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THE STRUCTURE OF POISSON COHOMOLOGY

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Guy Kass passed away while working on the present joint project.
The two other authors finalized the note so
to publish it in his memory and, through this,
to thank him for his friendship and commitment.

Abstract. In this paper, we survey the general approach to Lichnerowicz–Poisson cohomology, which was developed in [**8, 2, 1**], and detail the structure of this cohomology. Further, we depict an advantageous alternative cohomological method and apply this new device to the quadratic Poisson tensor Λ_3 of the Dufour–Haraki classification.

1. Introduction. Poisson Geometry was initiated by André Lichnerowicz, who wrote the definition of a Poisson manifold by means of a bivector field: a Poisson manifold is a smooth manifold M endowed with a Poisson tensor, i.e. with a skew-symmetric contravariant 2-tensor field Λ satisfying $[\Lambda, \Lambda] = 0$, where $[\cdot, \cdot]$ is the Schouten–Nijenhuis bracket. An antisymmetric contravariant 2-tensor Λ of a manifold M induces an operator $\partial_\Lambda := [\Lambda, \cdot]$ on the contravariant Grassmann algebra of M (that is on the associative graded commutative algebra of multivector fields of M). If Λ is a Poisson tensor, the operator ∂_Λ is a differential, i.e. $\partial_\Lambda^2 = 0$, called the Lichnerowicz–Poisson differential (or simply LP-differential) and the corresponding cohomology is the Lichnerowicz–Poisson cohomology (or LP-cohomology) of the considered Poisson manifold M.

LP-cohomology is a useful tool in Poisson Geometry, it plays an important role in Deformation and Quantization Theory, and attracts more and more

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interest among algebraists. For symplectic manifolds, the LP-cohomology is naturally isomorphic to the usual de Rham cohomology. Generally, computation of the LP-cohomology is quite demanding: in the Euclidean setting, while the de Rham cohomology is trivial, the LP-cohomology spaces of nonsymplectic Poisson structures are rather large.

It is well known that in some neighborhood of each point a Poisson manifold can be viewed as the product of a part of the usual symplectic manifold and a Poisson manifold, whose tensor vanishes at the considered point. As regards local classification, we thus deal with Poisson tensors in $\mathbb{R}^n$ that vanish at the origin. If we denote the coordinates by $(x_1, \dots, x_n)$, such a tensor, say Λ, reads

$$\Lambda(x) = \sum_{1\le i<j\le n} \Lambda_{ij}(x)\partial_i\wedge\partial_j = \sum_{1\le i<j\le n}\left(\sum_{1\le k\le n} c_{ij}^k x_k + \sum_{1\le k,\ell\le n} c_{ij}^{k\ell} x_k x_\ell + \dots\right)\partial_i\wedge\partial_j,$$

where $\partial_i = \frac{\partial}{\partial x_i}$. If the terms of order ≥ 2 all vanish, the considered Poisson tensors are in one-to-one correspondence with the Lie algebra structures on $(\mathbb{R}^n)^*$; the features of these Poisson structures were investigated by many authors during the last decades, see [**18**] for a historical survey. Having studied these linear Poisson structures, it is natural to take an interest in quadratic tensors:

$$\Lambda(x) = \sum_{1\le i<j\le n} \sum_{1\le k,\ell\le n} c_{ij}^{k\ell} x_k x_\ell \;\; \partial_i \wedge \partial_j.$$

Quadratic Poisson structures were classified by Dufour and Haraki [**4**] in the three-dimensional Euclidian setting; apart from some partial results in $\mathbb{R}^4$, the classification in higher dimension is still an open question. The cohomology of quadratic structures was computed in the Euclidian plane [**9**] or for specific cases [**7, 11**]. Recently, we suggested in [**8, 2**] a cohomological approach that allows to deal with the formal Poisson cohomology of all Poisson tensors of the Dufour–Haraki classification (DHC), and in [**1**] we proposed a generic cohomological technic for a large class of quadratic Poisson structures in arbitrary dimension.

The present note is organized as follows. In Section 2, we summarize the cohomological methods devised in the aforementioned papers [**8, 2, 1**]; in Section 3, we present a new, quite "user-friendly," cohomological modus operandi, and apply it to the tensor Λ_3 of the DHC.

2. Cohomological techniques and structure of Poisson cohomology. Masmoudi and Poncin developed in [**8**] a general approach to the formal Poisson cohomology of a broad set of isomorphism classes of the DHC, namely those that can be written as a linear combination of wedge products of mutually commuting linear vector fields. In the three-dimensional Euclidean context this means that

$$\Lambda = aY_2 \wedge Y_3 + bY_3 \wedge Y_1 + cY_1 \wedge Y_2, \quad a, b, c \in \mathbb{R}, \tag{1}$$

where the vector fields Y_1, Y_2, Y_3 are linear and satisfy the condition $[Y_i, Y_j] = 0$, for all $i, j \in \{1, 2, 3\}$. One of the most important features of such quadratic Poisson tensors is that they are induced by classical r-matrices, and so they can be quantized according to Gerstenhaber and Drinfeld without any use of Kontsevich's formula. More precisely, a quadratic Poisson tensor Λ of type (1) is implemented by a classical r-matrix that belongs to its stabilizer $\mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$ for the canonical matrix action, see [**1**] for more details. Below, such Poisson structures will be called "strongly r-matrix induced structures" (SRMI).

As for cohomology computations, the main advantage of SRMI Poisson tensors is related with the following observations. If we substitute the Y_i-vector fields for the standard basic vector fields ∂_i, the LP-cochains (multivector fields) read $C = \sum f\mathbf{Y}$, where f is a function and where $\mathbf{Y}$ is a wedge product of some Y_i. As Λ is a linear combination $\Lambda = \sum c\mathbf{Y}$, $c \in \mathbb{R}$, and since the vector fields Y_i are mutually commuting, the LP-differential then assumes a *simplified shape*:

$$\partial_\Lambda(f\mathbf{Y}) = [\Lambda, f\mathbf{Y}] = [\Lambda, f] \wedge \mathbf{Y} = \sum_{i=1}^{3} X_i(f) Y_i \wedge \mathbf{Y}, \tag{2}$$

where

$$X_1 = aY_2 - cY_3, \quad X_2 = bY_3 - aY_1, \quad X_3 = cY_1 - bY_2$$

and where the sums have been omitted.

