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ON COMPUTING THE DIMENSION OF A SYSTEM OF PLANE CURVES

by Łucja Farnik

Abstract. We consider systems of plane curves passing through a given configuration of points. For $s \leq 9$ we show how to compute the dimension of a system of curves of degree d passing through an arbitrary set of s simple points, or an arbitrary set of $s-1$ simple points and one double point, using simple methods.

1. Introduction. The classification of systems of curves in $\mathbb{P}^2$ with respect to their speciality began with Nagata in 1959. However, methods used in [**8**] were very complicated.

The problem has been studied by many authors. Recently, in papers [**3**] and [**4**] – the second paper answers the questions raised in [**5**] – it was shown by means of Hilbert functions how to compute the dimensions of the systems $\mathcal{L}_d(m_1p_1, \ldots, m_sp_s)$ for up to eight points with arbitrary multiplicities m_i. The dimensions of the systems $\mathcal{L}_d(m_1p_1, \ldots, m_sp_s)$ are also known for the configurations with all the points on a conic (see [**1**] and [**4**]).

The problem of classifying the systems $\mathcal{L}_d(m_1p_1, \ldots, m_sp_s)$ for an arbitrary number of points and arbitrary multiplicities has not been solved yet.

We show a solution to part of this problem (for $s \leq 8$ the results are already known from the papers cited above) but using simple tools.

2. Notation and basic facts. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero.

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By $L = \mathcal{L}_d(m_1p_1, \dots, m_sp_s)$ we denote a system of curves of degree $d \in \mathbb{N}$ passing through given points $p_1, \dots, p_s \in \mathbb{K}^2$ with multiplicities at least $m_1, \dots, m_s \in \mathbb{N}$.

Notion of a "system" comes from one-to-one correspondence between the vector space L over the field $\mathbb{K}$ and the linear system of a divisor $d\widetilde{H} - \sum_{j=1}^{s} m_jE_j$, where E_j are exceptional lines on a blow-up of $\mathbb{K}^2$ in $p_1, \dots, p_s$, and $\widetilde{H}$ is a pull-back of an arbitrary line omitting points $p_1, \dots, p_s$. A polynomial from L defines a curve passing through points p_j with multiplicities at least m_j if and only if the corresponding function from the linear system of divisor vanishes along exceptional lines with the same multiplicities.

We define the *dimension* of a system of curves to be

$$\dim L := \dim_{\mathbb{K}} L - 1.$$

The *expected dimension* is defined by the formula

$$\operatorname{edim} L := \max\left\{ \frac{(d+2)(d+1)}{2} - 1 - \sum_{j=1}^{s} \binom{m_j+1}{2}, -1 \right\}.$$

We call the dimension above "expected," because for points $p_1, \dots, p_s$ in general position (see Definition 8) we expect the dimension of a system $\mathcal{L}_d(m_1p_1, \dots, m_sp_s)$ to be equal to the number of all monomials of degree less than or equal to d, minus one (projectivisation) and minus the number of conditions imposed by these points. Clearly, a point of multiplicity m imposes $\binom{m+1}{2}$ conditions.

By linear algebra we get:

Remark 1. For every system of curves L

$$\dim L \geq \operatorname{edim} L.$$

Definition 2. We say that a system of curves L is *special*, if

$$\dim L > \operatorname{edim} L.$$

Otherwise we say that L is *non-special*.

Now we recall a few facts about lines, conics and cubics in $\mathbb{P}^2 = \mathbb{P}^2\mathbb{K}$:

Facts 3. *There always exists exactly one line through two distinct simple points $p_1, p_2 \in \mathbb{P}^2$.*

Five pairwise distinct simple points $p_1, \dots, p_5 \in \mathbb{P}^2$, with no three lying on a line, uniquely determine an irreducible conic.

A family of cubics passing through eight simple points $p_1, \dots, p_8 \in \mathbb{P}^2$, with no four of them collinear and no seven lying on one conic, is one-dimensional (it can be proved e.g. using methods presented in section 4).

