0}(H)$ is minimizing functional (26), then its support *supp* $\mu \subset \mathcal{P}^{(\sigma)}_{t_0}$ and all its ergodic components $\{\bar{\mu}\}$ are minimizing this functional too, and the convex hull of the corresponding homologies $conv\{\rho_{t_0}(\bar{\mu})\}$ is a linear domain $Z^{(\sigma)}_{t_0}(H)$ of the Mather type β–function (25). These results are of much inportance for a number of applications in dynamics. In particular, ergodic measures are well known to possess the crucial property that every invariant Borel set has measure either 0 or 1, giving rise to the following important equality:

$$\lim_{\tau\to\infty} \mathcal{A}^{(\tau)}_{t_0}(\gamma) = \mathcal{A}_{t_0}(\bar{\mu})) \tag{30}$$

uniformly on $(\gamma_{t_0},(0), \dot{\gamma}_{t_0}(0); t_0) \in \mathcal{P}_{t_0} \cap$ *supp* $\bar{\mu}$, where $\gamma \in \Sigma_{t_0}(H)$. All of the properties formulated above are inferred from the following theorem modelling the similar one in [**10**].

Theorem 3.1. *Let a measure $\mu \in \mathcal{M}_{t_0}(H)$ be minimizing functional* (26) *and satisfying the condition $\beta^{(a)}_{t_0}(\rho_{t_0}(\mu)) = \mathcal{A}_{t_0}(\mu)$. Then supp $\mu \subset \Sigma_{t_0}(H)$ and the convex hull of the set of homologies $\rho_{t_0}(\bar{\mu}) \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$, where $\{\bar{\mu}\} \subset \mathcal{M}_{t_0}(H)$ are the corresponding ergodic components of the measure $\mu \in \mathcal{M}_{t_0}(H)$, is a linear domain $Z_{t_0}(H)$ of Mather type β–function* (25).

Sketch of a proof. Let $h_{t_0} : \pi_1(\mathcal{L}_{t_0}(H)) \to H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ be the corresponding Hurewicz homomorphism and take some basis $\sigma_k \in im\ h_{t_0} \subset H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$, $k = \overline{1,r}$, where $r = \dim im\ h_{t_0}$, with its dual basis $a^{(1)}_j \in H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$, $j = \overline{1,r}$. Then for any points $\tilde{x}, \tilde{y} \in \overline{\mathcal{L}}_{t_0}(H) \times \mathbb{S}$ one can define an element $\xi^{(\tau)}(\tilde{x}, \tilde{y}|\tilde{\gamma}) \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ as the sum

$$\xi^{(\tau)}(\tilde{x}, \tilde{y}|\tilde{\gamma}) := \frac{1}{\tau}\sum_{j=1}^{r} \sigma_j \int_0^\tau \tilde{a}^{(1)}_j(\tilde{\gamma}), \tag{31}$$

where $\gamma : [0,\tau] \to \mathcal{L}_{t_0}(H) \times \mathbb{S}$ is any continuous arc joining these two chosen points $\tilde{x}, \tilde{y} \in \overline{\mathcal{L}}_{t_0}(H) \times \mathbb{S}$, and $\tilde{a}^{(1)}_j \in H^1(\overline{\mathcal{L}}_{t_0}(H);\mathbb{R})$ are the corresponding liftings of 1–forms $a^{(1)}_j \in H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$, $j = \overline{1,r}$, to $\bar{\mathcal{L}}_{t_0}(H)$. One can then show that if $\mu \in \mathcal{M}_{t_0}(H)$ is ergodic and *supp* $\mu \subset \Sigma_{t_0}(H)$, the measure μ is minimizing functional (26). Put $\sigma := \rho_{t_0}(\mu)$ and let a set $Z_{t_0}(H) \subset H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ be a supporting domain containing this homology class $\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$. Thus, one can see that the extremal points of the convex set $Z_{t_0}(H)$ are also extremal points of Mather type β–function (25). Next expand the homology class $\sigma = \rho_{t_0}(\mu)$ as a convex combination of extremal points

