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ON INFINITESIMAL ORBIT TYPES OF A NORMALIZABLE ISOMETRIC ACTION ON A LORENTZ MANIFOLD

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Abstract. Let $G \times M \to M$ be an isometric action of a Lie group G on a semi-Riemannian manifold (M, g). If M is Riemannian, or M is Lorentzian but the action is normalizable, then there is a unique infinitesimal orbit type, such that the orbits belonging to this type build an open and dense set in M. Moreover in the Lorentzian case a non-normalizable orbit $G(x)$ has lightlike tangent spaces and for every point $p \in G(x)$ there is a 1-parameter subgroup in G such that its orbit at p yield a lightlike geodesic segment through p, which is contained in $G(x)$.

1. Introduction. The celebrated principal orbit type theorem due to D. Montgomery, H. Samelson and C. T. Yang [**2**] says, that if we have a compact Lie group G and a differentiable action of this group on a connected differentiable manifold M, then among the orbit types there is a unique one, the so called principal type, for which the orbits belonging to this type build an open, dense and connected set in M. In particular orbits of maximal dimension have principal orbit type. Now this theorem doesn't hold in general if the Lie group is not compact. For example if we take the pseudo-orthogonal group $SO(2,1)$ and its action on the 3-dimensional Minkowski space, then the orbits in the interior of the lightcone and the orbits in the exterior of the lightcone will belong to different orbit types, since the stabilizer G_v of a timelike vector v in the interior of the lightcone is conjugated to the compact group $O(2)$ and the stabilizer G_w of a spacelike vector w in the exterior of the lightcone is conjugated to the non-compact group $SO(1,1)$. In this example the action is isometric. So it is not true in general that for an isometric action of an

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arbitrary Lie group on a semi-Riemannian manifold the principal orbit type theorem holds. But if we use the concept of proper action introduced by R. S. Palais then it can be shown, that for a proper isometric action of an arbitrary connected Lie group on a connected Riemannian manifold, the principal orbit type theorem holds.

Recently, J. Szenthe [**4**] gave an analogue of the principal orbit type theorem using the definition of infinitesimal orbit type, where the concept of the orbit type was modified the slightliest way, namely Szenthe didn't use in the definition the isometry subgroup G_x of a point but its Lie algebra $\mathfrak{g}_x$, see Definition 1. below. In the paper of Szenthe [**4**] the concept of the stable and unstable infinitesimal types are introduced, and the objective of his paper is to show that the union of orbits of stable infinitesimal types build an open and dense set. Szenthe gave criteria under which this will be true. Now our point of view will be different. We will make the assumption that the action is normalizable; this means that for every orbit $G(x)$ we have a G-invariant bundle $\widetilde{N}G(x) \subset TM$ over the orbit $G(x)$, the so called normalizer bundle, such that for every $x' \in G(x)$ the fiber of this bunle at x', the normal space $\widetilde{N}_{x'}G(x)$, yields a decomposition $\widetilde{N}_{x'}G(x) \oplus T_{x'}G(x) = T_{x'}M$ of the tangent space at x', see Definition 3. This assumption obviously holds in the Riemannian case and also in the Lorentzian case if there are no lightlike orbits. However, there are normalizable isometric actions on Lorentz manifolds which have lightlike orbits, such is the Schwarzschild space-time [**2**]. The main result of the paper is the theorem which asserts that if the isometric action of an arbitrary Lie group on a Lorentz manifold is normalizable, then there is an infinitesimal principal type such that the orbits belonging to this type build an open and dense set. The Riemannian case will be only a starting point for us to examine the Lorentzian case. Since we assume that the isometric action in the Lorentzian case is normalizable, therefore, we will investigate the non-normalizable orbits in the last section and prove the following. If $G(x)$ is a non-normalizable orbit of an isometric action of a Lie group on a Lorentzian manifold, then $G(x)$ is a lightlike submanifold, such that for every point $p \in G(x)$ there is a 1-parameter subgroup such that its orbit at p yield a lightlike geodesic segment through p, which is contained in $G(x)$.

2. The Riemannian case.

Definition 1. Let $\alpha : G \times M \to M$ be an isometric action of a Lie group G on a semi-Riemannian manifold (M, g). The infinitesimal orbit type of an orbit $G(x)$ is the whole conjugacy class of the identity component G_x^0 of the isotropy subgroup (also called stabilizer) G_x of x, i.e. the **infinitesimal orbit type** of $G(x)$ is $\left\{ gG_x^0 g^{-1} \,|\, g \in G \right\}$. This type will be denoted by $\operatorname{inftyp} G(x)$.

It is clear that in the above definition we could take the conjugacy class of the Lie algebra $\mathfrak{g}_x$ of the stabilizer G_x instead of the identity component of G_x. This explains the notation *infinitesimal* orbit type. This definition gives rise to a partial ordering in the set of the orbits:

Definition 2. The infinitesimal orbit type of the orbit $G(x)$ is greater than or equal to that of $G(y)$ if

$$G^0_{x'} \subseteq G^0_{y'} \text{ for some } x' \in G(x) \text{ and } y' \in G(y)$$

or equivalently

$$\mathfrak{g}_{x'} \subseteq \mathfrak{g}_{y'} \text{ for some } x' \in G(x) \text{ and } y' \in G(y).$$

This fact will be denoted by $\operatorname{inftyp} G(x) \geq \operatorname{inftyp} G(y)$.

We will need later the following definition.

Definition 3. Let $\alpha : G \times M \to M$ be an isometric action of a Lie group G on a semi-Riemannian manifold (M, g). The orbit $G(x)$ is **normalizable** if there is a subspace $\widetilde{N_x}G(x) \subset T_xM$ for which the following holds:

- $\widetilde{N_x}G(x) \oplus T_xG(x) = T_xM$ is a decomposition (not necessarily orthogonal);
- $\widetilde{N_x}G(x)$ is invariant under the action of G_x.
  Moreover the action α is called normalizable, if every orbit is normalizable.