Note that for such a family of cubics there always exists a uniquely determined point p_9 such that p_9 lies on every cubic passing through $p_1, \dots, p_8$ – this fact can be derived from Bézout's Theorem (Theorem 10). (It may happen that p_9 is the same point as one of the points $p_1, \dots, p_8$. Then we get that the curve passing through p_9 is tangent to some fixed line, which passes through this point. See "infinitely near point" in [**7**, Chapter V.3] for details).

Therefore our family of cubics passing through $p_1, \dots, p_9$ is one-dimensional if the ninth point lies on every cubic of this family, and zero-dimensional otherwise.

Definition 4. If a family of cubics passing through pairwise distinct points $p_1, \dots, p_9$ (such that no four of them are collinear and no seven lie on a conic) is one-dimensional, the position of points is said to be a *non-general position on a cubic.*

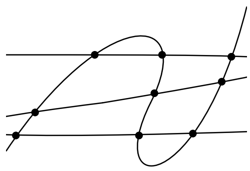


Figure 1. Nine points in a non-general position on a cubic

A non-general position of points on a cubic is independent of position of points on conics or lines. There exist points, e.g. $(0,0), (1,1), (-1,1), (2,4), (3,9), (3,11), (-3,7), (0,10), (-\frac{6212}{15159}, \frac{115138}{15159})$ – the last point was computed by a computer program Singular (see [**6**]), such that no three are collinear and no six lie on a conic, with the property that the first eight determine the ninth one.

Definition 5. *A configuration of s points*, where $s \leq 9$, is an assignment of a three-element sequence $(a_1, a_2, a_3) \in \{0,1\}^3$ to every subset of given s points in $\mathbb{P}^2$ in such a way that $a_1 = 1$ iff a subset is nine-element and points are in non-general position on a cubic; $a_2 = 1$ iff all points of our subset lie on an irreducible conic; $a_3 = 1$ iff they all lie on a line.

Note that in the definition above, is not necessary to require irreducibility of a conic. If we know that some points lie on an arbitrary conic (reducible or not), the third elements of the sequences assigned to the subsets of the set of points on the conic will distinguish these two situations.

In this paper we consider configurations of pairwise distinct, ordered points $p_1, \dots, p_s \in \mathbb{P}^2$ for $s \leq 9$. Our definition of a configuration makes sense for no more than nine points. For example ten points we would also have to know whether the points lie on an irreducible cubic.

We do not consider systems of curves passing through more than nine points since number of cases to solve becomes large, and a method of omitting a chosen point (Theorem 14) becomes much more complicated.

Remark 6. From now on, while dealing with a configuration of s points ($s \leq 9$), we will not look at all the subsets of these points, but only at subsets maximal with respect to number of points on irreducible conics and lines, taking into account – if the number of points is nine – whether all the points are in non-general position on a cubic or not.

We do not consider all the subsets of our points to avoid redundancy. So we will not say that e.g. the subsets of collinear points are also collinear, as it should be mentioned according to the definition.

For example if we consider a configuration with "five points collinear and two out of this line" we mean that exactly five points lie on a line. Our configuration looks like that:

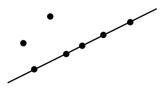


Figure 2. Five points collinear and two out of this line

Remark 7. In the example above, it may happen that the two points which do not lie on a line are collinear with one given point on the line. We will show that for simple points it does not make any difference for the dimension of a system of curves (of an arbitrary degree) passing through these points. If the point in the intersection of those lines is a double point, we will have to solve two cases separately.

Definition 8. Given a configuration, we say that *points $p_1, \dots, p_s$ are in general position* (for $s \leq 9$) if no three of them lie on a line, no six lie on an irreducible conic and points are not in the non-general position on a cubic.

