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ON THE MULTIPLICITY OF A QUASI-HOMOGENEOUS ISOLATED SINGULARITY

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Abstract. We give a formula for the multiplicity of a quasi-homogeneous isolated singularity in terms of its weights.

Let $f = f(x_1, \ldots, x_n) \in \mathbb{C}\{x_1, \ldots, x_n\}$ be a convergent power series. We call f an *isolated singularity* at the origin $0 \in \mathbb{C}^n$ if $f(0) = 0$ and $0 \in \mathbb{C}^n$ is an isolated solution of the system of equations $\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$. By the *multiplicity* $\mathrm{ord} f$ of a series f, we mean the lowest degree of a monomial which appears in f with nonzero coefficient. Moreover, let us recall that f is *quasi-homogeneous* of type $(w_1, \ldots, w_n)$ if it is a polynomial of the form

$$f = \sum_{\frac{i_1}{w_1} + \cdots + \frac{i_n}{w_n} = 1} c_{i_1 \ldots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

for some positive rationals $w_1, \ldots, w_n$.

The quasi-homogeneous isolated singularities have been studied by many authors. Milnor and Orlik ([**2**], Theorem 1) proved that the Milnor number of a quasi-homogeneous isolated singularity of type $(w_1, \ldots, w_n)$ equals $\prod_{i=1}^{n}(w_i - 1)$. Thus this product is an integer, even though the $w_i's$ themselves may not be integers.

The main result of this note is

Theorem. *If f is a quasi-homogeneous isolated singularity of type $(w_1, \ldots, w_n)$ then*

$$\mathrm{ord} f = \min\{m \in \mathbb{N} : m \geqslant \min\{w_i : i = 1, \ldots, n\}\}.$$

S. S.-T. Yau proved the above formula for $n = 3$ (see [**4**], Theorem 6). His proof is based on the classification of quasi-homogeneous isolated singularities given in [**1**] and in [**3**] and it does not generalize to the case of an arbitrary n.

PROOF. Since $\mathrm{ord} f$ is an integer, it suffices to show that

$$\min\{w_i : i = 1, \dots, n\} \leqslant \mathrm{ord} f < \min\{w_i : i = 1, \dots, n\} + 1.$$

To check the first inequality, let us note that

$$\mathrm{ord} f = \min\{i_1 + \cdots + i_n : c_{i_1 \dots i_n} \neq 0\}.$$

For any $i_1, \dots, i_n$ such that $c_{i_1, \dots, i_n} \neq 0$, there holds

$$1 = \frac{i_1}{w_1} + \cdots + \frac{i_n}{w_n} \leqslant \frac{i_1 + \cdots + i_n}{\min\{w_i : i = 1, \dots, n\}},$$

hence

$$1 \leqslant \frac{\mathrm{ord} f}{\min\{w_i : i = 1, \dots, n\}}$$

and the first inequality follows.

In order to prove the inequality $\mathrm{ord} f < \min\{w_i : i = 1, \dots, n\} + 1$, we need the following observation due to Arnold (see [**1**]).

LEMMA. *Fix an $i \in \{1, \dots, n\}$. For an isolated singularity f, at least one of the monomials of the form $x_i^a x_j$, $a \geqslant 1$, $j = 1, \dots, n$ appears in the series f with a nonzero coefficient.*

PROOF. We may assume that $i = 1$. Let us write

$$f(x_1, \dots, x_n) = a_0(x_2, \dots, x_n) + x_1 a_1(x_2, \dots, x_n) + \cdots .$$

There is $\mathrm{ord}\, a_0 \geqslant 2$ and $\mathrm{ord}\, a_1 \geqslant 1$ as $\mathrm{ord}\, f \geqslant 2$. We will show that there exists a $k \geqslant 1$ such that $\mathrm{ord}\, a_k = 0$ or $\mathrm{ord}\, a_k = 1$.

To obtain a contradiction, suppose that $\mathrm{ord}\, a_k \geqslant 2$ for all $k \geqslant 1$. This gives $\mathrm{ord} \frac{\partial a_k}{\partial x_j} \geqslant 1$ for $j = 2, \dots, n$ and hence

$$a_k(0, \dots, 0) = 0 \quad \text{and} \quad \frac{\partial a_k}{\partial x_j}(0, \dots, 0) = 0 \text{ for all } k \geqslant 1 \text{ and } j \geqslant 2,$$

thus

$$\left.\begin{array}{l} \frac{\partial f}{\partial x_1}(x_1, 0, \dots, 0) = a_1(0) + 2x_1 a_2(0) + \cdots = 0, \\ \frac{\partial f}{\partial x_j}(x_1, 0, \dots, 0) = \frac{\partial a_0}{\partial x_j}(0) + x_1 \frac{\partial a_1}{\partial x_j}(0) + \cdots = 0 \text{ for } j = 2, \dots, n \end{array}\right. \quad \text{in } \mathbb{C}\{x_1\}$$

and this implies the inclusion $\{x_2 = \cdots = x_n = 0\} \subset \left\{ \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0 \right\}$. We get a contradiction because $0 \in \mathbb{C}^n$ is an isolated critical point of f. $\square$

Now let us suppose that $w_1 = \min\{w_i : i = 1, \dots, n\}$. According to Lemma, at least one of the monomials of the form $x_1^a x_j$, $a \geqslant 1$, $j = 1, \dots, n$

appears in f with nonzero coefficient. Thus $\mathrm{ord} f \leqslant a+1$ and for some $j \in \{1,\ldots,n\}$ there is $\frac{a}{w_1}+\frac{1}{w_j}=1$. This gives

$$\mathrm{ord} f \leqslant w_1\left(1-\frac{1}{w_j}\right)+1 = w_1+1-\frac{w_1}{w_j} < w_1+1$$

and the proof is complete. □

References

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