Note that we could use the weaker condition in the definition, that $\widetilde{N_x}G(x)$ *is invariant under the action of* G^0_x, since we can work in the proofs only with the local Lie group action, that is we can consider an arbitrary small neighbourhood U of the identity element in G and the local orbit of x by U. We used the above definition to make the proofs easier to follow, and to be able to speak about the *normal bundle* $\widetilde{N}G(x)$ which is the bundle over $G(x)$ obtained from the normal space $\widetilde{N_x}G(x)$ by the action of $T\alpha : TM \to TM$, i.e. the **normal bundle** is given by

$$\widetilde{N}G(x) = \cup_{g \in G} T\alpha_g\left(\widetilde{N_x}G(x)\right).$$

It is clear that thus a bundle is obtained which is invariant under $T\alpha$.

It is important to note that the concept of the normal bundle and the orthogonal bundle are not the same. The fiber $\widetilde{N_z}G(x)$ over $z \in G(x)$ of the normal bundle $\widetilde{N}G(x)$ is not necessarily orthogonal to the tangent space $T_zG(x)$. In the Lorentzian case the orthogonal space $(T_zG(x))^{\perp}$ does not give a decomposition of T_zM by $(T_zG(x))^{\perp} \oplus T_zG(x)$ in general. Also the normal space $\widetilde{N_x}G(x)$ is not necessarily the same as the orthogonal space $(T_xG(x))^{\perp}$, since the normal space gives a decomposition $\widetilde{N_x}G(x) \oplus T_xG(x) = T_xM$.

LEMMA 4. *Let $\alpha : G \times M \to M$ be an isometric action of a Lie group G on a semi-Riemannian manifold (M, g) and $G(x)$ a normalizable orbit. Then the zero section in the normal bundle $\widetilde{N}G(x)$ has a neighbourhood U for which the following assertions hold:*

1. *U is invariant under the action $T\alpha$;*
2. *For a sufficiently small neighbourhood $V \subset G(x)$ of $x \in G(x)$, if $U|_V$ is the restricted normal bundle of U over V, i.e. $U|_V = \pi^{-1}(V) \cap U$, where $\pi : \widetilde{N}G(x) \to G(x)$ is the projection, the restricted exponential mapping $\exp : U|_V \to M$ is a diffeomorphism;*
3. *The mapping $\exp : U \to M$ is G-equivariant, with respect to the action $T\alpha$ of G restricted to U and the action α of G on M.*

PROOF. Let $P \subset \widetilde{N_x}G(x)$ be a starlike open neighbourhood of 0_x where the tangent linear maps of the exponential map $\exp|_{\widetilde{N}G(x)}$ are non-degenerate. Put

$$U \stackrel{def}{=} \cup \{T\alpha_g(P) \mid g \in G\},$$

where $\alpha_g : M \to M$ is the isometry corresponding to the element g by the action α. Then the first property is obviously satisfied by U, and if P is chosen to be suitably small, then the second property holds also. The third one can be proved by the properties of the exponential map. $\square$

Now, if we use in Definition 2 the Lie algebra to define the infinitesimal types, it is clear by the above lemma, that if we take a vector $v \in U$ then the infinitesimal type of the orbit of v by the action $T\alpha : G \times U \to U$ and the infinitesimal type of the orbit of $\exp(v)$ is the same. That is U appears as a model for the action α near to the orbit $G(x)$.

LEMMA 5. *Let $G(x)$ be a normalizable orbit and $U \subset \widetilde{N}G(x)$ a neighbourhood such as in the above lemma. If $v \in \widetilde{N_x}G(x) \cap U$ then* $\operatorname{inftyp} G(v) \geq \operatorname{inftyp} G(0_x)$ *for the action $T\alpha|_U$.*

PROOF. By the invariance of the normal bundle we have that for every $g \in G$ *either* $\alpha_g(x) = x$ and then $T\alpha_g\left(\widetilde{N_x}G(x) \cap U\right) = \widetilde{N_x}G(x) \cap U$, i.e. $\widetilde{N_x}G(x) \cap U$ is invariant, *or* $\alpha_g(x) \neq x$ and then $T\alpha_g\left(\widetilde{N_x}G(x) \cap U\right) = \widetilde{N_{\alpha_g(x)}}G(x) \cap U$, but then $\left(\widetilde{N_x}G(x) \cap U\right) \cap \left(\widetilde{N_{\alpha_g(x)}}G(x) \cap U\right) = \emptyset$. So we have that if $g \in G_v$ then $\left(\widetilde{N_x}G(x) \cap U\right) \cap \left(\widetilde{N_{\alpha_g(x)}}G(x) \cap U\right) \neq \emptyset$, thus $\widetilde{N_x}G(x) \cap U$ must be invariant and $\alpha_g(x) = x$. So we proved that $G_v \subseteq G_x$. Since $G_x = G_{0_x}$ we have $G_v \subseteq G_{0_x}$. But then $G_v^0 \subseteq G_{0_x}^0$, i.e. $\operatorname{inftyp} G(v) \geq \operatorname{inftyp} G(0_x)$. $\square$

Using our remark preceding this lemma we have that:

Corollary 6. *Let $G\,(x)$ be a normalizable orbit and $U \subset \widetilde{N}G\,(x)$ a neighbourhood such as in Lemma 4 then for every point $y \in \exp(U)$*

$$\operatorname{inftyp} G\,(y) \geq \operatorname{inftyp} G\,(x)$$

holds.

The following useful definition was introduced by J. Szenthe.

Definition 7. Let $\alpha : G \times M \to M$ be an isometric action of a Lie group G on a semi-Riemannian manifold (M, g). The orbit $G\,(x)$ is locally stable, if it has a G-invariant neighbourhood $\widetilde{U}$ such that for every $y \in \widetilde{U}$ the equality $\operatorname{inftyp} G\,(x) = \operatorname{inftyp} G\,(y)$ holds.