If we take an interest in the *formal* cohomology, i.e. if we substitute the space $\mathcal{R} := \mathbb{R}[[x, y, z]] \otimes \wedge\mathbb{R}^3$ of multivector fields with coefficients in the formal series in the coordinates x, y, z for the usual LP-cochain space $\mathcal{X}(\mathbb{R}^3) = C^\infty(\mathbb{R}^3) \otimes \wedge\mathbb{R}^3$, the LP-cochains are actually of the form

$$C = \sum \frac{p}{D}\mathbf{Y}, \tag{3}$$

where p belongs to $\mathbb{R}[[x, y, z]]$ and D is a *fixed* degree 3 homogenous polynomial in x, y, z. Of course a sum as in (3) is a cochain in space $\mathcal{R}$ if and only if precise divisibility conditions are satisfied. Hence, a canonical injection

$$\mathcal{R} \hookrightarrow \mathcal{P} \tag{4}$$

of the space $\mathcal{R}$ into the space $\mathcal{P}$ that is constituted by all the sums of type (3). The elements of the space $\mathcal{R}$ are the real cochains, those of the space $\mathcal{P}$ are called the potential cochains. The LP-differential ∂_Λ extends in a natural way, as in the simplified version (2), to the potential cochain space $\mathcal{P}$, thus making it a differential space – whose cohomology is less intricate than the $\mathcal{R}$-cohomology. The remaining task is then to deduce the $\mathcal{R}$-cohomology from the $\mathcal{P}$-cohomology. This was done in the following way:

Let $p_{\mathcal{R}}$ and $p_{\mathcal{S}}$ be the projections that correspond to a splitting $\mathcal{P} = \mathcal{R} \oplus \mathcal{S}$. Define a differential on $\mathcal{S}$ by $\partial_{\mathcal{S}} = p_{\mathcal{S}}\partial_\Lambda$, and a homomorphism $\phi : \mathcal{S} \to \mathcal{R}$ of differential spaces by $\phi = p_{\mathcal{R}}\partial_\Lambda$. Eventually, we deal with a short exact sequence

$$(\mathcal{R}, \partial_\Lambda) \overset{i}{\hookrightarrow} (\mathcal{P}, \partial_\Lambda) \overset{p_{\mathcal{S}}}{\twoheadrightarrow} (\mathcal{S}, \partial_{\mathcal{S}})$$

of differential spaces and an *exact triangle in cohomology* (where the connecting homomorphism is the linear map $\phi_\sharp$ induced by ϕ). The triangle allows thus to compute the highly intricate LP-cohomology $H(\mathcal{R}, \partial_\Lambda)$ via the more accessible cohomologies $H(\mathcal{P}, \partial_\Lambda)$ and $H(\mathcal{S}, \partial_{\mathcal{S}})$. This cohomological method was applied in [**8**] to explicitly compute the cohomology of the structures Λ_2 and Λ_7 of the DHC.

In [**2**] we established a cohomological modus operandi for the isomorphism classes of the DHC that are not SRMI. We showed that any structure Λ of the DHC decomposes into the sum of a major SRMI structure and a small compatible Poisson tensor:

$$(5) \quad \Lambda = \Lambda_I + \Lambda_{II} = aY_2 \wedge Y_3 + bY_3 \wedge Y_1 + cY_1 \wedge Y_2 + \Lambda_{II}, \quad [\Lambda_I, \Lambda_{II}] = 0.$$

This splitting differs from the decomposition suggested in [**6**] in the sense that we incorporate the biggest possible part of the structure into the strongly induced term. We privilege decomposition (5), since Λ_{II} vanishes in many cases and the computing technique of [**8**] then allows to deal with the cohomology of Λ_I. If Λ is a SRMI tensor twisted by a small Poisson structure as in (5), the decomposition

$$\partial_\Lambda = [\Lambda, \cdot] = [\Lambda_I, \cdot] + [\Lambda_{II}, \cdot] = \partial_{\Lambda_I} + \partial_{\Lambda_{II}}, \quad \partial^2_{\Lambda_I} = \partial^2_{\Lambda_{II}} = \partial_{\Lambda_I}\partial_{\Lambda_{II}} + \partial_{\Lambda_{II}}\partial_{\Lambda_I} = 0,$$

where the last anticommutation relation corresponds to the compatibility condition $[\Lambda_I, \Lambda_{II}] = 0$, is nothing but the germ of a double complex. Of course, in order to get a double complex, we must define a bidegree on cochains. As a real formal Poisson cochain of order c, $c \in \{0, 1, 2, 3\}$, can be written in the form $C = \sum p\, \partial^c$, where p belongs to $\mathbb{R}[[x, y, z]]$ and ∂^c is a wedge product of c factors ∂_i, the degrees j_1, j_2, j_3 of x, y, z in the monomials of p and the cochain degree c induce a bigrading $r = j_1 + j_2 + c$, $s = j_3$ of the formal LP-cochain space $\mathcal{R}$. Of course, depending on the investigated Poisson tensor, different combinations of the natural degrees j_1, j_2, j_3, c are possible, but the preceding model encompasses (almost) all the twisted structures of the DHC. This "algebraic" bidegree differs from the "geometric" bigrading defined by Vaisman [**15**] for a regular Poisson manifold by means of the choice of a transversal distribution to the symplectic foliation. It turns out that the differentials ∂_I and ∂_{II} have the weights $(1,0)$ and $(-1,2)$ respectively, so that we obtain a vertically positive double complex. Such double complexes were partially studied in [**13**] and are graded filtered differential spaces; the associated spectral sequence (SpecSeq) provides the cohomology of Λ via computation of the "impact" of the small twist Λ_{II} on the second term of the SpecSeq, i.e. on the cohomology of the SRMI part Λ_I of Λ – which is accessible by the method of [**8**]. However, richness of Poisson cohomology requires computation through the whole SpecSeq, so that we had to construct a complete model of that sequence. Furthermore, we depicted in full detail all the isomorphisms involved in the theory of SpecSeq, because the quest for true upshots coerces us into reading our results through all these isomorphisms. Finally, the work [**2**] also provides quite useful practical insight into the operating mode of spectral sequences.