$\bar{\sigma}_j \in Z_{t_0}(H)$, $j=\overline{1,m}$, for some $m \in \mathbb{Z}_+$. Then, since elements $\bar{\sigma}_j \in Z_{t_0}(H)$, $j=\overline{1,m}$, are extremal, there exist ergodic measures $\bar{\mu}_j \in \mathcal{M}_{t_0}^{(\sigma_j)}(H)$, $j=\overline{1,m}$, such that $\rho_{t_0}(\bar{\mu}_j)=\bar{\sigma}_j$, $j=\overline{1,m}$. Moreover, since $Z_{t_0}^{(\sigma)}(H)$ is a linear domain, there easily follows that

$$\beta_{t_0}^{(a)}(\sigma)=\sum_{j=1}^{m} c_j\beta_{t_0}^{(a)}(\bar{\sigma}_j)=\sum_{j=1}^{m} c_j\mathcal{A}_{t_0}^{(a)}(\bar{\mu}_j), \tag{32}$$

where $\sigma=\sum_{j=1}^{m} c_j\bar{\sigma}_j$ with some real coefficients $c_j \in \mathbb{R}$, $j=\overline{1,m}$. Due to the ergodicity of the measure $\mu \in \mathcal{M}_{t_0}(H)$ from the Birkhoff–Khinchin ergodic theorem [**1**], one derives that there exists such an orbit $\tilde{\gamma}:[0,\tau\} \to \overline{\mathcal{L}}_{t_0}(H)\times\mathbb{S}$ with the *supp* $\gamma \subset$ *supp* μ such that property (30) and the equality

$$\sigma := \rho_{t_0}(\mu)=\lim_{\tau\to\infty}\xi^{(\tau)}(\tilde{x},\tilde{y}|\tilde{\gamma}) \tag{33}$$

hold. Further, there exist curves $\tilde{\gamma}_j \in \Sigma_{t_0}(H)$, *supp* $\gamma_j \subset$ *supp* $\bar{\mu}_j$, $j=\overline{1,m}$, such that the equalities

$$\bar{\sigma}_j := \rho_{t_0}(\bar{\mu}_j)=\lim_{\tau\to\infty}\xi^{(\tau)}(\tilde{x},\tilde{y}|\tilde{\gamma}_j) \tag{34}$$

as well as $\beta_{t_0}^{(a)}(\bar{\sigma}_j)=\mathcal{A}_{t_0}^{(a)}(\bar{\mu}_j)=\lim_{\tau\to\infty}\mathcal{A}_{t_0}^{(\tau)}(\tilde{\gamma}_j)$ hold for every $j=\overline{1,m}$. Under conditions (14b) applied to the invariant neighborhood $\mathcal{L}_{t_0}(H)$, one shows that for any measure $\mu \in \mathcal{M}_{t_0}(H)$ such that $\rho_{t_0}(\mu)=\sigma$, the inequality $\mathcal{A}_{t_0}^{(a)}(\mu) \le \beta_{t_0}^{(a)}(\rho_{t_0}(\mu))$ holds, thereby proving its minimality. Suppose now that the measure $\mu \in \mathcal{M}_{t_0}(H)$ has all its ergodic components with supports contained in $\Sigma_{t_0}(H)$ and the convex hull of its homologies is a linear domain of Mather type function (25). One can approximate a measure $\mu \in \mathcal{M}_{t_0}(H)$ (in the weak topology) by means of a convex combination $\hat{\mu} := \sum_{j=1}^{m}\hat{c}_j\bar{\mu}_j$, where $\hat{c}_j \in \mathbb{R}$ and $\bar{\mu}_j \in \mathcal{M}_{t_0}(H)$, $j=\overline{1,m}$, are ergodic components of the measure $\mu \in \mathcal{M}_{t_0}(H)$. Then *supp* $\bar{\mu}_j \subset \Sigma_{t_0}(H)$, implying that all $\bar{\mu}_j \in \mathcal{M}_{t_0}(H)$, $j=\overline{1,m}$, are minimizing (26), that is they are minimal. Therefore, since the convex hull of homologies $\{\rho_{t_0}(\bar{\mu}_j) \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R}) : j=\overline{1,m}\}$ is a linear domain due to its minimality, then

$$\begin{aligned}\mathcal{A}_{t_0}^{(a)}(\hat{\mu}) &= \sum_{j=1}^{m}\hat{c}_j\mathcal{A}_{t_0}^{(a)}(\bar{\mu}_j)=\sum_{j=1}^{m}\hat{c}_j\beta_{t_0}^{(a)}(\rho_{t_0}(\bar{\mu}_j)) \\ &= \beta_{t_0}^{(a)}(\rho_{t_0}(\sum_{j=1}^{m}\hat{c}_j\bar{\mu}_j))=\beta_{t_0}^{(a)}(\rho_{t_0}(\mu),\end{aligned} \tag{35}$$

obviously meaning that the measure $\hat{\mu} \in \mathcal{M}_{t_0}(H)$ is minimal too. Now making use of the fact that limits of minimizing measures are minimizing too, one finally obtains that the measure $\mu \in \mathcal{M}_{t_0}(H)$ is minimizing the functional (26), thereby proving the theorem. $\square$