Corollary 8. *Let $\alpha : G \times M \to M$ be an isometric action of a Lie group G on a semi-Riemannian manifold (M, g) which is normalizable. Then the locally stable orbits build an open and dense set in M.*

Proof. Assume that the orbit $G\,(x)$ is not locally stable. Corollary 6 yields that there is a G-invariant neighbourhood $V \stackrel{def}{=} \exp(U)$ of $G\,(x)$ such that $\operatorname{inftyp} G\,(x) \leq \operatorname{inftyp} G\,(y)$ holds for every $y \in V$, where U is the neighbourhood in Corollary 6. Since $G\,(x)$ is not locally stable there is an orbit $G\,(x_1) \subset V$ arbitrary close to $G\,(x)$ for which $\operatorname{inftyp} G\,(x) < \operatorname{inftyp} G\,(x_1)$. If $G\,(x_1)$ is not locally stable then the above argument yields that there is an orbit $G\,(x_2)$ arbitrary close to $G\,(x_1)$ for which $\operatorname{inftyp} G\,(x_1) < \operatorname{inftyp} G\,(x_2)$. Thus we get a sequence $\operatorname{inftyp} G\,(x) < \operatorname{inftyp} G\,(x_1) < \operatorname{inftyp} G\,(x_2) < \ldots$, i.e. $\mathfrak{g}_x \supsetneqq \mathfrak{g}_{x_1} \supsetneqq \mathfrak{g}_{x_2} \supsetneqq \ldots$ if x, $x_1, \ldots$ are suitably points of the orbits. But such a sequence must be finite, thus there is an orbit $G\,(x_n)$ which must be locally stable and which can be chosen arbitrary close $G\,(x)$. So the locally stable orbits build a dense set, which is open by their definition. $\square$

Let us consider now the Riemannian case. Note that if (M, g) is Riemannian, then every orbit is normalizable, since the orthogonal complement $T_x^{\perp} G\,(x)$ of $T_x G\,(x)$ is invariant under the action of $T\alpha$.

Theorem 9. *If $\alpha : G \times M \to M$ is an isometric action of a Lie group on a connected Riemannian manifold (M, g) then there is a unique maximal infinitesimal orbit type, called infinitesimal principal, and the union of the infinitesimal principal orbits is an open, dense and connected set in M.*

Proof. Proof by induction on the dimension of the manifold. If $\dim M = 1$ then either there is only one orbit which is M, or for every point $x \in M$ the unit component of its stabilizer G_x^0 is the same, which will be the unit component of the Lie group G^0. Let us assume that the lemma is true for every isometric action where the dimension of the Riemannian manifold is less than or equal to k. Consider an isometric action $\alpha : G \times M \to M$ on a Riemannian manifold, where $\dim M = k+1$, and an orbit $G(x)$.

First we describe below some properties **(A)**, **(B)**, **(C)**, of the induced action $T\alpha|_{\widetilde{N}G(x)} : G \times \widetilde{N}G(x) \to \widetilde{N}G(x)$. Note that in the proof of the properties **(A)**, **(B)**, **(C)** we won't need that the metric is Riemannian, so the properties **(A)**, **(B)**, **(C)** hold also in the case when (M, g) is a semi-Riemannian manifold and the action is isometric which we will need later.

- **(A)** If we consider the restricted action $T\alpha|_{G_x, \widetilde{N}_x G(x)} : G_x \times \widetilde{N}_x G(x) \to \widetilde{N}_x G(x)$ then it is a linear action on the vector space $\widetilde{N}_x G(x)$. So the infinitesimal orbit type of two vectors $v, w \in \widetilde{N}_x G(x)$ are the same if $v = c \cdot w$ for some non-zero constant c, i.e. if we take a line through the origin then every point on this line, except of the origin, has the same infinitesimal orbit type. (It is also true that their orbit type is the same).
- **(B)** Moreover, as we have shown it in the proof of Lemma 5, the isotropy subgroup G_v of v by the action $T\alpha|_{\widetilde{N}G(x)}$ is contained in G_x so it is the same as the isotropy subgroup of v by the restricted action $T\alpha|_{G_x, \widetilde{N}G(x)}$.
- **(C)** If we take an open set V in $\widetilde{N}_x G(x)$ then the set
$$G(V) \stackrel{\text{def}}{=} \{T\alpha_g(v) \mid v \in V,\, g \in G\}$$
is an open set in $\widetilde{N}G(x)$ by a simple obvious argument.

Now since the action α is isometric $T\alpha|_{G_x, \widetilde{N}_x G(x)}$ acts on the unit-sphere $\widetilde{S}_x G(x)$ of $\widetilde{N}_x G(x)$, which has a dimension $\le k$, so by the induction, if we assume that $\dim \widetilde{S}_x G(x) \neq 0$, then there is a unique minimal infinitesimal orbit type (infinitesimal principal type) by the restricted action $T\alpha|_{G_x, \widetilde{S}_x G(x)}$ which build an open, dense and connected set in $\widetilde{S}_x G(x)$. But since the action is linear by **(A)** on $\widetilde{N}_x G(x)$ this yields that by the action $T\alpha|_{G_x, \widetilde{N}_x G(x)}$ the lemma holds and we have a principal infinitesimal type on $\widetilde{N}_x G(x)$ such that the orbits of this type build an open, dense and connected set in $\widetilde{N}_x G(x)$. Note that if $\dim \widetilde{S}_x G(x) = 0$ then $\dim \widetilde{N}_x G(x) = 1$ so by induction the lemma holds by the action $T\alpha|_{G_x, \widetilde{N}_x G(x)}$. Since every orbit by the action $T\alpha|_{\widetilde{N}G(x)}$ intersects $\widetilde{N}_x G(x)$ by **(B)** we have that if $v \in \widetilde{N}_x G(x)$ belongs to the unique minimal infinitesimal type κ by the action $T\alpha|_{G_x \widetilde{N}_x G(x)}$, then the type of v by $T\alpha|_{\widetilde{N}G(x)}$ is the same. Using **(C)** it yields that the orbits belonging to the type κ by the action $T\alpha|_{\widetilde{N}G(x)}$build an open set in $\widetilde{N}G(x)$, and since this set is G-invariant, the proof of **(C)** yields that this set is dense and connected. Moreover the above yield that κ is a unique minimal infinitesimal orbit type.

Now consider a neighbourhood $U \subset \widetilde{N}G\,(x)$ and a connected neighbourhood $V \subset G\,(x)$ of x such as in Lemma 4. Then that lemma yields that in the connected open set $\exp\,(U|_V)$ there is a minimal infinitesimal type and the points belonging to this type build an open, dense and connected set.