An extension of the computing technique of [**8**] to SRMI tensors of arbitrary dimensional vector spaces could eventually be given in [**1**]. More precisely, we investigated the formal Poisson cohomology associated with the quadratic Poisson tensors of $\mathbb{R}^n$ that read as real linear combination

$$\Lambda = \sum_{i<j} \alpha^{ij} Y_i \wedge Y_j, \quad \alpha^{ij} \in \mathbb{R}, \tag{6}$$

of wedge products of n commuting linear vector fields $Y_1, \ldots, Y_n$, such that $Y_1 \wedge \ldots \wedge Y_n \neq 0$.

Using some nontrivial combinatorics, we were able to inject the space $\mathcal{R} := \mathbb{R}[[x_1, \ldots, x_n]] \otimes \wedge\mathbb{R}^n$ of real LP-cochains into the larger space $\mathcal{P}$ of potential cochains

$$C = \sum \frac{p}{D} \mathbf{Y}, \tag{7}$$

where p belongs to $\mathbb{R}[[x_1, \ldots, x_n]]$ and D is the determinant $\det(Y_1, \ldots, Y_n)$. This injection generalizes the abovementioned much more obvious injection in the 3-dimensional setting. The LP-differential extends in a natural way from $\mathcal{R}$ to $\mathcal{P}$; this extension turns out to be a Koszul differential, which is defined by means of n commuting endomorphisms $X_i - (\operatorname{div} X_i)\,\mathrm{id}$, $X_i = \sum_j \alpha^{ij} Y_j$, $\alpha^{ji} = -\alpha^{ij}$, of the spaces $\mathcal{P}^r$ made up by the potential cochains (7), whose numerators p are homogeneous polynomials of degree r. We choose then again a space $\mathcal{S}$ that is supplementary to $\mathcal{R}$ in $\mathcal{P}$ and we show that the LP-differential induces a relative differential on $\mathcal{S}$. This allows decomposing the LP-cohomology into, basically, a Koszul cohomology and a relative cohomology:

The Lichnerowicz–Poisson cohomology of arbitrary dimensional SRMI Poisson tensors splits into a Koszul cohomology and a relative cohomology.

Furthermore, by means of a quite universal homotopy formula and the recent characterization of the joint spectrum of commuting operators established in [**3**], we could show that the Koszul cohomology – in our finite-dimensional setting $\mathcal{P}^r$ – is made up by "weak" joint eigenvectors. This upshot is subliminal in Spectral Theory and could so far not be made precise in the general infinite-dimensional context. For SRMI tensors, it is possible to supply a convergent algorithm that computes these joint eigenvectors and reduces the corresponding Koszul cohomology, i.e. the main building block of Poisson cohomology, to a problem of linear algebra.

Finally, these upshots and the previous works [**8**] and [**2**] deepen our insight into the structure of the LP-cohomology, in particular as concerns Casimir functions, i.e. functions that are constant along the symplectic leaves, and the cohomological impact of the singularities and the stabilizer of the considered Poisson tensor. Let us also mention that two relevant concepts of exactness are involved in the interpretation of three-dimensional Poisson cohomology. More precisely, on the one hand, a Poisson tensor is said to be Koszul exact, if it is a cycle and a boundary for the homology operator, which we obtain by just pulling the usual de Rham differential back to skew-symmetric contravariant tensor fields by means of a volume form (the image of a Poisson tensor Λ by this homology operator is the so-called curl of Λ, which is a Poisson 1-cocycle and whose class is the well-known modular class). On the other hand a Poisson tensor is of course a 2-cocycle for the Poisson cohomology operator and may thus be Poisson exact.

Richness of Poisson cohomology is proportional to the "distance" of the considered Poisson tensor to Poisson exactness and inversely proportional to its "distance" to Koszul exactness.

3. Advantageous approach to the cohomology of tensor Λ_3 of the DHC. As explained above, the computation of the LP-cohomology of SRMI tensors of the DHC is deduced from the more accessible, potential and relative cohomologies, i.e. the $\mathcal{P}$- and $\mathcal{S}$-cohomologies. In this Section, we provide a new cohomological method that allows, roughly speaking, to deduce the LP-cohomology directly from the $\mathcal{P}$-cohomology, i.e. without computing the $\mathcal{S}$-cohomology. We apply this new devise to compute the formal LP-cohomology of the structure

$$\Lambda_3 = (2x - ay)z\partial_2 \wedge \partial_3 + axz\partial_3 \wedge \partial_1 + x^2\partial_1 \wedge \partial_2, \quad a \in \mathbb{R},$$

of the DHC. In the following, we assume that $a \neq 0$ and denote the structure Λ_3 simply by Λ, if no confusion can arise. Some of the aforementioned notations and facts will be shortly and appropriately recalled or further developed to make this Section self-contained and to increase its readability.

3.1. Geometric description of potential cochains. Let us recall that zero-, one-, two-, and three-cochains may be written

$$C^0 = q,\;\; C^1 = q_1\partial_1 + q_2\partial_2 + q_3\partial_3,\;\; C^2 = q_1\partial_{23} + q_2\partial_{31} + q_3\partial_{12},\;\; C^3 = q\partial_{123},$$

where $q, q_1, q_2, q_3 \in \mathbb{R}[[x, y, z]]$ are formal series in x, y, z, and where $\partial_{ij} = \partial_i \wedge \partial_j$ for $i, j \in \{1, 2, 3\}$ and $\partial_{123} = \partial_1 \wedge \partial_2 \wedge \partial_3$. We will call such cochains *real cochains*.

If we set the family of commuting vector fields

$$Y_1 = x\partial_1 + y\partial_2,\;\; Y_2 = x\partial_2,\;\; Y_3 = z\partial_3, \tag{8}$$

the tensor Λ reads

$$\Lambda = 2Y_{23} + aY_{31} + Y_{12},$$

where $Y_{ij} = Y_i \wedge Y_j$ for $i, j \in \{1, 2, 3\}$. Obviously, the LP-differential $\partial_\Lambda = [\Lambda, -]$ simplifies if we also write the real cochains in terms of the Y_i-vector fields (8) since these fields are mutually commuting. As it is easily seen that

$$\partial_1 = \frac{xz}{D}Y_1 - \frac{yz}{D}Y_2,\;\; \partial_2 = \frac{xz}{D}Y_2,\;\; \partial_3 = \frac{x^2}{D}Y_3,$$

where $D = x^2 z$, the real cochains may be written as

$$C^0 = \frac{p}{D},\;\; C^1 = \frac{p}{D}Y_1 + \frac{p'}{D}Y_2 + \frac{p''}{D}Y_3,\;\; C^2 = \frac{p}{D}Y_{23} + \frac{p'}{D}Y_{31} + \frac{p''}{D}Y_{12},\;\; C^3 = \frac{p}{D}Y_{123},$$

where $p, p', p'' \in \mathbb{R}[[x, y, z]]$. Any such expression will be called a potential cochain.

