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DIFFEOMORPHISMS WITH MILDLY WILD FRAME OF SEPARATRICES

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Abstract. In the paper there is constructed a structural stable diffeomorphism on sphere $\mathbb{S}^3$ whose nonwandering set consists of $n+1$ sinks, n saddles of index 1, one source and the frame of one-dimensional separatrices is mildly wild.

Introduction. Let's consider a class $\Psi_n(\mathbb{S}^3), n \in \mathbb{N}$, of diffeomorphisms $f : \mathbb{S}^3 \to \mathbb{S}^3$ with the following properties:

1) the non-wandering set $\Omega(f)$ consists of $2n+2$ hyperbolic fixed points: $\omega_1, \dots, \omega_n, \omega$ are sinks, $\sigma_1, \dots, \sigma_n$ are saddles of index 1, α is a source;

2) one-dimensional unstable separatrices ℓ_{σ_i} and l_{σ_i} of saddle point $\sigma_i, i \in \{1, \dots, n\}$ satisfy the conditions: $\bar{\ell}_{\sigma_i} = \{\omega\} \cup \ell_{\sigma_i} \cup \{\sigma_i\}$ and $\bar{l}_{\sigma_i} = \{\omega_i\} \cup l_{\sigma_i} \cup \{\sigma_i\}$ (here $\bar{A}$ is closure of the set A).

In Figure 1 the phase portrait of diffeomorphism $f \in \Psi_n(\mathbb{S}^3)$ is represented.

According to [**5**] (Theorem 2.3) the closure $\bar{\ell}_{\sigma_i}$ of separatrix ℓ_{σ_i} is everywhere smooth except, maybe, at ω. So the topological embedding of $\bar{\ell}_{\sigma_i}$ may be complicated in a neighborhood of the sink. According to [**1**], ℓ_{σ_i} is called *tame* (or *tamely embedded*) if there is a homeomorphism $\psi : W^s(\omega) \to \mathbb{R}^3$ such that $\psi(\omega) = O$, where O is the origin and $\psi(\bar{\ell}_{\sigma_i} \setminus \sigma_i)$ is a ray starting from O. In the opposite case ℓ_{σ_i} is called *wild*. Artin and Fox constructed an example of a wild arc in $\mathbb{R}^3$ which was not connected with dynamic. Using that example D. Pixton [**4**] constructed a diffeomorphism from the class $\Psi_1(\mathbb{S}^3)$ such that arc ℓ_{σ_1} is exactly Artin–Fox wild arc. In Figure 2 the phase portrait of this diffeomorphism is represented. Later, in [**2**] was established that the class $\Psi_1(\mathbb{S}^3)$ is

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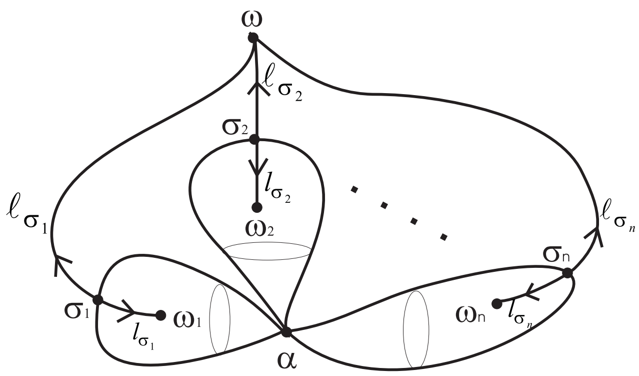

FIGURE 1. The phase portrait of diffeomorphism $f \in \Psi_n(\mathbb{S}^3)$

divided into infinitely many classes of topological conjugacy and the complete topological invariant in each class is the type of embedding of separatrix ℓ_{σ_1}. Moreover, in that paper the construction of a model diffeomorphism in each class of topological conjugacy was described.