So we proved that for every $x \in M$ there is a connected open neighbourhood F_x, where our lemma holds. If two such neighbourhoods intersect, then in the open intersection $F_x \cap F_y$ the set of points of the unique minimal infinitesimal type belonging to the set F_x and the set of points of the unique minimal infinitesimal type belonging to the set F_y are both open and dense. Hence these two types must be the same, so on $F_x \cup F_y$ there is a unique minimal infinitesimal type which build an open dense and connected set. As the sets F_x cover M we have that our lemma holds on M. □

A similar proof by induction can be found in K. Jänich [**1**] for the principal orbit type theorem.

3. The Lorentzian case. Note that as the example of the action $SO(2,1)$ on the 3-dimensional Minkowski space shows, the infinitesimal principal orbit type theorem does not hold in general in the Lorentzian case. In the Lorentzian case, we will reshape the above Riemannian proof, where some essential modifications will be made. We will need the following definition.

Definition 10. Let (M, g) be a Lorentz manifold, $P \subset M$ a submanifold, and $U \subset M$ an arbitrary open submanifold. Let us restrict the metric tensor to U, and consider the Lorentz manifold (U, g), then the union of the chronological future and past of $P \cap U$ in (U, g) will be denoted by $I\,(P, U)$, which will be considered as a subset in M.

In what follows $(G\,(x) \cap A)_x$ will denote the path connected component of $G\,(x) \cap A$ containing x, in case of any set $A \subset L$ and orbit $G\,(x)$.

Lemma 11. *Let (M, g) be a Lorentz manifold and $\alpha : G \times M \to M$ an isometric action of a Lie group G on M which is normalizable. Then for every orbit $G\,(x)$ there is an open neighbourhood $U \subset M$ of x such that there is a unique infinitesimal type in $I\,((G\,(x) \cap U)_x\,, U)$ for which the orbits belonging to this type build an open and dense set in $I\,((G\,(x) \cap U)_x\,, U)$.*

Proof. Let us consider the normal space $\widetilde{N}_xG\,(x)$ and the normal bundle $\widetilde{N}G\,(x)$ which is G-invariant. Recall our remark in the proof of Theorem 9 that the properties **(A)**, **(B)**, **(C)** hold also in the semi-Riemannian case for the action $T\alpha|_{\widetilde{N}G(x)} : G \times \widetilde{N}G\,(x) \to \widetilde{N}G\,(x)$. Now let us consider first the reduced action $T\alpha|_{G_x, \widetilde{N}_xG(x)} : G_x \times \widetilde{N}_xG\,(x) \to \widetilde{N}_xG\,(x)$. This action leaves the metric $g|_{\widetilde{N}_x G(x)}$ invariant on the vector space $\widetilde{N}_x G(x)$. There are 3 cases according as the space $\widetilde{N}_x G(x)$ is spacelike, timelike or lightlike.

(s) If $\widetilde{N}_x G(x)$ is spacelike consider the submanifold

$$S_s \stackrel{\text{def}}{=} \left\{ v \in \widetilde{N}_x G(x) \mid g(v,v) = 1 \right\} \subset \widetilde{N}_x G(x)$$

which is a Riemannian manifold, since the metric $g|_{\widetilde{N}_x G(x)}$ is Riemannian. Since the metric $g|_{\widetilde{N}_x G(x)}$ is invariant under the action $T\alpha|_{G_x, \widetilde{N}_x G(x)}$ the set S_s is also invariant, moreover if we restrict this action to S_s we get an isometric action $T\alpha|_{G_x, S_s} : G_x \times S_s \to S_s$ on a compact Riemannian manifold. Using Theorem 9 we obtain that there is a unique maximal infinitesimal type under this restricted action, and the orbits of this type build an open and dense set in S_s. As the action $T\alpha|_{G_x, \widetilde{N}_x G(x)}$ is linear on $\widetilde{N}_x G(x)$, this was property **(A)**, we obtain that this type is a unique maximal infinitesimal type on $\widetilde{N}_x G(x)$ for the action $T\alpha|_{G_x, \widetilde{N}_x G(x)}$ and the orbits belonging to this type build an open and dense set. As in Theorem 9 using properties **(B)**, **(C)** we see that this is a unique maximal infinitesimal type on the normal bundle $\widetilde{N}G(x)$ of the action $T\alpha$ and the orbits belonging to this type build an open and dense set in $\widetilde{N}G(x)$. Let $\widetilde{U} \subset \widetilde{N}G(x)$ be an open neighbourhood of 0_x such that the exponential map is a diffeomorphism on $\widetilde{U}$. Since exp is G-equivariant and the orbits belonging to the unique maximal infinitesimal type build an open and dense set in $\widetilde{U}$, we have that on the set $\exp\left(\widetilde{U}\right)$ there is a unique maximal infinitesimal type, for the action α, and the points belonging to this type build an open and dense set in $U \stackrel{def}{=} \exp\left(\widetilde{U}\right)$. So the assertion holds also on $I\left(\left(U \cap G(x)\right)_x, U\right) \subset U$.

(t) If $\widetilde{N}_x G(x)$ is timelike, then consider

$$S_t \stackrel{\text{def}}{=} \left\{ v \in \widetilde{N}_x G(x) \mid g(v,v) = -1 \right\}.$$

Now if $g|_{\widetilde{N}_x G(x)}$ is negative definite, and then $\widetilde{N}_x G(x)$ is 1-dimensional, then we can repeat essentially the same argument as above in case (s), since $-g|_{\widetilde{N}_x G(x)}$ is positive definite. If $g|_{\widetilde{N}_x G(x)}$ is non-degenerate and indefinite, then $\widetilde{N}_x G(x)$ is at least 2-dimensional and $(S_t, \widetilde{g})$ is a Riemannian submanifold in $\left(\widetilde{N}_x G(x), g|_{\widetilde{N}_x G(x)}\right)$, where the Riemannian metric $\widetilde{g}$ is the one induced by $g|_{\widetilde{N}_x G(x)}$ on S_t. As in the case (s), here also the invariance of $g|_{\widetilde{N}_x G(x)}$ by the action $T\alpha|_{G_x, \widetilde{N}_x G(x)}$ gives that S_t is invariant under the action of $T\alpha|_{G_x, \widetilde{N}_x G(x)}$, moreover $T\alpha|_{G_x, S_t} : G_x \times S_t \to S_t$ is an isometric action on the Riemannian manifold S_t. Thus by Lemma 9 there is a unique maximal infinitesimal type and the points of this type build an open and dense set in S_t. Now if we define the G_x-invariant set