For a fixed $r \in \mathbb{N}$, we identify any potential cochain coefficient

$$\frac{p}{D} = \frac{1}{D}\sum_{k+\ell \le r} p_{k,l}\, x^{r-k-l} y^k z^\ell, \quad p_{k,\ell} \in \mathbb{R},$$

with the point $(k, \ell) \in \mathbb{R}^2$. Thus such coefficients may be represented as elements of the space generated by points of a first quadrant triangle in the (k, ℓ)-plane:

$$\frac{1}{D}\sum_{k+\ell \le r} p_{k,l}\, x^{r-k-l} y^k z^l \leftrightarrow (p_{k,\ell}) \in \oplus_{k+\ell \le r}\mathbb{R}^{k\ell},$$

where $\mathbb{R}^{k\ell} = \langle (k, \ell) \rangle$ is the space generated by the point (k, ℓ). In order to simplify explanations, we introduce the following geometrically motivated notations. We denote by $\triangle^r$ the space $\oplus_{k+\ell \le r}\mathbb{R}^{k\ell}$ of r-triangles, by $\triangle^r_s$ for all $s \le r$ the space $\oplus_{k+\ell \le s}\mathbb{R}^{k\ell}$ of lower triangles, by $\mathcal{D}^r_s$ the diagonals subspaces $\oplus_{k+\ell=s}\mathbb{R}^{k\ell}$, by $\mathcal{L}^r_s$ the lines subspace $\oplus_{k=0}^{r-s}\mathbb{R}^{ks}$, and by $\mathcal{C}^r_s$ the columns subspace $\oplus_{\ell=0}^{r-s}\mathbb{R}^{s\ell}$. Finally, to simplify notation, we denote $\lozenge^{k\ell}_r$ the quotient $\frac{x^{r-k-l}y^k z^\ell}{D}$ for any (k, ℓ).

Lemma 1. *For a fixed $r \in \mathbb{N}$, let*

(9) $C^0 = \pi,\;\; C^1 = \pi_1 Y_1 + \pi_2 Y_2 + \pi_3 Y_3,\;\; C^2 = \pi_1 Y_{23} + \pi_2 Y_{31} + \pi_3 Y_{12},\;\; C^3 = \pi Y_{123},$

be potential cochains of degree r, and

$$\pi_1 \leftrightarrow (p_{k,\ell}) \in \langle \triangle^r \rangle, \;\; \pi_2 \leftrightarrow (p'_{k,\ell}) \in \langle \triangle^r \rangle, \;\; \pi_3 \leftrightarrow (p''_{k,\ell}) \in \langle \triangle^r \rangle.$$

The real coefficients $p_{k,\ell}, p'_{k,\ell}$ and $p''_{k,\ell}$ are assumed to be zero if at least one index is strictly negative.

1. The 0-cochain C^0 is real if and only if $r \geq 3$ and $\pi \in \langle \triangle^r_{r-2} - \mathcal{L}^r_0 \rangle$.

2. The 1-cochain C^1 is real if and only if $r \geq 2$ and

$$\pi_1 \in \langle \triangle^r_{r-1} - \mathcal{L}^r_0 \rangle, \; \pi_2 \in \langle \triangle^r - \mathcal{L}^r_0 - \{(0,r)\} \rangle, \; \pi_3 \in \langle \triangle^r_{r-2} \rangle$$

$$p'_{k,\ell} = -p_{k-1,\ell} \;\; if \;\; k+\ell = r.$$

3. The 2-cochain C^2 is real if and only if $r \geq 1$ and

$$\pi_1 \in \langle \triangle^r - \{(0,r)\} \rangle, \; \pi_2 \in \langle \triangle^r_{r-1} \rangle, \; \pi_3 \in \langle \triangle^r - \mathcal{L}^r_0 \rangle$$

$$p'_{k,\ell} = p_{k+1,\ell} \;\; if \;\; k+\ell = r-1.$$

4. The 3-cochain C^3 is always real.

Proof. The 0-cochain C^0 writes

$$C^0 = \sum_{k+\ell \leq r} p_{k,\ell} \Diamond^{k\ell} = \sum_{k+\ell \leq r} p_{k,\ell} x^{r-k-l-2} y^k z^{\ell-1},$$

so it is real if and only if $r \geq 3$ and $p_{k,\ell} = 0$ if $k+\ell > r-2$ or $\ell = 0$.

The 1-cochain C^1 writes $C^1 = x\pi_1\partial_1 + (y\pi_1 + \pi_2 x)\partial_2 + z\pi_3\partial_3$, so it is real if and only if $x\pi_1$, $y\pi_1 + \pi_2 x$, $z\pi_3$ are polynomials. Clearly $x\pi_1 = \sum_{k+\ell \leq r} p_{k,\ell} x^{r-k-l-1} y^k z^{\ell-1}$ is polynomial if and only if $p_{k,\ell} = 0$ if $k+\ell = r$ or $\ell = 0$. In that case we have

$$y\pi_1 + x\pi_2 = \sum_{k+\ell \leq r} (p_{k-1,\ell} + p'_{k,\ell}) x^{r-k-l-1} y^k z^{\ell-1},$$

so $y\pi_1 + x\pi_2$ is polynomial if and only if $p'_{k,\ell} + p_{k-1,\ell} = 0 \;\; if \;\; k+\ell = r$ or $\ell = 0$, that is $p'_{k,\ell} = 0$ if $\ell = 0$ and $p'_{k,\ell} + p_{k-1,\ell} = 0 \;\; if \;\; k+\ell = r$ (in particular, $p'_{0r} = 0$). Eventually, $z\pi_3 = \sum_{k+\ell \leq r} p''_{k,\ell} x^{r-k-l-2} y^k z^\ell$ is polynomial if and only if $k+\ell \geq r-2$.

The proof of point 3 is based upon similar arguments.