Here is an example of six points in general position:

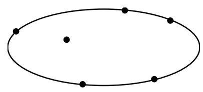


Figure 3. Six points in general position

Remark 9. For given points we can check algorithmically if they are in non-general position on a cubic, and how many of them maximally lie on a conic or a line, e.g. using computer program Singular.

3. Main tools.

3.1. Bézout's Theorem. We will state Bézout's Theorem, as it plays an important role in this paper. Recall that for a Hilbert polynomial of a set V, which is of the form

$$\mathrm{HP}_V(d) = c_m d^m + c_{m-1} d^{m-1} + \ldots + c_0, \quad c_m \neq 0,$$

we define a *degree of* V by the formula $\deg V := m! \cdot c_m$.

Theorem 10 (Bézout). *Let V be an algebraic set of positive dimension. Let $f \in \mathbb{K}\,[x_0, \ldots, x_n]$ be such a homogeneous polynomial that no component of V is contained in $V(f)$. Then* $\deg(V \cap V(f)) = \deg V \cdot \deg f$.

Corollary 11. *Let C_1 and C_2 be curves in $\mathbb{P}^2$ with no common component. Let d_1, d_2 be the degrees of homogeneous polynomials f_1, f_2 defining these curves. Then*

$$(C_1.C_2) = d_1 \cdot d_2,$$

where $(C_1.C_2)$ denotes the intersection number defined by the formula $(C_1.C_2) = \sum_{P \in C_1 \cap C_2} \dim_{\mathbb{K}} \mathcal{O}_{\mathbb{P}^2,P}/(f_1, f_2)$.

Hence, $\#(C_1 \cap C_2) \leq d_1 \cdot d_2$.

For more information and the proof see [**2**, Chapter 5.3].

3.2. Omitting points. We are going to formulate a theorem (cf. [**7**, Chapter V.4]) which is useful in checking whether a given system is special or not. To state it, we need one more definition:

Definition 12. We say that a point $p \in \mathbb{K}^2$ is *a base point* of a system $\mathcal{L}_d(m_1p_1, \ldots, m_sp_s)$ if every curve of the system passes through p.

Following Chapter V.4 of [**7**], two types of base points may be distinguished. The first type contains of *assigned base points*. These are exactly points $p_1, \ldots, p_s$. All the other base points belong to the second type – *unassigned base points*.

For example, in the system $\mathcal{L}_1(p_1, p_2, p_3)$ where points p_1, p_2, p_3 are collinear, every point of this line different from p_i's is an unassigned base point.

Remark 13. Since now, we will use the notion of *base point* only for unassigned base points.

Here is the theorem:

Theorem 14. *Let $p_1, \ldots, p_s \in \mathbb{K}^2$ and let $m_1, \ldots, m_{s-1} \in \mathbb{N}$. If the system $\mathcal{L}_d(m_1p_1, \ldots, m_{s-1}p_{s-1}, p_s)$ is special, then the system $\mathcal{L}_d(m_1p_1, \ldots, m_{s-1}p_{s-1})$ has a base point p_s or is special as well.*

Proof. Let $\mathcal{L}_d(m_1p_1,\dots,m_{s-1}p_{s-1},p_s)$ be a special system. Assume that this system without the simple point p_s is non-special. We will show that p_s is a base point of the system $\mathcal{L}_d(m_1p_1,\dots,m_{s-1}p_{s-1})$.

By assumption, the dimension of $\mathcal{L}_d(m_1p_1,\dots,m_{s-1}p_{s-1})$ is equal to the expected dimension of $\mathcal{L}_d(m_1p_1,\dots,m_{s-1}p_{s-1})$.