$$I\left(\widetilde{N}_x G(x)\right) \stackrel{\text{def}}{=} \left\{v \in \widetilde{N}_x G(x) \mid g(v,v) < 0\right\},$$

then by property **(A)** we have that there is a unique maximal infinitesimal type in $I\left(\widetilde{N}_x G(x)\right)$ by the action $T\alpha|_{G_x, \widetilde{N}_x G(x)}$. Since the set

$$I\left(\widetilde{N} G(x)\right) = \cup_{g \in G} T\alpha_g \left(I\left(\widetilde{N}_x G(x)\right)\right)$$

is G-invariant, by the properties **(B)** and **(C)** we have that there is a unique maximal infinitesimal type in $I\left(\widetilde{N} G(x)\right)$ by the action

$$T\alpha|_{I\left(\widetilde{N}G(x)\right)} : G \times I\left(\widetilde{N} G(x)\right) \to I\left(\widetilde{N} G(x)\right)$$

and the points belonging to this type build an open and dense set in $I\left(\widetilde{N} G(x)\right)$. As in case (s) let us take a suitable small neighbourhood $\widetilde{U} \subset \widetilde{N} G(x)$ of 0_x such that the exponential map is a diffeomorphism on $\widetilde{U}$ then we have that there is a unique maximal infinitesimal type in $\exp\left(I\left(\widetilde{N} G(x)\right) \cap \widetilde{U}\right)$ by the action α and the points belonging to this type build an open and dense set in

$$\exp\left(I\left(\widetilde{N} G(x)\right) \cap \widetilde{U}\right).$$

Now if $\widetilde{U}$ is suitably small and $U \stackrel{\text{def}}{=} \exp\left(\widetilde{U}\right)$, then it can be shown that

$$\exp\left(I\left(\widetilde{N} G(x)\right) \cap \widetilde{U}\right) = I\left((G(x) \cap U)_x, U\right).$$

(l) If there is only lightlike normal space $N_x G(x)$ then $T_x G(x)$ has to be lightlike. Then let $F_{\widetilde{N}_x G(x)}$ and $F_{T_x G(x)}$ denote the unique 1-dimensional lightlike subspaces of $\widetilde{N}_x G(x)$ and $T_x G(x)$. Since $T_x G(x)$ and $\widetilde{N}_x G(x)$ are G_x-invariant the lightlike spaces $F_{\widetilde{N}_x G(x)}$ and $F_{T_x G(x)}$ are also G_x-invariant, moreover the 2-dimensional timelike subspace N_x^2 spanned by $F_{\widetilde{N}_x G(x)}$ and $F_{T_x G(x)}$ is also G_x-invariant. So the $(n-2)$-dimensional spacelike subspace $\left(N_x^2\right)^\perp \subset T_x M$ is also G_x-invariant. The intersection of G_x-invariant subspaces

$$R \stackrel{\text{def}}{=} \widetilde{N}_x G(x) \cap \left(N_x^2\right)^\perp$$

is G_x-invariant and it is easy to see that this is a 1-codimensional subspace in $\widetilde{N}_x G(x)$ which is spacelike, since the following is true. For the lightlike subspaces $\widetilde{N}_x G(x) \subset F^{\perp}_{\widetilde{N}_x G(x)}$ holds, moreover $\left(N_x^2\right)^{\perp} \subset F^{\perp}_{\widetilde{N}_x G(x)}$ is true. Now a simple calculation shows that

$$
\begin{aligned}
(1)\ \dim\Big(\widetilde{N}_x G(x) \cap \left(N_x^2\right)^{\perp}\Big) &\geq \dim\left(\widetilde{N}_x G(x)\right) + \dim\left(N_x^2\right)^{\perp} - \dim\Big(F^{\perp}_{\widetilde{N}_x G(x)}\Big) \\
&= \dim\left(\widetilde{N}_x G(x)\right) + (\dim M - 2) - (\dim M - 1) \\
&= \dim\left(\widetilde{N}_x G(x)\right) - 1.
\end{aligned}
$$

Since $R = \widetilde{N}_x G(x) \cap \left(N_x^2\right)^{\perp}$ is spacelike and $\widetilde{N}_x G(x)$ is lightlike $R \subsetneqq \widetilde{N}_x G(x)$, thus

$$
\dim\left(\widetilde{N}_x G(x)\right) > \dim R,
$$

and this by the above inequality (1) gives that R must be 1-codimensional in $\widetilde{N}_x G(x)$.

We can assume that $\dim \tilde{N}_x G(x) > 1$, because if $\dim \tilde{N}_x G(x) = 1$ then the following holds. The action of G_x on the one dimensional lightlike subspace is trivial, i.e. every vector in $\tilde{N}_x G(x)$ is fixed, or the action is non-trivial and there are three orbits, the vector 0_x and the two half lines. In both cases there is a unique infinitesimal type, such that the orbits belonging to that infinitesimal type in $\tilde{N}G(x)$ by $T\alpha|_{G_x, \tilde{N}_x G(x)}$ build an open and dense set. But then the properties **(A)**, **(B)**, **(C)** yield that there is a unique infinitesimal type by $T\alpha|_{\tilde{N}G(x)}$ such that the orbits of this infinitesimal type build an open and dense set in $\tilde{N}G(x)$, thus by the exponential map, we get that the orbit $G(x)$ has a G-invariant neighbourhood, for which the orbits of the above infinitesimal type build an open and dense set, so the lemme is true. Now we can assume that $\dim \tilde{N}_x G(x) > 1$ and let us consider the spacelike subspace R, which can be considered as a Riemannian manifold. So by Theorem 9 we have a unique maximal infinitesimal type in R by the restricted action $T\alpha|_{G_x,R} : G_x \times R \to R$ and the points belonging to this type build an open and dense set in R. Now since

$$
\widetilde{N}_x G(x) = F_{\widetilde{N}_x G(x)} \oplus R
$$

is a G_x-invariant decomposition, every vector $v \in \widetilde{N}_x G(x)$ can be written in the form $v = v_l + v_R$, where v_l is a lightlike vector and v_R is a spacelike vector.