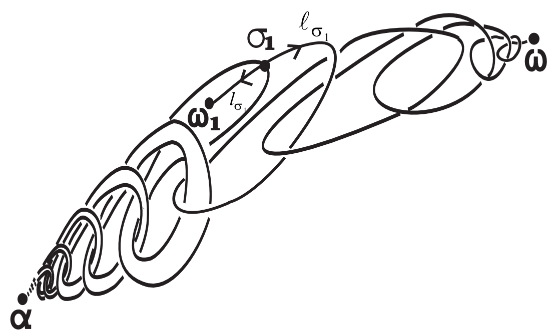

FIGURE 2. Pixton's example

Set $L_n(\omega) = \bigcup\limits_{i=1}^{n} \ell_{\sigma_i} \cup \omega$. For $n > 1$ the set $L_n(\omega)$ is *a frame of arcs* that is a union of arcs with exactly one common point. According to [**1**] the set $L_n(\omega)$ is called *tame* if there is a homeomorphism $\psi : W^s(\omega) \to \mathbb{R}^3$ such that $\psi(\omega) = O$ and $\psi(\bar{\ell}_{\sigma_i} \setminus \sigma_i)$ is a ray starting from O for each $i = 1, \dots, n$. In the opposite case $L_n(\omega)$ is called *wild*. It is clear that a frame is wild if at least one its arc is wild. However, in [**3**] there is an example of wild frame of arcs in

$\mathbb{R}^3$ for which each arc is tame. Moreover, if we delete any arc from such frame then residuary frame is tame. Such a frame is called *mildly wild*. Using this example we construct for $n \geq 2$ a diffeomorphism $f \in \Psi_n(\mathbb{S}^3)$ with mildly wild frame of separatrices $L_n(\omega)$.

1. Debruner–Fox example. Let's consider in $\mathbb{R}^3$ a three-dimensional ring V^0 given in spherical coordinates as follows: $\frac{1}{2} \leq \rho \leq 1$ and homothety $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ given by the formula $\phi(\rho, \varphi, \theta) = (\frac{1}{2}\rho, \varphi, \theta)$. Let $\ell_1^0, \dots, \ell_n^0$ be simple pairwise disjoint arcs in V^0 satisfying to the following conditions:

(1) ℓ_i^0 has the beginning at the point $(1, \frac{2\pi i}{n}, 0)$ and end at the point $(\frac{1}{2}, \frac{2\pi i}{n}, 0)$;
(2) the rotation by angle $\frac{2\pi}{n}$ around of the axis OZ translates an arc ℓ_i^0 in an arc ℓ_{i+1}^0 ($\ell_{n+1}^0 = \ell_1^0$);
(3) the arcs $\ell_1^0, \dots, \ell_n^0$ are smooth and have a regular normal projection to the plane XOY (just as is shown in Fig. 3a) for case $n = 6$).

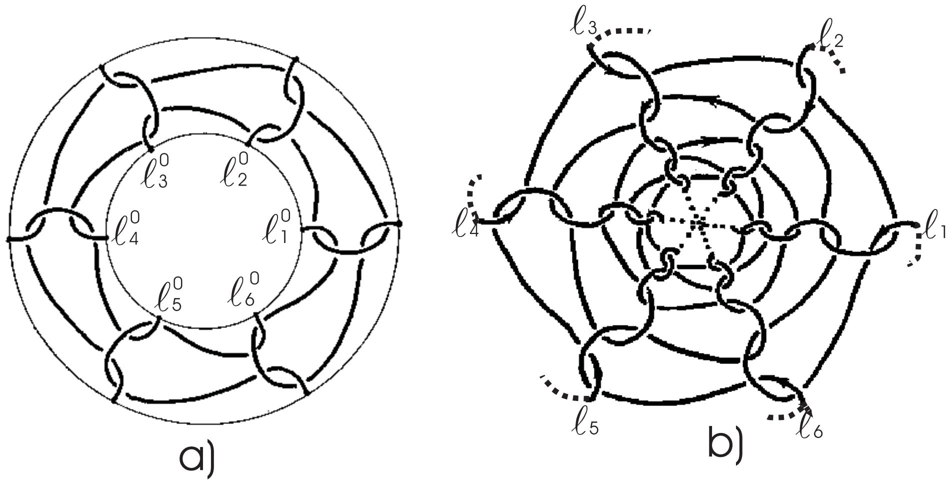

FIGURE 3. Debrunner–Fox example

Set $\ell_i = \bigcup_{k\in\mathbb{Z}} \phi^k(\ell_i^0)$ and $L_n = \bigcup_{i=1}^{n} \ell_i \cup O$. The projection of the frame L_n for the case $n = 6$ on the plane XOY is represented in Fig. 3b. According to [**3**], the frame L_n is mildly wild.