Consider two cases:

1. $\dim\mathcal{L}_d(m_1p_1,\dots,m_{s-1}p_{s-1}) = -1$.
   For an arbitrary point p we have that $\dim\mathcal{L}_d(m_1p_1,\dots,m_{s-1}p_{s-1},p) = -1$, and therefore the system $\mathcal{L}_d(m_1p_1,\dots,m_{s-1}p_{s-1},p)$ is non-special, which gives a contradiction.
2. $\operatorname{edim}\mathcal{L}_d(m_1p_1,\dots,m_{s-1}p_{s-1}) > -1$.
   Let us add a point p_s. The expected dimension decreased by one, while by assumption that system $\mathcal{L}_d(m_1p_1,\dots,m_{s-1}p_{s-1},p_s)$ is special, the dimension has not changed. This means that there are the same curves in the system $\mathcal{L}_d(m_1p_1,\dots,m_{s-1}p_{s-1})$ as in the system $\mathcal{L}_d(m_1p_1,\dots,m_{s-1}p_{s-1},p_s)$. Therefore p_s is a base point of the system $\mathcal{L}_d(m_1p_1,\dots,m_{s-1}p_{s-1})$. □

From now on, we will frequently use the theorem is this way: if a system $\mathcal{L}_d(m_1p_1,\dots,m_{s-1}p_{s-1})$ is non-special and does not have a base point then $\mathcal{L}_d(m_1p_1,\dots,m_{s-1}p_{s-1},p_s)$ is non-special. To see that a system $\mathcal{L}_d(m_1p_1,\dots,m_{s-1}p_{s-1})$ does not have a base point we will show how to omit an arbitrary simple point p_s with curves of degree d.

3.3. Remarks. If for fixed points $p_1,\dots,p_s\in\mathbb{K}^2$ and for fixed integers $m_1,\dots,m_s\in\mathbb{N}$ the system $\mathcal{L}_d(m_1p_1,\dots,m_sp_s)$ is non-empty and non-special for some $d\in\mathbb{N}$, then for every $d'\geqslant d$ the system $\mathcal{L}_{d'}(m_1p_1,\dots,m_sp_s)$ is also non-special. So while examining if, for a given configuration of points with multiplicities, systems $\mathcal{L}_d(m_1p_1,\dots,m_sp_s)$ are special or not we may finish on the first d for which the dimension of the system is non-negative and equal to the expected dimension.

If we apply Bézout's Theorem or Theorem 14 we get a system with fewer points. Therefore to show that a system is non-special, we can use induction on number of points. (The degree of curve which passes through our points does not change and the number of points to omit decreases.)

4. How to compute the dimension of a system of curves.

4.1. Examples of computing the dimension of systems $\mathcal{L}_d(p_1,\dots,p_s)$. We will show that the dimensions of systems $\mathcal{L}_d(p_1,\dots,p_s)$ may be computed using two tools: Bézout's Theorem and Theorem 14.

First we give a few examples, which will be the part of the proof of Theorem 18.

Example 15. Let us compute the dimensions of systems $L_d = \mathcal{L}_d(p_1, \ldots, p_8)$, where points $p_1, \ldots, p_5$ lie on a line, points p_6, p_7, p_8 lie out of this line and are collinear.

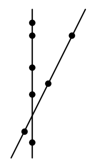


Figure 4. The considered configuration of points

By Bézout's Theorem we know that up to $d = 4$ the line passing through points $p_1, \ldots, p_5$ is a component of the system L_d, if only the system is non-empty.

For $d = 1$, a system of curves passing through given eight points is empty (because not all the points lie on one line), so its dimension is -1.

For $d = 2$, by Bézout's Theorem there are two components of L_2: the line passing through points $p_1, \ldots, p_5$ and the line passing through points p_6, p_7, p_8. These curves are uniquely determined, therefore $\dim L_2 = 0$. The expected dimension of this system is equal to -1.

For $d = 3$, by Bézout's Theorem the line passing through points $p_1, \ldots, p_5$ and the line passing through points p_6, p_7, p_8 are components of a system. The dimension of L_3 is therefore equal to the dimension of curves of degree one with no conditions, which is 2. So $\dim L_3 = 2$, while $\operatorname{edim} L_3 = 1$.

