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A NOTE ON ALEXANDER'S THEOREM

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Abstract. The aim of this note is to extend a result of H. Alexander [**1**] from the case of scalar functions to the case of functions with values in topological vector spaces.

Let $\mathbb{B} := \{z \in \mathbb{C}^N ; \|z\| < 1\}$ be the unit ball in $\mathbb{C}^N$ with respect to a complex norm $\|\cdot\|$. Given a subset E of the unit sphere $\partial\mathbb{B}$, let $\rho = \rho(E)$ be the radius of the maximal ball $r\mathbb{B}$ contained in the set $Int(\bigcap\Omega)$, where the intersection is taken over all balanced domains of holomorphy Ω containing E. It is known [**3, 4**] that ρ is a Choquet capacity characterizing non-pluripolar complex cones in $\mathbb{C}^N$. Namely, if V is a complex cone in $\mathbb{C}^N$ with vertex at 0 then V is pluripolar if and only if $E := V \cap \partial\mathbb{B}$ is pluripolar, if and only if $\rho(E) = 0$.

Let F be a sequentially complete topological vector space over $\mathbb{C}$. Let Γ be a set of continuous seminorms determining the topology of F.

In 1974 H. Alexander [**1**] proved (among others) that if $\{f_n\}$ is a sequence of holomorphic functions on the unit ball $\mathbb{B}$ such that the restriction of $\{f_n\}$ to each complex line L through the center 0 of $\mathbb{B}$ is uniformly convergent in a neighborhood of 0 in L then $\{f_n\}$ converges uniformly in a neighborhood of 0 in $\mathbb{B}$.

The goal of this note is to extend this result to the case where the target space $\mathbb{C}$ is replaced by any sequentially complete complex topological vector space F.

The main result of this article is given by the following theorem.

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THEOREM A. *Let E be a circled non-pluripolar subset of the unit sphere $\partial\mathbb{B}$ in $\mathbb{C}^N$. Let $\mathcal{X}$ be a family of F-valued holomorphic functions in the unit ball $\mathbb{B}$ such that* $\forall_{a\in E}\exists_{r_a>0}\forall_{q\in\Gamma}\exists_{M_q>0}$

$$\text{(a)} \qquad q(f(\lambda a)) \le M_q, \quad |\lambda| \le r_a, \quad f \in \mathcal{X}.$$

Then there exists $r > 0$ such that $\forall_{q\in\Gamma}\exists_{M_q>0}$ *such that*

$$\text{(b)} \qquad q(f(z)) \le M_q, \quad \|z\| \le r, \quad f \in \mathcal{X}.$$

COROLLARY 1. *Let V be a non-pluripolar complex cone in $\mathbb{C}^N$ with vertex at 0. Then for every family $\mathcal{X}$ of F-valued holomorphic functions on $\mathbb{B}$ such that for every complex line $L \subset V$ with $0 \in L$ the family $\mathcal{X}_L := \{f_{|\mathbb{B}\cap L}; f \in \mathcal{X}\}$ of holomorphic functions of a complex variable in the disk $\mathbb{B} \cap L$ is uniformly bounded on a neighborhood (dependent on L) of $0 \in \mathbb{C}$, then there exists $r > 0$ such that $\mathcal{X}$ is uniformly bounded on the ball rB.*

This and Vitali's theorem [**2**] imply the following Corollary 2 which is the Alexander theorem in the case of functions with values in sequentially complete topological vector spaces.

COROLLARY 2. *Let V be a non-pluripolar complex cone in $\mathbb{C}^N$. If $\mathcal{X} = \{f_n\}$ is a sequence of F-valued holomorphic functions in the unit ball $\mathbb{B} \subset \mathbb{C}^N$ such that for every complex line $L \subset V$ with $0 \in L$ the sequence $\{f_n|_{L\cap\mathbb{B}}\}$ is uniformly convergent on a neighborhood (dependent on L) of $0 \in \mathbb{C}$, then there exists $r > 0$ such that the sequence $\mathcal{X}$ is uniformly convergent on the ball $r\mathbb{B}$.*