Consider some properties of a so called [**10**] supporting domain

$$Z_{t_0}^{(a)}(H) := \{\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R}) : \beta_{t_0}^{(a)}(\sigma) = \prec a^{(1)}, \sigma \succ + c_{t_0}^{(a)}\} \tag{36}$$

for Mather type β–function (25) at some fixed $a^{(1)} \in H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$ with $c_{t_0}^{(a)} \in \mathbb{R}$ properly defined by (27). Define also by $\mathcal{P}_{t_0}^{(a)} := \cup_{\sigma \in Z_{t_0}^{(a)}(H)}$ *supp* $\mu(\sigma)$, where $\mu(\sigma) \in \mathcal{M}_{t_0}(H)$ and $\rho_{t_0}(\mu(\sigma)) = \sigma \in Z_{t_0}^{(a)}(H)$. Now using expression (27), describe a supporting domain $Z_{t_0}^{(a)}(H) \subset H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ as follows:

$$Z_{t_0}^{(a)}(H) = \{\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R}) : \beta_{t_0}^{(0)}(\sigma) = c_{t_0}^{(a)}\}, \tag{37}$$

where the function $\beta_{t_0}^{(0)} : H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ being bounded from below is chosen in such a way that $\beta_{t_0}^{(0)}(\sigma) \geq c_{t_0}^{(a)}$ for all $\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$. Now take a measure $\mu \in \mathcal{M}_{t_0}(H)$ and suppose that *supp* $\mu \subset \Sigma_{t_0}(H)$. Since $\beta_{t_0}^{(0)}(\sigma) \geq c_{t_0}^{(a)}$ for all $\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ and due to (37) $Z_{t_0}^{(a)}(H) = (\beta_{t_0}^{(0)})^{-1}\{c_{t_0}^{(a)}\}$ at some fixed $a^{(1)} \in H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$, which evidently implies that the measure $\mu \in \mathcal{M}_{t_0}(H)$ is minimizing functional (26) and $\rho_{t_0}(\mu) \in Z_{t_0}^{(a)}(H)$. Thereby the following theorem has been proved.

THEOREM 3.2. *Suppose that $Z_{t_0}^{(a)}(H) \subset H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$ is a supporting domain of Mather type function* (27) *and a measure $\mu \in \mathcal{M}_{t_0}(H)$ satisfies the condition supp $\mu \subset \Sigma_{t_0}(H)$. Then this measure $\mu \in \mathcal{M}_{t_0}(H)$is minimizing and $\rho_{t_0}(\mu) \in Z_{t_0}^{(a)}(H)$.*

The following corollaries from the Theorem 3.2 hold. (cf. [**10**])

COROLLARY 3.3. *The minimizing measure $\mu \subset \mathcal{M}_{t_0}(H)$ with supp $\mu \subset \Sigma_{t_0}(H)$ satisfies the condition $\mathcal{A}_{t_0}^{(0)}(\mu) = c_{t_0}^{(a)}$. By means of choosing the element $a^{(1)} \in H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$ one can assume the value $c_{t_0}^{(a)} = 0$.*

COROLLARY 3.4. *For any strictly extremal closed curve $\sigma \in H_1(\mathcal{L}_{t_0}(H);\mathbb{R})$, the following properties take place:*

i) *there exists an ergodic measure $\bar{\mu}(\sigma) \in \mathcal{M}_{t_0}(H)$ whose support is a minimal set and $\rho_{t_0}(\bar{\mu}(\sigma)) = \sigma$;*
ii) *for every closed 1–form $a^{(1)} \in H^1(\mathcal{L}_{t_0}(H);\mathbb{R})$ the equality $\prec a^{(1)}, \sigma \succ = \lim_{\tau\to\infty} \frac{1}{\tau}\int_{t_0}^{t_0+\tau} a^{(1)}(\dot{\gamma})ds$ holds uniformly for all $(\gamma_{t_0(0)}, \dot{\gamma}_{t_0(0)}; t_0) \in \mathcal{P}_{t_0} \cap$ supp $\bar{\mu}(\sigma)$, $\rho_{t_0}(\bar{\mu}(\sigma)) = \sigma$ and $\gamma \in \Sigma_{t_0}(H)$;*