For $d = 4$, by Bézout's Theorem the line passing through points $p_1, \ldots, p_5$ is a component of a system. Consider the system $\mathcal{L}_3(p_6, p_7, p_8)$, in which p_6, p_7, p_8 are collinear. After taking out one point, e.g. p_8, we see that the system $\mathcal{L}_3(p_6, p_7)$ is non-special and does not have a base point – it is easy to omit every point using three lines. Therefore the non-speciality of the system $\mathcal{L}_3(p_6, p_7, p_8)$ implies that L_4 is also non-special and we have that $\dim L_4 = \operatorname{edim} L_4 = 6$. So we already know that for $d \geq 4$ the systems L_d are non-special.

To sum up, for the given configuration of points the system $\mathcal{L}_d(p_1, \ldots, p_8)$ is special for $d = 2, 3$.

Note that in the example above we do not have to know whether or not one of the points $p_1, \ldots, p_5$ lies on a line passing through points p_6, p_7 and p_8.

Example 16. Let us compute the dimensions of systems $L_d = \mathcal{L}_d(p_1, \ldots, p_8)$, where points $p_1, \ldots, p_6$ lie on an irreducible conic and points p_7, p_8 lie out of this conic.

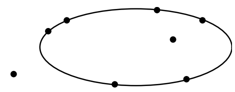


FIGURE 5. The considered configuration of points

By Bézout's Theorem, systems L_1 and L_2 are empty.

For $d = 3$, Bézout's Theorem gives us no information about the system. (Any curve of degree three passing through $p_1, \ldots, p_6$ and not containing our conic intersects the conic in exactly $2 \cdot 3$ points. These are $p_1, \ldots, p_6$. The assertion of Bézout's Theorem is fulfilled so we cannot say anything about the assumption.)

We use Theorem 14. We want to show that the system L_3 without one chosen point p_i is non-special and does not have a base point.

Take out one point from $p_1, \ldots, p_5$, say p_1, of the configuration. The system $L_3(p_2, \ldots, p_8)$ is non-special by remarks on page 234.

Now we will show how to omit p_1 using a curve of degree three. There exists a line passing through p_7 and one of the points $p_2, \ldots, p_6$ such that p_1 does not lie on it. Let this line be a component of a desired cubic. Let the conic passing through four remaining points of $p_2, \ldots, p_6$ and the point p_8 be the second component. By Bézout's Theorem we know that this conic does not pass through p_1. Therefore we have shown that L_3 is non-special. Its dimension and expected dimension equals 1.

So for the considered configuration of points, systems $\mathcal{L}_d(p_1, \ldots, p_8)$ are non-special for every d.

EXAMPLE 17. Let us compute the dimensions of systems $L_d = \mathcal{L}_d(p_1, \ldots, p_9)$, where points $p_1, \ldots, p_6$ lie on an irreducible conic and points p_7, p_8, p_9 lie out of this conic and are collinear, and no four points among $p_1, \ldots, p_9$ lie on a line.

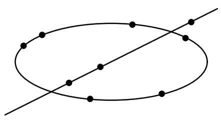


FIGURE 6. The considered configuration of points

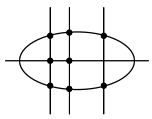
Bézout's Theorem implies emptiness of the systems L_1 and L_2.

For $d = 3$ Bézout's Theorem cannot be used. Also the method of omitting a chosen point that comes from Theorem 14 gives no result unless we see that we have to consider two cases.

The first case: points are in non-general position on a cubic (see Definition 4). The dimension of the system L_3 is then equal to 1 and the expected dimension is equal to 0. So the system is special.

In the second, complementary, case a chosen point may be omitted by a curve of degree three. Non-speciality of this system can be acquired as it is said in remarks on page 234. Therefore $\dim L_3 = \operatorname{edim} L_3 = 0$.