PROOF OF THEOREM A. We have

$$f(z) = \sum_{n=0}^{\infty} P_n(z, f), \quad \|z\| < 1, \quad f \in \mathcal{X},$$

where $P_n(z, f) := \sum_{|\alpha|=n} \frac{f^{(\alpha)}(0)}{\alpha!} z^\alpha$ is the nth homogeneous polynomial of the Taylor series development of f around 0. In particular, $f(\lambda a) = \sum_0^\infty P_n(a, f)\lambda^n$, $|\lambda| < 1$, $a \in E$, $f \in \mathcal{X}$. Hence, by (a),

$$\text{(1)} \qquad q(P_n(a, f)) \le \frac{M_q}{r_a^n},\ n \ge 0,\ a \in E,\ f \in \mathcal{X}.$$

The function

$$\varphi_n(z) := \frac{1}{n} \log \sup_{f\in\mathcal{X}} q(P_n(z, f)),\ z \in \mathbb{C}^N,\ n \ge 1,$$

is a continuous PSH function of the Lelong class $\mathcal{L}$.

Put $E_s := \{a \in E; \varphi_n(a) \le s,\ n \ge 1\}$. By (1) $\cup_1^\infty E_s = E$ and $E_s \subset E_{s+1}$ for all $s \ge 1$. Therefore $\lim_{s\to\infty} \rho(E_s) = \rho \equiv \rho(E)$.

Fix $0 < \theta < 1$ and take $s = s_q$ so large that $\rho(E_s) \geq \theta\rho$. Then by the Bernstein–Walsh inequality for the homogeneous functions of Lelong class we get

$$\varphi_n(z) \leq s_q + \log \frac{\|z\|}{\theta\rho},\ n \geq 1,\ z \in \mathbb{C}^N.$$

Put $\varphi(z) := \limsup_{n\to\infty} \varphi_n(z)$. The sequence $\{\varphi_n\}$ is locally uniformly upper bounded in $\mathbb{C}^N$. Therefore φ^* is a homogeneous function of the Lelong class. By Bedford–Taylor theorem on negligible sets there exists a circled non-pluripolar subset E_0 of E such that $\rho(E_0) = \rho(E)$ and $\varphi^*(z) = \varphi(z)$ for all $z \in E_0$. Put $A_s := \{a \in E_0; \varphi(a) \leq s\}$. By (1) there exists s such that $\rho(A_s) \geq \theta\rho$. Hence, by Bernstein–Walsh inequality, we get

$$\varphi(z) \leq \varphi^*(z) \leq s + \log \frac{\|z\|}{\theta\rho}, \quad z \in \mathbb{C}^N.$$

Observe that the number s does not depend on $q \in \Gamma$. It depends only on θ and on the function $E \ni a \to r_a \in (0, \infty)$.

By the Hartogs Lemma for every $q \in \Gamma$ there is n_q such that

$$\varphi_n(z) \leq s + 1 + \log \frac{1}{\theta\rho},\ \|z\| \leq 1,\ n > n_q.$$

Hence

$$\varphi_n(z) \leq \log\left(\frac{e^{s+1}\|z\|}{\theta\rho}\right),\ z \in \mathbb{C}^N,\ n > n_q. \tag{2}$$

Put

$$B_m := \{a \in E; q(P_n(a, f)) \leq m,\ 0 \leq n \leq n_q,\ f \in \mathcal{X}\}.$$

By (1) there is $m = m_q > 0$ such that $\rho(B_m) \geq \theta\rho$. Then

$$q(P_n(z, f)) \leq m_q \left(\frac{\|z\|}{\theta\rho}\right)^n,\ 0 \leq n \leq n_q, z \in \mathbb{C}^N,\ f \in \mathcal{X}. \tag{3}$$

From (2) and (3) one gets

$$q(P_n(z, f)) \leq m_q \left(\frac{e^{s+1}\|z\|}{\theta\rho}\right)^n,\ n \geq 0,\ f \in \mathcal{X},\ z \in \mathbb{C}^N.$$

It follows that

$$q(f(z)) \leq \frac{m_q}{1-\theta}, \quad \|z\| \leq \theta^2\rho e^{-s-1}, \quad f \in \mathcal{X}.$$

Hence $q(f(z)) \leq M_q$ for all $f \in \mathcal{X}$ and $\|z\| \leq r$, where $M_q := m_q/(1-\theta)$, $r := \theta^2\rho e^{-s-1}$. □

Corollary from the proof. *If a family $\mathcal{X}$ satisfies* (a) *with $r_a = r_0 = const$, $a \in E$ where $0 < r_0 \leq 1$ then the family is locally uniformly bounded in the ball $r\mathbb{B}$ with $r := r_0\rho$, $\rho = \rho(E)$.*

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