2. Inclusion of mildly wild frame in dynamic. Set $\mathbb{D}^2 = \{(y, z) \in \mathbb{R}^2 : y^2 + z^2 \leq 1\}$. Let's present solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ as the space of orbits of an action of the group $G = \{g^n, n \in \mathbb{Z}\}$ on $\mathbb{D}^2 \times \mathbb{R}^1$ where $g : \mathbb{D}^2 \times \mathbb{R}^1 \to \mathbb{D}^2 \times \mathbb{R}^1$ is a diffeomorphism given by the formula $g(x, y, z) = (x - 1, y, z)$. For each arc $\ell_i \in L_n$ we choose a ϕ-invariant tubular neighborhood $V(\ell_i)$ and a diffeomorphism $\zeta_i : V(\ell_i) \to \mathbb{D}^2 \times \mathbb{R}^1$ conjugating the diffeomorphisms $\phi|_{V(\ell_i)}$ and g.

The required diffeomorphism f we will construct modifying the diffeomorphism ϕ in the style of classical Cherry-flow whose support is contained in a compact 3-ball inside $V(\ell_i)$. Let's describe details.

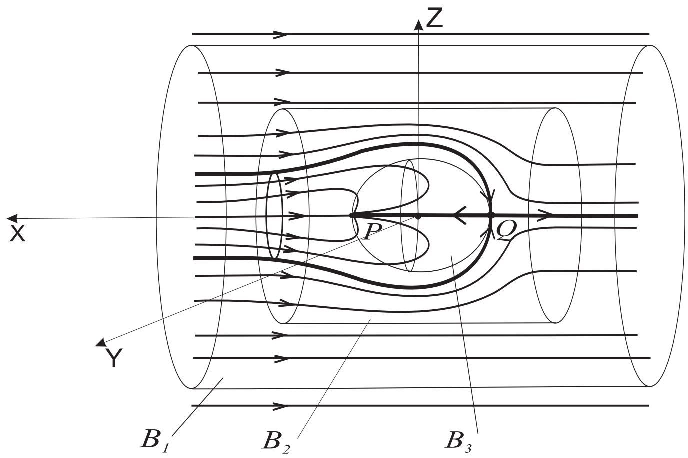

FIGURE 4. Cherry-flow

Let $\psi_1 : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function such that:

(a) $\psi_1(x, y, z) = -1$ for any points (x, y, z) outside of the cylinder $B_1 = \{(x, y, z) \in \mathbb{R}^3 : |x| \leq 1, y^2 + z^2 \leq \frac{1}{2}\}$;
(b) $\psi_1(x, y, z) < 0$ for any points (x, y, z) outside of a 3-ball $B_3 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1/16)\}$;
(c) $\psi_1(x, y, z) > 0$ for any points $(x, y, z) \in int\ B_3$;
(d) $\psi_1(x, y, z)$ is regular at points of 2-sphere ∂B_3;
(e) $|\frac{\partial \psi_1}{\partial x}(\pm\frac{1}{4}, 0, 0)| = 1$.

Let $\psi_2 : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function satisfying to the following conditions:

(a) $\psi_2(x, y, z) = 0$ for any points (x, y, z) outside of the cylinder B_1;
(b) $\psi_2(x, y, z) < 0$ for any points $(x, y, z) \in int\ B_1$;
(c) $\psi_2(x, y, z) = -1$ for any points (x, y, z) belonging to the cylinder $B_2 = \{(x, y, z) \in \mathbb{R}^3 : |x| \leq \frac{1}{2}, y^2 + z^2 \leq \frac{1}{4}\}$.