The 3-cochain C^3 writes $x^2 z\pi\partial_{123}$, so it is real if and only if $x^2 z\pi$ is polynomial, which is always the case. $\square$

3.2. Fundamental operators. It is easily seen that the coboundaries of zero-, one-, two-potential cochains are respectively given by

$$\begin{aligned}
[\Lambda, \pi] &= X_1(\pi)Y_1 + X_2(\pi)Y_2 + X_3(\pi)Y_3,\\
[\Lambda, \pi_1 Y_1 + \pi_2 Y_2 + \pi_3 Y_3] &= (X_2(\pi_3) - X_3(\pi_2))Y_{23} + (X_3(\pi_1) - X_1(\pi_3))Y_{31}\\
&\quad + (X_1(\pi_2) - X_2(\pi_1))Y_{12},\\
[\Lambda, \pi_1 Y_{23} + \pi_2 Y_{31} + \pi_3 Y_{12}] &= (X_1(\pi_1) + X_2(\pi_2) + X_3(\pi_3))Y_{123},
\end{aligned}$$

where

$$X_1 = Y_2 - aY_3, \quad X_2 = -Y_1 + 2Y_3, \quad X_3 = aY_1 - 2Y_2.$$

These operators are called the fundamental operators. For a fixed $r \in \mathbb{N}$, they satisfy the following relations

$$\begin{aligned}
X_1\left(\sum_{k+\ell\le r} p_{k,\ell}\lozenge_r^{k\ell}\right) &= \sum_{k+\ell\le r-1}((k+1)p_{k+1,\ell} - a(\ell-1)p_{k,\ell})\lozenge_r^{k\ell}\\
&\quad - \sum_{k+\ell=r} a(\ell-1)p_{k,\ell}\lozenge_r^{k\ell},\\
X_2\left(\sum_{k+\ell\le r} p_{k,\ell}\lozenge_r^{k\ell}\right) &= \sum_{k+\ell\le r}(3\ell - r)p_{k,\ell}\lozenge_r^{k\ell},\\
X_3\left(\sum_{k+\ell\le r} p_{k,\ell}\lozenge_r^{k\ell}\right) &= \sum_{k+\ell\le r-1}(a(r-\ell-2)p_{k,\ell} - 2(k+1)p_{k+1,\ell})\lozenge_r^{k\ell}\\
&\quad + \sum_{k+\ell=r} a(r-\ell-2)p_{k,\ell}\lozenge_r^{k\ell}.
\end{aligned} \tag{10}$$

These Equations show that the LP-differential preserves the total degree, i.e. the sum of the just defined individual degrees in x, y, z. Hence the cohomology can be computed part by part and the numerators p, p', p'' of the cochain coefficients, see (9), can be viewed as homogeneous polynomials of total degree $r \in \mathbb{N}$. Let us remark that the LP-differential conserves not only the space of r-triangles $\triangle^r$ but also every lines subspace $\mathcal{L}_s^r$.

From Equation (10) we easily deduce the following

Lemma 2. *Let $r \in \mathbb{N}$.*

1. *X_1 is an automorphism of every lines subspace $\langle \mathcal{L}_s^r \rangle$ with $s \neq 1$. For $r \ge 1$ we have*

$$\ker(X_1) = \langle (0,1) \rangle, \quad \mathrm{Im}(X_1) = \langle \triangle^r - \{(r-1,1)\} \rangle.$$

2. *If $r \notin 3\mathbb{N}$, X_2 is an automorphism of the space $\triangle^r$.*
If $r = 3n$ with $n \in \mathbb{N}$, X_1 is an automorphism of every lines subspace $\langle \mathcal{L}_s^{3s} \rangle$ with $s \neq n$; furthermore
$$\ker(X_2) = \langle \mathcal{L}_n^{3n} \rangle, \ \ \mathrm{Im}(X_2) = \langle \triangle^{3n} - \mathcal{L}_n^{3n} \rangle.$$
3. *X_3 is an automorphism of every lines subspace $\langle \mathcal{L}_s^r \rangle$ with $s \neq 2$. For $r \geq 2$ we have*
$$\ker(X_3) = \langle (0, r-2) \rangle, \ \ \mathrm{Im}(X_3) = \langle \triangle^r - \{(2, r-2)\} \rangle.$$

3.3. 0-Cohomology group. Let $C = \pi = \frac{p}{D}$ be a real 0-cochain of degree r: $r \geq 3$ and $\pi \in \langle \triangle_{r-2}^r - \mathcal{L}_0^r \rangle$. The cocycle condition $[\Lambda, \pi] = 0$ is $\pi \in \cap_{i=1}^3 \ker(X_i)$. For $r \neq 3$, we have $\ker(X_1) \cap \ker(X_3) = \{0\}$, and for $r = 3$, $\cap_{i=1}^3 \ker(X_i) = \mathbb{R}$, so we get the 0-cohomology space.

Proposition 1. *The 0-cohomology space of the Poisson structure Λ_3 is*
$$H^0(\Lambda_3) = \mathbb{R}.$$

3.4. 1-Cohomology group. A real 1-cocycle $C = \pi_1 Y_1 + \pi_2 Y_2 + \pi_3 Y_3$ of degree r, $r \geq 1$, is a 1-cochain that satisfies both, the divisibility condition
$$\pi_1 \leftrightarrow (p_{k,\ell}) \in \langle \triangle_{r-1}^r - \mathcal{L}_0^r \rangle, \quad \pi_2 \leftrightarrow (p'_{k,\ell}) \in \langle \triangle^r - \mathcal{L}_0^r - \{(0,r)\} \rangle,$$
$$\pi_3 \leftrightarrow (p''_{k,\ell}) \in \langle \triangle_{r-2}^r \rangle, \qquad p'_{k,\ell} = -p_{k-1,\ell} \ \text{ if } \ k + \ell = r,$$
and the cocycle condition
$$X_2(\pi_3) = X_3(\pi_2), \ X_3(\pi_1) = X_1(\pi_3), \ X_1(\pi_2) = X_2(\pi_1). \tag{11}$$

A real 1-coboundary of degree r, $r \geq 3$, is a real 1-cocycle that reads
$$X_1(\varpi) Y_1 + X_2(\varpi) Y_2 + X_3(\varpi) Y_3, \quad \varpi \in \langle \triangle_{r-2}^r - \mathcal{L}_0^r \rangle. \tag{12}$$

Let $C = \pi_1 Y_1 + \pi_2 Y_2 + \pi_3 Y_3$ be a real 1-cocycle of degree r.