iii) *if* $(\gamma_{t_0(0)}, \dot{\gamma}_{t_0(0)}; t_0) \in \mathcal{P}_{t_0} \cap$ *supp* $\bar{\mu}(\sigma)$, $\rho_{t_0}(\bar{\mu}(\sigma)) = \sigma$ *and* $\gamma \in \Sigma_{t_0}(H)$ *is the corresponding orbit in* $\mathcal{L}_{t_0}(H) \times \mathbb{S}$, *then* $\beta^{(a)}_{t_0}(\sigma) = \lim_{\tau \to \infty} \mathcal{A}^{(\tau)}_{t_0}(\gamma)$ *uniformly.*

The above formulated statements can be effectively used for studying dynamics of many perturbed integrable Hamiltonian flows and their regularity properties. As it is well known, they are strongly based on the intersection theory of stable and unstable manifolds related with either hyperbolic closed orbits or singular points of a Hamiltonian system under regard. These aspects of our study of ergodic measure and homology properties of such Hamiltonian flows will be assumed in another paper under preparation.

4. Acknowledgements. The author is grateful to the Organizers of the International conference “Geometry and Topology of Manifolds” held in Krynica, 2001, for invitation to deliver a lecture on the topic of the paper. Some crucial results of the paper were previously discussed with Profs. B. Szafirski, Jagellonian University, and Alexander S. Mishchenko, Moscow State University, whose comments and help in preparing the manuscript were of great importance. The author is also thankful to participants of the seminar on differential equations at the Jagellonian University headed by Profs. A. Pelczar and J. Ombach and of the “Nonlinear Analysis” seminar held at the Dept. of Applied Mathematics of the AGH University of Science and Technology of Kraków for valuable discussions. The last but not least thanks are addresed with pleasure to the anonimous referees for many valuable comments and remarks which helped to improve the final draft of the paper.

References

1. Abraham R., Marsden J., *Foundations of Mechanics,* Cummings, NY, 1978.
2. Arnold V.I., *Mathematical methods of classical mechanic,* Springer, 1978.
3. Salamon D. and Zehnder E., *Morse theory for periodic solutions of Hamiltonian systems and the Maslov index,* Comm. Pure Appl. Math., **45** (1992), 1303–1360.
4. Floer A., *A relative Morse index for the symplectic action,* Comm. Pure Appl. Math., **41** (1988), 393–407.
5. Aebischer B., Borer M. and others, *Symplectic geometry: Introductory course*, Birkhauses Verlag, Basel, 1992, 79–165.
6. Floer A., *Morse theory for Lagrangian intersections,* J. Differential Geom., **28** (1988), 513–547.
7. Hofer H., *Lusternik–Schnirelman theory for Lagrangian intersections,* Ann. Inst. H. Poincaré, **5**, No. **5** (1988), 465–499.
8. Eliashberg Y., Givental A. and Hofer H., *Introduction to symplectic field theory,*. //arXive: math.SG/0010059 6 Oct 2000, 1–102.
9. Mather J. N., *Action minimizing invariant measures for positive definite Lagrangian systems,* Math. Z., **2017** (1991), 169–207.

10. Mane R., *On the minimizing measures of Lagrangian dynamical systems,* Nonlinearity, **5** (1992), 623–638.
11. Prykarpatsky A.K., *On invariant measure structure of a class of ergodic discrete dynamical systems,* Nonlin. Oscillations, **3**, No. **1** (2000), 78–83.
12. McDuff D., *Elliptic methods in symplectic geometry,* Bull. Amer. Math. Soc., **23** (1990), 311–358.
13. Edwards R.E., *Functional analysis*, Holt, Rinehart and Winston Publ., New York, 1965.
14. Kryloff N.M. and Bogoliubov N.N., *La theorie generale de la mesure et son application a l'etude des systemes dynamiques de la mechanique nonlineaire,* Ann. Math., II, **38** (1937), 65–113.
15. Niemycki V.V. and Stepanov V.V., *Qualitive theory of differential equations,* Princeton, Univ. Press, 1960.

Received January 2, 2002

Akademia Górniczo–Hutnicza
Dept. of Applied Mathematics
Al. Mickiewicz 30
30–059 Kraków, Poland
e-mail: prikarpa@wms.mat.agh.edu.pl

IAPMM of NAS
Dept. of Nonlinear Mathematical Analysis
Lviv 79601 Ukraina
e-mail: prykanat@cybergal.com