Flow X we define by the following system of the equations:

$$\begin{cases} \dot{x} = \psi_1(x, y, z) \\ \dot{y} = y \cdot \psi_2(x, y, z) \\ \dot{z} = z \cdot \psi_2(x, y, z). \end{cases}$$

The flow X has exactly two fixed points: a saddle at the point $Q = (-1/4, 0, 0)$ and a sink at the point $P = (1/4, 0, 0)$, both points are hyperbolic (see Fig. 4). Unstable separatrices of the point Q are the interval $\{|x| < \frac{1}{4}, y = 0, z = 0\}$, (laying in the basin of the attracting point P) and the set $\{x < -\frac{1}{4}, y = 0, z = 0\}$.

Let g_X be a shift of unit time of the flow X. By the construction g_X coincides with shift onf time unit of a flow $\frac{\partial}{\partial t}$ outside of the cylinder B_1, belonging to the interior $\mathbb{D}^2 \times \mathbb{R}$. Let's define the diffeomorphism $\bar{f} : \mathbb{R}^3 \to \mathbb{R}^3$ as follows:

(a) $\bar{f}$ coincides with ϕ outside $V(\ell_i)$.
(b) $\bar{f}$ coincides with $\zeta_i^{-1} \circ g_X \circ \zeta_i$ inside $V(\ell_i)$.

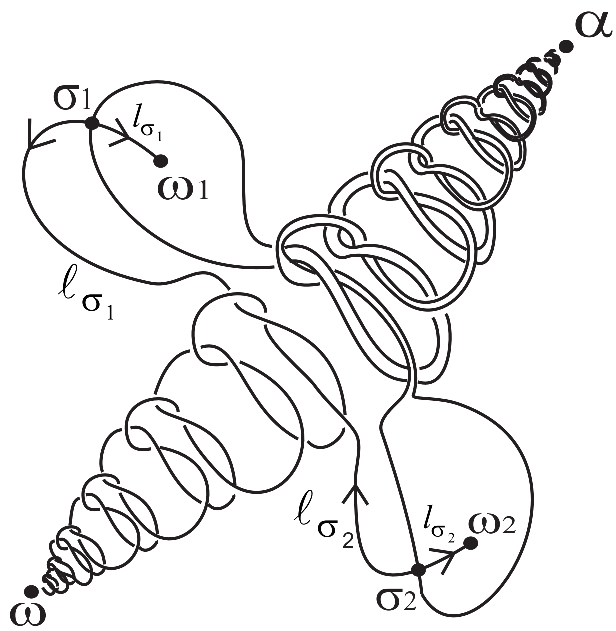

FIGURE 5. Phase portrait of the diffeomorphism $\Psi_2(\mathbb{S}^3)$ with mildly wild frame of separatrices

3. Projection on $\mathbb{S}^3$. Let's define the stereographic projection $\vartheta : \mathbb{S}^3 \setminus (0,0,0,1) \to \mathbb{R}^3$ by the formula $\vartheta(x_1,x_2,x_3,x_4) = (\frac{x_1}{1-x_4}, \frac{x_2}{1-x_4}, \frac{x_3}{1-x_4})$.
Set $f(x) = \begin{cases} \vartheta^{-1}(\bar{f}(\vartheta(x))), & x \in \mathbb{S}^3 \setminus (0,0,0,1); \\ x, & x = (0,0,0,1). \end{cases}$

By construction the non-wandering set of the constructed diffeomorphism f is fixed, hyperbolic and consists of $(n+1)$ sinks $\omega_1 = \vartheta^{-1}(\zeta_1^{-1}(P)), \dots, \omega_n = \vartheta^{-1}(\zeta_n^{-1}(P)), \omega = \vartheta^{-1}(O)$, n saddles $\sigma_1 = \vartheta^{-1}(\zeta_1^{-1}(Q)), \dots, \sigma_n = \vartheta^{-1}(\zeta_n^{-1}(Q))$ of index 1 and one source $(0,0,0,1)$. Thus, the diffeomorphism f belongs to the class $\Psi_n(\mathbb{S}^3)$ and has Debruner–Fox frame of separatrices. The phase portrait of the diffeomorphism for the case $n = 2$ is represented in Figure 5.

Remark 1. Notice that the construction above can be used for any frame $L_n(\omega)$, not only Debruner–Fox. For instance, by this construction we can realize Pixton's example (see Fig. 2).

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