First we prove the following:

If $v, w \in \widetilde{N}_x G(x)$ and $v = v_l + v_R$, $w = w_l + w_R$ *and* $v_l \neq 0$, $w_l \neq 0$ *moreover* v_R, w_R *belong to the unique maximal infinitesimal type by* $T\alpha|_{G_x,R}$ *then* v *and* w *belong to the same type by* $T\alpha|_{G_x, \widetilde{N}_x G(x)}$.

Let $u \in F_{\widetilde{N}_x G(x)}$, $u \neq 0$ be a non-zero element in the 1-dimensional lightlike subspace and $(G_x)_u$ its isotropy subgroup by $T\alpha|_{G_x, \widetilde{N}_x G(x)}$. It is clear that

$(G_x)_u$ does not depend on the choice of $u \in F_{\widetilde{N}_x G(x)} \setminus \{0_x\}$ only on the action $T\alpha|_{G_x, \widetilde{N}_x G(x)}$, moreover $H \overset{def}{=} (G_x)_u$ is a *normal subgroup* of G_x. So the isotropy subgroup of v_l and w_l for $T\alpha|_{G_x, \widetilde{N}_x G(x)}$ is H. Since v_R and w_R belong to the same infinitesimal type for $T\alpha|_{G_x,R}$, there is an element $g \in G_x$ such that $g^{-1} (G_x)^0_{v_R} g = (G_x)^0_{w_R}$ holds for the isotropy subgroups for the action $T\alpha|_{G_x,R}$. Note that the isotropy subgroup of every vector in R is the same for $T\alpha|_{G_x,R}$ and $T\alpha|_{G_x, \widetilde{N}_x G(x)}$. By the G_x-invariant decomposition $\widetilde{N}_x G(x) = F_{\widetilde{N}_x G(x)} \oplus R$ we have that

$$(G_x)^0_v = \left((G_x)_{v_l} \cap (G_x)^0_{v_R} \right)^0 = \left(H \cap (G_x)^0_{v_R} \right)^0 ,$$

where the zeros denote the identity components of the groups. But then

$$g^{-1} (G_x)^0_v \, g = \left(g^{-1} H g \cap g^{-1} (G_x)^0_{v_R} \, g \right)^0 = \left(H \cap (G_x)^0_{w_R} \right)^0 = (G_x)^0_{w_R} ,$$

thus v and w have the same infinitesimal type.

So we have proved that if κ is the unique maximal infinitesimal type of the action $T\alpha|_{G_x,R}$ and we take the vectors $v \in \widetilde{N}_x G(x)$ for which $v_l \neq 0$ and $\operatorname{inftyp}(v_R) = \kappa$, then these vectors in $\widetilde{N}_x G(x)$ have the same infinitesimal type for $T\alpha|_{G_x, \widetilde{N}_x G(x)}$. Let ω denote this infinitesimal type. The vectors of infinitesimal type ω build a dense set, since the points of type κ build a dense and open set in R. *We will prove that this set in $\widetilde{N}_x G(x)$ is also open.*

Let us consider first:

Case (I): *When $(G_x)^0_{v_R} \subset H$ for a vector $v_R \in R$ which belongs to the type κ in R by $T\alpha|_{G_x,R}$.* Since H is a normal subgroup in G_x we have that $g^{-1} (G_x)^0_{v_R} \, g \subset H$ for every $g \in G_x$ and since

$$(G_x)^0_v = \left((G_x)_{v_l} \cap (G_x)^0_{v_R} \right)^0 = \left(H \cap (G_x)^0_{v_R} \right)^0 = (G_x)^0_{v_R}$$

we have that $v \in \widetilde{N}_x G(x)$ *belongs to the above mentioned infinitesimal type ω if and only if v_R belongs to the infinitesimal type κ*, thus the infinitesimal type of v depends only on the infinitesimal type of v_R. Since the points belonging to the type κ build an open and dense set in R we have that the points belonging to the type ω in S_l build an open and dense set in S_l. Of course in case (I) $\omega = \kappa$ holds also.

Case (II): *When the above case* (I) *does not hold.* Then we have the following:

Consider a vector $v \in \widetilde{N}_x G(x)$ for which $v_l \neq 0$, $\operatorname{inftyp}(v_R) = \kappa$ holds, thus $\operatorname{inftyp}(v) = \omega$. Since all the vectors in a suitable small neighbourhood of v_R in R have the same infinitesimal type κ, see Corollary 6, we can take a suitable small neighbourhood $A \subset \widetilde{N}_x G(x)$ of v such that for every vector $w \in A$, we have $w_l \neq 0$ and $\operatorname{inftyp}(w_R) = \kappa$. Thus every vector in A will belong to the infinitesimal type ω, i.e. v is in the interior of the set of vectors of infinitesimal type ω.

So it remains to prove the following two subcases. *If a vector* $v \in \widetilde{N}_x G(x)$ *belongs to the infinitesimal type* ω *but*

(*IIa*) $\operatorname{inftyp}(v_R) \neq \kappa$ *or*

(*IIb*) $\operatorname{inftyp}(v_R) = \kappa$ *but* $v_l = 0$,

then in both subcases there is a suitable small neighbourhood A *of* v *in* $\widetilde{N}_x G(x)$ *such that all the vectors in* A *have infinitesimal type* ω.

Since case (I) does not hold, there is a vector $z_R \in R$ for $\operatorname{inftyp}(z_R) = \kappa$ but $(G_x)^0_{z_R} \subsetneqq H$. Then the points of infinitesimal type κ in R of the restricted action $T\alpha|_{G_x,R}$ can not belong to the infinitesimal type ω of the action $T\alpha|_{G_x,\widetilde{N}_x G(x)}$, thus (*IIb*) cannot occur since the following is true. If $\operatorname{inftyp}(v_R) = \kappa$ then we have that $(G_x)^0_{v_R} = g^{-1}(G_x)^0_{z_R} g$ for some $g \in G_x$ and that

$$(G_x)^0_{v_R} = g^{-1}(G_x)^0_{z_R} g \subsetneqq g^{-1}Hg = H \tag{2}$$

so for a vector $v = v_R + v_l$, $v_l \neq 0$ the following holds:

$$(G_x)^0_v = \left(H \cap (G_x)^0_{v_R}\right)^0 = \left(H \cap g^{-1}(G_x)_{z_R} g\right)^0 =$$

$$g^{-1}\left(H \cap (G_x)^0_{z_R}\right)^0 g \subsetneqq g^{-1}(G_x)^0_{z_R} g = (G_x)^0_{v_R},$$

thus $\operatorname{inftyp}(v) \gneqq \operatorname{inftyp}(v_R)$. So κ and ω are of different infinitesimal types, which gives that the subcase (*IIb*) can not occur.