Assume that $r = 2$. Then $\pi_1 \in \langle (0,1) \rangle$, $\pi_2 \in \langle (0,1), (1,1) \rangle$, $\pi_3 \in \langle (0,0) \rangle$. The cocycle condition $X_3(\pi_1) = X_1(\pi_3)$ implies that $X_3(\pi_1) = X_1(\pi_3) = 0$, so $\pi_1 = \pi_3 = 0$. By using the condition $X_1(\pi_2) = X_2(\pi_1) = 0$, we get $\pi_2 = 0$. So there is no real nontrivial 1-cocycle of degree 2.

Assume that $r \geq 3$. The cocycle condition $X_3(\pi_1) = X_1(\pi_3)$ implies that $\pi_3 \in \langle \triangle_{r-2}^r - \mathcal{L}_0^r \rangle$. By subtracting from the cocycle C every coboundary of the form (12), the remaining real 1-cocycles are $C' = \pi_1 Y_1 + \pi_2 Y_2 + \pi_3 Y_3$ with $\pi_3 \in \langle (0, r-2) \rangle$. Clearly $X_1(\pi_3)$ and $X_2(\pi_3)$ belong to $\langle (0, r-2) \rangle$, and the cocycle conditions $X_3(\pi_1) = X_1(\pi_3)$ and $X_2(\pi_3) = X_3(\pi_2)$ imply that π_1 and π_2 belong to $\langle \{(0, r-2), (1, r-2)\} \rangle$. Consequently, we get from the divisibility condition $p'_{k,\ell} = -p_{k-1,\ell}$ (for any $k + \ell = r$) that π_1 belongs to $\langle (0, r-2) \rangle$, and from the condition $X_1(\pi_2) = X_2(\pi_1) \in \langle (0, r-2) \rangle$ that π_2 belongs also to $\langle (0, r-2) \rangle$. Hence, the remaining real 1-cocycles are therefore $C' = \pi_1 Y_1 + \pi_2 Y_2 + \pi_3 Y_3$ with $\pi_1, \pi_2, \pi_3 \in \langle (0, r-2) \rangle$.

If $r \geq 4$, the condition $X_1(\pi_3) = X_3(\pi_1)$ implies that $X_1(\pi_3) = 0$, so $\pi_3 = 0$. By subtracting from the cocycle C' every coboundary of the form $X_1(\varpi)Y_1 + X_2(\varpi)Y_2$ with $\varpi \in \langle(0, r-2)\rangle$, the remaining cocycles are $\pi_2 Y_2$ with $\pi_2 \in \langle(0, r-2)\rangle$. Finally, as the cocycle condition $X_1(\pi_2) = X_2(\pi_1) = 0$ implies that $\pi_2 = 0$, then every real 1-cocycle of degree $r \geq 4$ is a real 1-coboundary.

If $r = 3$, then the remaining real 1-cocycles are $C' = \pi_1 Y_1 + \pi_2 Y_2 + \pi_3 Y_3$ with $\pi_1, \pi_2, \pi_3 \in \langle(0,1)\rangle$. Obviously, the cocycles C' are real 1-cochains of the space $\mathbb{R}Y_1 \oplus \mathbb{R}Y_2 \oplus \mathbb{R}Y_3$. Note that real 1-coboundaries of the form (12) are trivial because, in this case, ϖ lies in the space $\langle(0,1)\rangle$ which is the intersection of the kernels of the operators X_1, X_2, X_3. We conclude therefore that the space $\mathbb{R}Y_1 \oplus \mathbb{R}Y_2 \oplus \mathbb{R}Y_3$ constitutes the 1-cohomology space.

Proposition 2. *The 1-cohomology space of the Poisson structure Λ_3 is*

$$H^1(\Lambda_3) = \mathbb{R}Y_1 \oplus \mathbb{R}Y_2 \oplus \mathbb{R}Y_3.$$

3.5. 2-Cohomology group. A real 2-cocycle $C = \pi_1 Y_{23} + \pi_2 Y_{31} + \pi_3 Y_{12}$ of degree r, $r \geq 1$, is a 2-cochain C that satisfies both, the divisibility condition

$$\begin{aligned} &\pi_1 \leftrightarrow (p_{k,\ell}) \in \langle\triangle^r - \{(0,r)\}\rangle, && \pi_2 \leftrightarrow (p'_{k,\ell}) \in \langle\triangle^r_{r-1}\rangle, \\ &\pi_3 \leftrightarrow (p''_{k,\ell}) \in \langle\triangle^r - \mathcal{L}^r_0\rangle, && p'_{k,\ell} = p_{k+1,\ell} \quad \text{if} \quad k+\ell = r-1, \end{aligned}$$

and the cocycle condition

$$X_1(\pi_1) + X_2(\pi_2) + X_3(\pi_3) = 0. \tag{13}$$

A real 2-coboundary of degree r, $r \geq 2$, is a real 2-cocycle that reads

$$(X_2(\varpi_3) - X_3(\varpi_2))Y_{23} + (X_3(\varpi_1) - X_1(\varpi_3))Y_{31} + (X_1(\varpi_2) - X_2(\varpi_1))Y_{12}, \tag{14}$$

where $\varpi_1 Y_1 + \varpi_2 Y_2 + \varpi_3 Y_3$ is a real 1-cochain of degree r, i.e.

$$\begin{aligned} &\varpi_1 \leftrightarrow (\varpi_{k,\ell}) \in \langle\triangle^r_{r-1} - \mathcal{L}^r_0\rangle, && \varpi_2 \leftrightarrow (\varpi'_{k,\ell}) \in \langle\triangle^r - \mathcal{L}^r_0 - \{(0,r)\}\rangle, \\ &\varpi_3 \leftrightarrow (\varpi''_{k,\ell}) \in \langle\triangle^r_{r-2}\rangle, && \varpi'_{k,\ell} = -\varpi_{k-1,\ell} \quad \text{if} \quad k+\ell = r. \end{aligned}$$

Let $C = \pi_1 Y_{23} + \pi_2 Y_{31} + \pi_3 Y_{12}$ be a real 1-cocycle of degree r.

