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NOTE ON THE NEWTON NUMBER

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Abstract. In his fundamental paper on Newton polyhedrons and Milnor numbers, Kouchnirenko introduced the notion of the Newton number. It is an integer-valued function on the set of convenient Newton polyhedrons. In this note, answering a question asked by Arnold, we present a simple proof of monotonicity of the Newton number.

In the collection of problems formulated at Arnold's seminars in Paris and Moscow [**1**], the following problem is posed.

> **1982-16.** Consider a Newton polyhedron Δ in $\mathbf{R}^n$ and the number $\mu(\Delta) = n!\,V - \sum(n-1)!\,V_i + \sum(n-2)!\,V_{ij} - \cdots$, where V is the volume under Δ, V_i is the volume under Δ on the hyperplane $x_i = 0$, V_{ij} is the volume under Δ on the hyperplane $x_i = x_j = 0$, and so on.
>
> Then $\mu(\Delta)$ grows (non strictly monotonically) as Δ grows (whenever Δ remains convex *and integer?*). *There is no elementary proof even for* $n = 2$.

Further, in [**1**] (page 417) S. K. Lando wrote that the monotonicity of $\mu(\Delta)$ follows from the semicontinuity of the spectrum of a singularity, proved independently by Varchenko in [**7**] and Steenbrink in [**5**].

The quantity $\mu(\Delta)$ is also called the Newton number of the polyhedron Δ. The reader can find an elementary geometrical proof of the monotonicity of the Newton number for $n = 2$ in [**4**]. In [**2**] (Corollary 5.6), Bivià-Ausina gives a proof using results on mixed multiplicities of ideals. In this note, we solve Arnold's problem using semicontinuity of the Milnor number in families of power series.

Recall first what we mean by a Newton polyhedron.

For every formal power series $f \in \mathbf{C}[[x_1, \ldots, x_n]]$

$$f(x_1, \ldots, x_n) = \sum_{\alpha \in \mathbf{N}^n} f_\alpha x^\alpha, \qquad \text{where } x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \tag{1}$$

we put $\operatorname{supp}(f) = \{\, \alpha \in \mathbf{N}^n : f_\alpha \neq 0 \,\}$ and define Δ_f to be the convex hull of the set $\{\, \alpha + v : \alpha \in \operatorname{supp}(f), v \in \mathbf{R}^n_+ \,\}$.

A set Δ is called *a Newton polyhedron* if $\Delta = \Delta_f$ for a power series f. A Newton polyhedron Δ is called *convenient* if it intersects all coordinate axes. Then the complement of Δ in $\mathbf{R}^n_+$ has finite volume and the Newton number $\mu(\Delta)$ is well defined.

Kouchnirenko Theorem. *If f is a convergent power series such that Δ_f is convenient, then $\mu_0(f) \geq \mu(\Delta_f)$. For almost every f, there is $\mu_0(f) = \mu(\Delta_f)$.*

Let Δ be a fixed convenient Newton polyhedron and let A be the (finite) set of lattice points which belong to the compact faces of Δ. The phrase *for almost every* in the statement of Kouchnirenko Theorem means that there exists a proper algebraic set D in the space of coefficients $(\mathbf{C})_{\alpha \in A}$ with the following property: *if f is a convergent power series such that $\Delta_f = \Delta$ and $(f_\alpha)_{\alpha \in A}$ does not belong to D, then $\mu_0(f) = \mu(\Delta)$.*

Let $\Delta' \subset \Delta$ be arbitrary convenient Newton polyhedra. Take convergent power series $f = \sum f_\alpha x^\alpha$, $g = \sum g_\alpha x^\alpha$ such that, $\Delta = \Delta_f$, $\Delta' = \Delta_g$ and $\mu(\Delta') = \mu_0(g)$. Assume that $(f_\alpha)_{\alpha \in A}$ does not belong to the D mentioned above, $f_\alpha \neq 0$ for $\alpha \in A$ and consider the family

$$F_t(x) = tf(x) + (1-t)g(x)$$

parameterized by $t \in \mathbf{C}$. Then:

(i) $F_0(x) = g(x)$, $F_1(x) = f(x)$,
(ii) $\Delta_{F_t} = \Delta$ for all but a finite number of values of $t \in \mathbf{C}$,
(iii) $\mu_0(F_t) = \mu(\Delta)$ for all but a finite number of values of $t \in \mathbf{C}$.

Recall that A is the set of lattice points which belong to the compact faces of Δ. In order to check (ii), consider the system of equations $tf_\alpha + (1-t)g_\alpha = 0$ for $\alpha \in A$. If t does not satisfy any of these equations, then $A \cap \operatorname{supp}(F_t) = A$, which gives $\Delta_{F_t} = \Delta$.

Now we check (iii). A family of coefficients $(F_{t,\alpha})_{\alpha \in A}$ for $t \in \mathbf{C}$ is a straight line $L \subset (\mathbf{C})_{\alpha \in A}$. The point $(f_\alpha)_{\alpha \in A}$ belongs to L and does not belong to D. Thus L intersects D at a finite number of points (because D is an algebraic set), which proves (iii).

By Kouchnirenko Theorem and (i)–(iii), we get:

$$\begin{aligned} \mu_0(F_t) &= \mu(\Delta) \qquad \text{for } t \neq 0, \text{ sufficiently close to } 0 \\ \mu_0(F_0) &= \mu(\Delta') \end{aligned}$$

By semicontinuity of the Milnor number (see [**6**], Proposition 5.3), there is $\mu_0(F_0) \geq \mu_0(F_t)$ for t close to 0. Hence $\mu(\Delta') \geq \mu(\Delta)$.

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