Thus the remaining subcase to prove is (*IIa*). Since we are in case (II), for any vector $z \in R$ for which $\operatorname{inftyp}(z_R) = \kappa$ is true, $(G_x)^0_{z_R} \subsetneqq H$ holds, this was inequality (2). Now let us take a vector $z \in \widetilde{N}_x G(x)$, $z_l \neq 0$, $\operatorname{inftyp}(z_R) = \kappa$, then $\operatorname{inftyp}(z) = \omega$. As we are in subcase (*IIa*) we consider a vector $v \in \widetilde{N}_x G(x)$, $\operatorname{inftyp}(v_R) \neq \kappa$ for which $\operatorname{inftyp}(v) = \omega$. Then since κ is the unique maximal infinitesimal type in R we have that $\operatorname{inftyp}(z_R) > \operatorname{inftyp}(v_R)$. Since $(G_x)^0_{z_R} \subsetneqq H$ we have that $(G_x)^0_z = \left(H \cap (G_x)^0_{z_R}\right)^0 \subsetneqq (G_x)^0_{z_R}$ by the definition of the infinitesimal type this means that $\operatorname{inftyp}(z) > \operatorname{inftyp}(z_R)$. Thus $\omega = \operatorname{inftyp}(z) > \operatorname{inftyp}(v_R)$, so $v_l \neq 0$. Since $\operatorname{inftyp}(v) = \operatorname{inftyp}(z) = \kappa = \omega$ there is a $g \in G_x$ for which $(G_x)^0_z = g^{-1}(G_x)^0_v g$. Moreover as $\operatorname{inftyp}(z_R)$ is maximal, for every $w_R \in R$ there is a $\widehat{g} \in G_x$ for which

$$(G_x)^0_{z_R} \subsetneqq \widehat{g}^{-1}(G_x)^0_{w_R} \widehat{g}. \tag{3}$$

Now since R is a Riemannian-manifold and G_x acts on it isometrically there is a neighbourhood $P \subset R$ of v_R such that for every vector $w_R \in P$, $\operatorname{inftyp}(w_R) \geq$

$\operatorname{inftyp}(v_R)$, see Corollary 6, i.e. for every $w_R \in P$ there are elements $f \in G_x$ for which

$$f^{-1}(G_x)^0_{w_R} f \subseteq (G_x)^0_{v_R}. \tag{4}$$

Now if $A \subset \widetilde{N}_x G(x)$ is a neighbourhood of v such that for every $w \in A$, $w_l \neq 0$ and $w_R \in P$ then the above inequalities (3) and (4) give that

$$(G_x)^0_{z_R} \subseteq \widehat{f}^{-1}(G_x)^0_{w_R} \widehat{f} \subseteq \widehat{j}^{-1}(G_x)^0_{v_R} \widehat{j} \text{ for some } \widehat{f}, \widehat{j} \in G_x,$$

$$\text{since } \operatorname{inftyp}(z_R) \geq \operatorname{inftyp}(w_R) \geq \operatorname{inftyp}(v_R), \text{ so}$$

$$(G_x)^0_z = \left(H \cap (G_x)^0_{z_R}\right)^0 \subseteq \left(H \cap \widehat{f}^{-1}(G_x)^0_{w_R} \widehat{f}\right)^0 = \widehat{f}^{-1}\left(H \cap (G_x)^0_{w_R}\right)^0 \widehat{f} =$$

$$\widehat{f}^{-1}(G_x)^0_w \widehat{f} \subseteq \left(H \cap \widehat{j}^{-1}(G_x)^0_{v_R} \widehat{j}\right)^0 = \widehat{j}^{-1}\left(H \cap (G_x)^0_{v_R}\right)^0 \widehat{j} = \widehat{j}^{-1}(G_x)^0_v \widehat{j}.$$

Therefore we get $(G_x)^0_z \subseteq \widehat{j}^{-1}(G_x)^0_v \widehat{j}$. Since $\operatorname{inftyp}(v) = \operatorname{inftyp}(z) = \omega$ we have that $(G_x)^0_z = \widehat{j}^{-1}(G_x)^0_v \widehat{j}$, so in the above inequality the equality holds. Thus in the first line of the above inequality

$$(G_x)^0_z = \widehat{f}^{-1}(G_x)^0_w \widehat{f}$$

is true. By the definition of the infinitesimal orbit type this means that $\omega = \operatorname{inftyp}(z) = \operatorname{inftyp}(w)$ for every $w \in A$. So the points in the open neighbourhood A of v belong to the infinitesimal type ω, thus v is an interior point in the set of points in $\widetilde{N}_x G(x)$ belonging to the type ω.

So far we have proved that there is an infinitesimal type for the action $T\alpha|_{G_x, \widetilde{N}_x G(x)}$ such that all the points belonging to this type in $\widetilde{N}_x G(x)$ build an open and dense set. Now using properties **(B)** and **(C)** as before we have that there is an infinitesimal type for the action $T\alpha|_{\widetilde{N}G(x)}$ such that all the points in the normal bundle $\widetilde{N}G(x)$ belonging to this type build an open and dense set. Taking an open neighbourhood $\widetilde{U} \subset \widetilde{N}G(x)$ of 0_x such that the exponential map is a diffeomorphism on $\widetilde{U}$, we get that there is an infinitesimal type for the action α such that all the points belonging to this type in $U \overset{def}{=} \exp\left(\widetilde{U}\right)$ build an open and dense set. Now the same will be true on the open set $I\left(\left(U \cap G(x)\right)_x, U\right)$.