Assume that $r = 1$. Then $\pi_1 \in \langle\{(0,0),(1,0)\}\rangle$, $\pi_2 \in \langle(0,0)\rangle$ and $\pi_3 \in \langle(0,1)\rangle$. Obviously, the cocycle condition $X_1(\pi_1) + X_2(\pi_2) + X_3(\pi_3) = 0$ shows that $\pi_1 \in \langle(0,0)\rangle$, and the divisibility condition implies that $\pi_2 = 0$. By using again the cocycle condition, we have $\pi_1 = 0$ and $\pi_3 = 0$. So there is no nontrivial real 2-cocycle of degree 1.

Assume that $r \geq 2$. As π_3 does not belong to $\mathcal{L}^r_0$, and as $p'_{r,0} = 0$, the cocycle condition implies that $p_{r,0} = 0$. The divisibility condition entails then that $p'_{r-1,0} = 0$, and so, by using again the cocycle condition, we have $p_{r-1,0} = 0$. By subtracting from the cocycle C every coboundary of the form

$$X_2(\varpi_3)Y_{23} - X_1(\varpi_3)Y_{31}, \quad \varpi_3 \in \langle\mathcal{L}^r_0 - \{(r,0),(r-1,0)\}\rangle,$$

the remaining cocycles are $C' = \pi_1 Y_{23} + \pi_2 Y_{31} + \pi_3 Y_{12}$ with $\pi_2 \in \langle \triangle^r_{r-1} - \mathcal{L}^r_0 \rangle$. Now the cocycle condition shows that π_1 vanishes on $\mathcal{L}^r_0$, so the remanning cocycles are $C' = \pi_1 Y_{23} + \pi_2 Y_{31} + \pi_3 Y_{12}$ with

$$\pi_1 \in \langle \triangle^r - \mathcal{L}^r_0 - \{(0,r)\} \rangle,\ \pi_2 \in \langle \triangle^r_{r-1} - \mathcal{L}^r_0 \rangle,\ \pi_3 \in \langle \triangle^r - \mathcal{L}^r_0 \rangle,$$
$$p'_{k,\ell} = p_{k+1,\ell} \quad \text{if} \quad k + \ell = r - 1.$$

We now subtract from the cocycle C' every 2-coboundary of the form

$$-X_3(\varpi_2) Y_{23} + X_3(\varpi_1) Y_{31} + (X_1(\varpi_2) - X_2(\varpi_1)) Y_{12},$$

where

$$\varpi_1 \leftrightarrow (\varpi_{k,\ell}) \in \langle \triangle^r_{r-1} - \mathcal{L}^r_0 \rangle,\ \varpi_2 \leftrightarrow (\varpi'_{k,\ell}) \in \langle \triangle^r - \mathcal{L}^r_0 - \{(0,r)\} \rangle,$$
$$\varpi'_{k,\ell} = -\varpi_{k-1,\ell} \ \text{ if } \ k + \ell = r.$$

The remaining cocycles are $C'' = \pi_1 Y_{23} + \pi_2 Y_{31} + \pi_3 Y_{12}$ with $\pi_1 \in \langle (2, r-2) \rangle$. Observe that the divisibility condition $p'_{k,\ell} = p_{k+1,\ell}$ (for any $k + \ell = r - 1$) entails then that π_2 belongs to $\langle \{\triangle^r_{r-2}, (1, r-2)\} - \mathcal{L}^r_0 \rangle$. If we now subtract from the cocycle C'' every coboundary of the form

$$X_3(\varpi_1) Y_{31} - X_2(\varpi_1) Y_{12}, \quad \varpi_1 \in \langle \triangle^r_{r-2} - \mathcal{L}^r_0 \rangle,$$

then the remanning cocycles are $C''' = \pi_1 Y_{23} + \pi_2 Y_{31} + \pi_3 Y_{12}$ with $\pi_2 \in \langle \{(0, r-2), (1, r-2)\} \rangle$. Obviously, $X_1(\pi_1) \in \langle \{(1, r-2), (2, r-2)\} \rangle$ and $X_2(\pi_2) \in \langle \{(0, r-2), (1, r-2)\} \rangle$, and the cocycle condition $X_1(\pi_1) + X_2(\pi_2) + X_3(\pi_3) = 0$ shows that $\pi_3 \in \langle \mathcal{L}^r_{r-2} \rangle$.

We conclude that the remaining cocycles are $C'' = \pi_1 Y_{23} + \pi_2 Y_{31} + \pi_3 Y_{12}$ with $\pi_1 \in \langle (2, r-2) \rangle$, $\pi_2 \in \langle \mathcal{L}^r_{r-2} - \{(2, r-2)\} \rangle$ and $\pi_3 \in \langle \mathcal{L}^r_{r-2} \rangle$.

If $r \geq 2$ and $r \neq 3$, the cocycle condition implies that $X_1(\pi_1) = 0$, so $\pi_1 = 0$. Consequently, the divisibility condition shows that π_2 belongs to $\langle (0, r-2) \rangle$, and the cocycle condition entails that $\pi_3 \in \langle \mathcal{L}^r_{r-2} - \{(2, r-2)\} \rangle$. As X_1 is an automorphism of $\langle \mathcal{L}^r_{r-2} \rangle$, then there exist $\varpi_2 \in \langle \mathcal{L}^r_{r-2} - \{(2, r-2)\} \rangle$ and $\varpi_3 \in \langle (0, r-2) \rangle$ such that $X_1(\varpi_2) = \pi_3$ and $X_1(\varpi_3) = \pi_2$. Now the cocycle condition $X_2(\pi_2) + X_3(\pi_3) = 0$ implies that $X_2(\varpi_3) - X_3(\varpi_2) = 0$, so the real 2-coboundary

$$0\, Y_{23} - X_1(\varpi_3) Y_{31} + X_1(\varpi_2) Y_{12}$$

is equal to the cocycle C'''. Consequently, every real 1-cocycle of degree $r \geq 2$ and $r \neq 3$ is a real 2-coboundary.

