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CONVERGENCE IN CAPACITY OF THE PLURICOMPLEX GREEN FUNCTION

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Abstract. In this paper we prove that if Ω is a bounded hyperconvex domain in $\mathbb{C}^n$ and if $\Omega \ni z_j \to \partial\Omega$, $j \to \infty$, then the pluricomplex Green function $g_\Omega(z_j, \cdot)$ tends to 0 in capacity, as $j \to \infty$.

A bounded open connected set $\Omega \subset \mathbb{C}^n$ is called hyperconvex if there exists negative plurisubharmonic function $\psi \in PSH(\Omega)$ such that $\{z \in \Omega : \psi(z) < c\} \subset\subset \Omega$ for all $c < 0$. Such ψ is called an exhaustion function for Ω. It was proved in [**6**] that for every hyperconvex domain there exists smooth exhaustion function ψ such that $\lim_{z\to\zeta} \psi(z) = 0$, for all $\zeta \in \partial\Omega$.

Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Let $z \in \Omega$. Recall that the pluricomplex Green function with the pole at z is defined as follows

$$g_\Omega(z, w) = \sup\{u(w) : u \in PSH(\Omega), u \le 0, |u(\xi) - \log|\xi - z|| \le C \ \text{ near } z\}.$$

It is well known that $g_\Omega(z, \cdot) \in PSH(\Omega) \cap \mathcal{C}(\Omega \setminus \{z\})$, $g_\Omega(z, w) = 0$ for $w \in \partial\Omega$ and $(dd^c g_\Omega(z, \cdot))^n = (2\pi)^n \delta_z$, where δ_z is the Dirac measure at z (see [**7**]). Carlehed, Cegrell and Wikstöm proved in [**4**] that for every $z_0 \in \partial\Omega$ there exists a pluripolar set $E \subset \Omega$ such that

$$\limsup_{z\to z_0} g_\Omega(z, w) = 0,$$

for every $w \in \Omega \setminus E$. Błocki and Pflug proved in [**3**] that if $\Omega \ni z_j \to \partial\Omega$ then $g_\Omega(z_j, \cdot) \to 0$ in L^p for every $1 \le p < +\infty$, as $j \to \infty$. By $z_j \to \partial\Omega$ we mean that $dist(z_j, \partial\Omega) \to 0$. This result was used in [**3**] to show Bergman completeness of the hyperconvex domain. Herbort proved in [**5**] that if a

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bounded hyperconvex domain $\Omega \subset \mathbb{C}^n$ admits a Hoelder continuous exhaustion function then the pluricomplex Green function $g_\Omega(z_j, \cdot)$ tends to zero uniformly on compact subsets of Ω if the pole $z_j \to z_0 \in \partial\Omega$. We prove the following theorem.

THEOREM 1. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $\Omega \ni z_j \to \partial\Omega$, $j \to \infty$. Then $g_\Omega(z_j, \cdot) \to 0$ in capacity as $j \to \infty$.*

First let us recall the definition of the relative capacity and of convergence in capacity.

DEFINITION 2. The relative capacity of the Borel set $E \subset \Omega \subset \mathbb{C}^n$ with respect Ω is defined in [**1**]

$$cap(E, \Omega) = \sup\left\{ \int_E (dd^c u)^n : u \in PSH(\Omega), -1 \le u \le 0 \right\}.$$

DEFINITION 3. Let $u_j, u \in PSH(\Omega)$. We say that a sequence u_j converges to u in capacity if for any $\epsilon > 0$ and $K \subset\subset \Omega$

$$\lim_{j\to\infty} cap(K \cap \{|u_j - u| > \epsilon\}) = 0.$$

REMARK. Convergence in capacity is stronger then convergence in L^p since the Lebesgue measure $(d\lambda)$ is dominated by the relative capacity, i.e. there exists constant $C(n, \Omega) > 0$ depends only on n and Ω such that

$$cap(E) \ge C(n, \Omega)\lambda(E).$$

To prove the last inequality observe that there exist constants $C_1, C_2 > 0$ depending only on Ω such that $-1 \le C_1|z|^2 - C_2 \le 0$ on Ω and $(dd^c(C_1|z|^2 - C_2))^n = 4^n n! C_1^n d\lambda$. Therefore the above inequality holds with $C(n, \Omega) = 4^n n! C_1^n$. Observe also that uniform convergence on compact sets is stronger then convergence in capacity, since the following inequality holds

$$cap(K \cap \{|u_j - u| > \epsilon\}) \le \epsilon^{-1} cap(K) \sup_K |u_j - u|.$$

To prove Theorem 1 we will need the following lemma proved in [**2**].