So in each case, in (s), (t), (l), we proved that there is an infinitesimal type such that the points in $I\left(\left(U \cap G(x)\right)_x, U\right)$ belonging to this type build an open and dense set. It is clear that this type is unique in $I\left(\left(U \cap G(x)\right)_x, U\right)$, since there can not be two disjoint, open and dense sets in the open set $I\left(\left(U \cap G(x)\right)_x, U\right)$. $\square$

Theorem 12. *Let (M,g) be a connected Lorentz-manifold and $\alpha : G \times M \to M$ an isometric action of a Lie group G. Assume that the action is normalizable, then there is a unique maximal infinitesimal type, called infinitesimal principal, such that all the points belonging to this type build an open and dense set.*

Proof. First note that if we take two open sets as in the above Lemma 11, and the intersection of these two open sets is non-empty, then the unique maximal infinitesimal type of this two open sets is the same, since in the open intersection there can not be two different maximal infinitesimal types. Moreover the union of the above sets, the set $\cup_{x \in M} I\left((U_x \cap G(x))_x, U_x\right)$ builds an open and dense set. So we have to prove first that *for every x, $y \in M$ the unique maximal infinitesimal type in $I\left((U_x \cap G(x))_x, U_x\right)$ and in $I\left((U_y \cap G(y))_y, U_y\right)$ is the same.*

There is a piecewise smooth curve

$$\varphi : [0 = \tau_0, \tau_1, \dots, \tau_n = 1] \to M, \varphi(0) = x,\ \varphi(1) = y$$

such that $\varphi([\tau_i, \tau_{i+1}])$ is a timelike geodesic. Now consider for every $t \in [0,1]$ a set $I\left((G(\varphi(t)) \cap U_t)_{\varphi(t)}, U_t\right)$ such as given in the above lemma, where we can assume that U_t is geodesically convex, see B. O'Neil [**3**] p. 129, Definition 5 and p. 130, Proposition 7. With this assumption, since φ is timelike, we can choose parameters a_t, $b_t \in [0,1]$, $a_t < t < b_t$ such that $\varphi((a_t,t)) \subset I\left((G(\varphi(t)) \cap U_t)_{\varphi(t)}, U_t\right)$ and $\varphi((t,b_t)) \subset I\left((G(\varphi(t)) \cap U_t)_{\varphi(t)}, U_t\right)$, where if $a_t = 0$ or $b_t = 1$ then we take the relative open set $[0,b_t)$ or $(a_t,1]$. Since $[0,1]$ is compact we get that there are parameters $t_0, t_1, \dots, t_k$ such that the relative open intervals (a_{t_i}, b_{t_i}) belonging to this parameters cover the whole $[0,1]$ interval. We can assume that the set $t_0, \dots, t_k$ also contains the points $\tau_0, \dots, \tau_n$. Since (a_{t_i}, b_{t_i}) and $\left(a_{t_{i+1}}, b_{t_{i+1}}\right)$ intersects for every $i = 0, 1, \dots, k-1$, we have that $I\left((G(\varphi(t_i)) \cap U_{t_i})_{\varphi(t_i)}, U_{t_i}\right) \cap I\left((G(\varphi(t_{i+1})) \cap U_{t_{i+1}})_{\varphi(t_{i+1})}, U_{t_{i+1}}\right)$ is non-empty, so as we mentioned at the beginning of the proof, the unique maximal infinitesimal type in $I\left((G(\varphi(t_i)) \cap U_{t_i})_{\varphi(t_i)}, U_{t_i}\right)$ and in $I\left((G(\varphi(t_{i+1})) \cap U_{t_{i+1}})_{\varphi(t_{i+1})}, U_{t_{i+1}}\right)$ is the same. As $\varphi(t_0) = x$ and $\varphi(t_k) = y$ we have that in $I\left((U_x \cap G(x))_x, U_x\right)$ and in $I\left((U_y \cap G(y))_y, U_y\right)$ the unique maximal infinitesimal type is the same. So we have that there is an infinitesimal type κ such that the points belonging to this type build a dense set in M. The orbits of infinitesimal type κ build also an open set, since every orbit of infinitesimal type κ must be locally stable or else according to Corollaries 6, 8 the orbits of infinitesimal type κ does not build a dense set. The same corollaries yield that the infinitesimal type is also maximal. Since there can not be two different disjoint, open and dense sets in M we have that this type is unique. □

The above theorem can be proved for the semi-Riemannian case or in a more general setting, this proof will be presented elsewhere.

Note that connectedness is not true in general as the following example shows.

Example 13. Consider the special orthogonal group $SO\,(1,1)$ on the two dimensional Minkowski space. Let v be a lightlike vector and T_v be the one parameter group generated by the translations in the direction of v. If we take the group of isometries generated by $SO\,(1,1)$ and T_v then by the action of this group there will be only 3 orbits. One is the lightlike line through the origin in the direction of v. This line cuts the plane into two half-planes and each half-plane will be an orbit. The two half-plane orbit will be infinitesimal principal, they build an open, dense but not connected set.

The normalizability is a sufficient condition but it is not necessary as the following example shows. So the natural question arouse, can we give an other condition which is weaker then normalizability? In the following example there is an infinitesimal type which is maximal, the orbits of this infinitesimal type build an open and dense set, but not every orbit is normalizable.

Example 14. Let v be a lightlike vector in the three dimensional Minkowski space $\mathbb{M}^3$. Let $H \subset SO\,(2,1)$ be the subgroup of those elements, which leave the lightlike line $\mathbb{R} \cdot v$ invariant. Moreover let T_v be the group of translations in the direction v. Consider the isometry group generated by H and T_v. The orbits will be the lightlike lines in the orthogonal space $v^\perp$ and the two half spaces will give two other orbits, which will have infinitesimal maximal type, so they build an open and dense set in $\mathbb{M}^3$. But the orbits which are lightlike lines will be non-normalizable.

4. Non-normalizable orbits. The above theorem implies that if there is no infinitesimal type such that the points belonging to this type build an open and dense set, then there must be a non-normalizable orbit. For a non-normalizable orbit in the Lorentzian-case we can prove the following.

Theorem 15. *If (M, g) is a Lorentz-manifold and $\alpha : G \times M \to M$ an isometric action of a Lie group G and the orbit $G\,(x)$ is non-normalizable, then it is a lightlike orbit such that for every $p \in G\,(x)$ there there is a 1-parameter subgroup in G such that its orbit at p yield a lightlike geodesic segment through p, which is contained in $G\,(x)$.*

Proof. The proof will be presented elsewhere. □

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