If $r = 3$ we have

$$\pi_1 \in \langle (2,1) \rangle, \quad \pi_2 \in \langle \{(0,1), (1,1)\} \rangle, \quad \pi_3 \in \langle \{(0,1), (1,1), (2,1)\} \rangle = \mathcal{L}^3_1,$$
$$p'_{11} = p_{2,1}.$$

The cocycle condition $X_1(\pi_1) + X_2(\pi_2) + X_3(\pi_3) = 0$ implies that $\pi_3 \in \langle\{(0,1),(2,1)\}\rangle$ and $p_{21} = 2p''_{2,1}$. Hence, the remanning cocycles C''' read

$$\begin{aligned}
C''' &= \pi_1 Y_{23} + \pi_2 Y_{31} + \pi_3 Y_{12} \\
&= 2p''_{21}\frac{y^2z}{D}Y_{23} + \left(p'_{01}\frac{x^2z}{D} + 2p''_{21}\frac{xyz}{D}\right)Y_{31} + \left(p''_{01}\frac{x^2z}{D} + p''_{21}\frac{y^2z}{D}\right)Y_{12} \\
&= p'_{01}Y_{31} + p''_{01}Y_{12} + p''_{21}(2yz\partial_{31} + y^2\partial_{12}).
\end{aligned}$$

As the operators X_1, X_2, and X_3 preserve each line subspace of Δ^3, and in particular $\mathcal{L}_1^3$, the coboundary condition for the cocycles C''' is given by

$$X_2(\varpi_3) - X_3(\varpi_2) = \pi_1, \;\; X_3(\varpi_1) - X_1(\varpi_3) = \pi_2, \;\; X_1(\varpi_2) - X_2(\varpi_1) = \pi_3,$$

where

$$\varpi_1 \in \langle \mathcal{L}_1^3 - \{(2,1)\}\rangle, \;\; \varpi_2 \in \langle \mathcal{L}_1^3 \rangle, \;\; \varpi_3 \in \langle\{(0,1)\}\rangle, \;\; \varpi_{11} = -\varpi'_{21}.$$

But, as $X_2 = 0$ on $\langle \mathcal{L}_1^3\rangle$, and as the kernels of the operators X_1 and X_3 are equal to $\langle(0,1)\rangle$, then the coboundary condition is actually given by

$$-X_3(\varpi_2) = \pi_1, \;\; X_3(\varpi_1) = \pi_2, \;\; X_1(\varpi_2) = \pi_3, \tag{15}$$

where

$$\varpi_1 \in \langle(1,1)\rangle, \;\; \varpi_2 \in \langle\{(1,1),(2,1)\}\rangle, \;\; \varpi_{11} = -\varpi'_{21}.$$

Observe that the unique solution of the first equality of (15) is given by $\varpi_2 = 0$ because π_1 belongs to $\langle(2,1)\rangle$ whereas $X_3(\varpi_2)$ belongs to $\langle\{(0,1),(1,1)\}\rangle$. Now the divisibility condition implies then that $\varpi_1 = 0$, so cocycles C''' belong to the cohomology.

Proposition 3. *The 2-cohomology space of the Poisson structure Λ_3 is*

$$H^2(\Lambda_3) = \mathbb{R}Y_{31} \oplus \mathbb{R}Y_{12} \oplus \mathbb{R}(2yz\partial_{31} + y^2\partial_{12}).$$

3.6. 3-Cohomology group. Let $C = \pi Y_{123}$, with $\pi = \frac{p}{D}$, be a potential 3-cochain of degree r. Then C is real and satisfies the cocycle condition $[\Lambda, C] = 0$.

Assume that $r = 0$. The remaining cocycles are πY_{123} with $\pi \in \langle(0,0)\rangle$. As there is no real 2-coboundary of degree 0, the cocycles $\pi Y_{123} = p\partial_{123}$, $p \in \mathbb{R}$, belong to the 3-cohomology space.

Assume that $r \geq 1$. The coboundary condition is

$$\pi = X_1(\varpi_1) + X_2(\varpi_2) + X_3(\varpi_3),$$

where $\varpi_1 Y_{23} + \varpi_2 Y_{31} + \varpi_3 Y_{12}$ is a real 2-cochain of degree r. By subtracting from C every coboundary of the form $X_3(\varpi_3)Y_{123}$ with $\varpi_3 \in \langle \Delta^r - \mathcal{L}_0^r\rangle$ we get the remaining cocycles $C' = \pi Y_{123}$, where $\pi \in \langle\{\mathcal{L}_0^r, (2, r-2)\}\rangle$. We now subtract from C' every coboundary of the form $X_1(\varpi_1)Y_{123}$ with $\varpi_1 \in \langle \mathcal{L}_0^r -$

$\{(r,0)\}\rangle$. The remaining cocycles are $C'' = \pi Y_{123}$ with $\pi \in \langle\{(r,0),(2,r-2)\}\rangle$. When considering the real 2-cochain

$$B := \frac{p_{r,0}}{a}\left(\frac{y^r}{D}Y_{23} + \frac{xy^{r-1}}{D}Y_{31}\right),$$

it is easy to check that $C'' = \widetilde{\pi}Y_{123} + [\Lambda, B]$ with $\widetilde{\pi} \leftrightarrow (\widetilde{p}) \in \langle(2, r-2)\rangle$.

If $r \neq 3$, it is easily seen that there exists a real 2-cochain B' such that $C'' = [\Lambda, B + B']$, i.e. C'' is a coboundary. In fact the real 2-cochain B' is given by

$$B' := \left(\alpha\frac{x^2z^{r-2}}{D} + \beta\frac{xyz^{r-2}}{D} + \frac{\widetilde{p}}{a(3-r)}\frac{y^2z^{r-2}}{D}\right)Y_{23} + \frac{\widetilde{p}}{a(3-r)}\frac{xyz^{r-2}}{D}Y_{31},$$

where

$$\alpha = a\beta(r-3), \quad \beta = \frac{2(2-r)\widetilde{p}}{a^2(r-3)^2}.$$

If $r = 3$, then $\widetilde{\pi} \in \langle(2,1)\rangle$. As $\langle(2,1)\rangle$ does not belong to the any of the image spaces of the operators X_i, then cocycles πY_{123} are not coboundaries. So the cocycles $\widetilde{p}\frac{y^2z}{D}Y_{123} = \widetilde{p}y^2z\partial_{123}$ belong to the 3-cohomology space.

Proposition 4. *The 3-cohomology space of the Poisson structure Λ_3 is*

$$H^3(\Lambda_3) = \mathbb{R}\,\partial_{123} \oplus \mathbb{R}\,y^2z\,\partial_{123}.$$

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