LEMMA 4. *Let Ω be a bounded domain $\mathbb{C}^n$. Assume that u, v are bounded negative plurisubharmonic functions such that $\lim_{z\to\zeta} v(z) = 0$, for all $\zeta \in \partial\Omega$. Then*

$$\int_\Omega (-v)^n (dd^c u)^n \le n! (\sup_\Omega |u|)^{n-1} \int_\Omega (-u)(dd^c v)^n.$$

PROOF OF THEOREM 1. Let us denote $u_j = g_\Omega(z_j, \cdot)$. Suppose that u_j does not converge in capacity to 0, $j \to \infty$. Then for some $\epsilon > 0$ and $K \subset\subset \Omega$

there exist a subsequence u_{j_k}, and constants $c>0$ and $N>0$ such that for $j_k \geq N$ we have

$$cap(K\cap\{-u_{j_k}>\epsilon\})\geq c. \tag{1}$$

From the definition of capacity there exists $v\in PSH(\Omega)$ such that $-1\leq v\leq 0$ and

$$\int_{K\cap\{-u_{j_k}>\epsilon\}}(dd^cv)^n\geq\frac{c}{2}. \tag{2}$$

Now we will show that $u_j\to 0$ on K in $L^n((dd^cv)^n)$. Since Ω is hyperconvex then there exist ψ a continuous exhaustion function for Ω and a constant $A>0$ such that $A\psi<v$ on K. Define the following bounded plurisubharmonic function $\varphi=\max(A\psi,v)$. Then $\lim_{z\to\zeta}\varphi(z)=0$, for all $\zeta\in\partial\Omega$ and

$$(dd^c\varphi)^n\geq\chi_K(dd^cv)^n,$$

where χ_K is the characteristic function of the set K. Observe that φ is an exhaustion function for Ω, which implies that $\varphi(z_j)\to 0$ if $dist(z_j,\partial\Omega)\to 0$.

Using the monotone convergence theorem and Lemma 4 we get

$$\int_K(-u_j)^n(dd^cv)^n=\int_\Omega(-u_j)^n(dd^c\varphi)^n=\lim_{k\to+\infty}\int_\Omega(-\max(u_j,-k))^n(dd^c\varphi)^n$$

$$\leq n!(\sup_\Omega|\varphi|)^{n-1}\lim_{k\to+\infty}\int_\Omega|\varphi|(dd^c\max(u_j,-k))^n=n!(2\pi)^n(\sup_\Omega|\varphi|)^{n-1}|\varphi(z_j)|,$$

which means that $u_j\to 0$ on K in $L^n((dd^cv)^n)$, since $\varphi(z_j)\to 0$, as $j\to\infty$.

Observe that inequality (2) implies that

$$\frac{c}{2}\leq\int_{K\cap\{-u_{j_k}>\epsilon\}}(dd^cv)^n\leq\epsilon^{-n}\int_K(-u_{j_k})^n(dd^cv)^n,$$

which is impossible since $u_{j_k}\to 0$ on K in $L^n((dd^cv)^n)$. This means that $u_j\to 0$ in capacity as $j\to\infty$. The proof is finished. □

Now we recall the definition of the multipolar Green function introduced by Lelong [**8**]. Let $A=\{(z^{(1)},\nu^{(1)}),\dots,(z^{(m)},\nu^{(m)})\}$ be a finite subset of $\Omega\times\mathbb{R}_+$. Let

$$g_\Omega(A,w)=\sup\{u(w):u\in\mathcal{L}_A,u\leq 0\},$$

where $\mathcal{L}_A$ denotes the family of plurisubharmonic functions on Ω having a logarithmic pole with weight $\nu^{(k)}$ at $w^{(k)}$, for $k=1,\dots,m$, i.e.

$$\mathcal{L}_A=\{u\in PSH(\Omega):|u(\xi)-\nu^{(j)}\log|\xi-z^{(j)}||\leq C_j\ \text{ near } z^{(j)},1\leq j\leq m\}.$$

We show that it is possible to generalize Theorem 1 for the multipolar Green function.

Corollary 5. *Let* Ω *be a bounded hyperconvex domain in* $\mathbb{C}^n$ *and let* $A_j = \{(z_j^{(1)}, \nu^{(1)}), \dots, (z_j^{(m)}, \nu^{(m)})\}$ *be a subset of* $\Omega \times \mathbb{R}_+$, *for* $j = 1, 2, \dots,$ *such that* $\Omega \ni z_j^{(k)} \to \partial\Omega$, $j \to \infty$ *for all* $k = 1, \dots, m$. *Then* $g_\Omega(A_j, \cdot) \to 0$ *in capacity as* $j \to \infty$.

Proof. Directly from the definition of the multipolar Green function we have

$$\sum_{k=1}^{m} \nu^{(k)} g_\Omega(z_j^{(k)}, \cdot) \le g_\Omega(A_j, \cdot) \le 0.$$

By Theorem 1 we have that $g_\Omega(z_j^{(k)}, \cdot) \to 0$ in capacity as $j \to \infty$ for all $k = 1, \dots, m$, so also $g_\Omega(A_j, \cdot) \to 0$ in capacity as $j \to \infty$. This ends the proof. □

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