Note that both cases can occur (which verifies the fact that non-general position on a cubic cannot be deduced if we know only maximal number of points on conics and lines). As an example consider such eight points, with six lying on a conic:



Eight points $p_1, \ldots, p_8$ do not all lie on a conic, no four of them are collinear. Therefore a family of curves of degree three passing through these points is one-dimensional. It is easy to point out its generators: the first one is a conic passing through six points and a "horizontal" line, the second one – three "vertical" lines. By Bézout's Theorem the generators intersect in exactly nine points, which can be seen in the picture. So if the ninth point is given as the point of intersection of the generators we get a non-general position of points on a cubic, and if the ninth point is any point different from this one and $p_1, \ldots, p_8$ we get a general position.

To know all the dimensions of systems L_d we have to compute $\dim L_4$ for non-general position on a cubic. We will show a curve of degree four that omits a point on a conic, say p_1. Non-speciality of $L_4(p_2, \ldots, p_9)$ can be done by induction.

Let the conic passing through p_2, p_3, p_4, p_5 and p_9 be the first component of the desired quartic curve. Now we consider lines through pairs of points p_6 and p_7, p_7 and p_8, p_6 and p_8. As the second component we take such one of those lines which does not pass through point p_1. It always exists because points p_6, p_7 and p_8 are not collinear. Let the third component be the line passing through a remaining point (p_6, p_7 or p_8) and omitting p_1.

Therefore $\dim L_4 = \operatorname{edim} L_4 = 5$ for a non-general position of points on a cubic.

Summing up, system $\mathcal{L}_d(p_1, \ldots, p_9)$ is special iff $d = 3$ and the points are in non-general position on a cubic.

4.2. Conclusions. Having computed the dimensions of all systems up to nine points, we state the theorem:

THEOREM 18. *If $s \leq 9$ then for an arbitrary d and for an arbitrary configuration of points $p_1, \ldots, p_s$ the dimension of a system $\mathcal{L}_d(p_1, \ldots, p_s)$ may be computed using two tools: Bézout's Theorem and Theorem 14.*

We do not write a complete proof because of its length. All the possible configurations of points may be dealt with analogously to the examples above. We consider systems of curves of consecutive degrees passing through: no fixed points, one point, two points, three collinear points, three non-collinear points etc.

Full proof is available at the internet site:
`http://www.im.uj.edu.pl/LucjaFarnik/Publications/Appendix.pdf`

Corollary 19. *For $s \leq 9$ the dimension of a system $\mathcal{L}_d(p_1, \ldots, p_s)$ is independent of the coordinates of points $p_1, \ldots, p_s$, but depends only on configuration of points.*

4.3. Examples of computing the dimension of systems $\mathcal{L}_d(2p_1, p_2, \ldots, p_s)$. Now we will show a few examples how to compute the dimension of systems $\mathcal{L}_d(2p_1, p_2, \ldots, p_s)$.

Example 20. Let us compute the dimensions of systems $L_d = \mathcal{L}_d(2p_1)$.

Obviously, the system L_1 is empty.

We know that $\operatorname{edim} L_2 = 2$. If $\dim L_2 \geq 3$ then after introducing three arbitrary points p_2, p_3 and p_4, the dimension of the system $\mathcal{L}_2(2p_1, p_2, p_3, p_4)$ would be non-negative. Bézout's Theorem gives a contradiction.

To sum up, the system $L_d = \mathcal{L}_d(2p_1)$ is non-special for every d.

Example 21. Let us compute the dimensions of systems $L_d = \mathcal{L}_d(2p_1, p_2, \ldots, p_9)$, where points $p_2, \ldots, p_9$ lie on an irreducible conic and a double point p_1 is out of this conic.

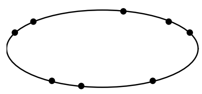


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Figure 7. The considered configuration of points

For $d = 1, 2$, the system of curves passing through our points is empty by Bézout's Theorem.

For $d = 3$, the conic is a component of the system, unless the system is empty. As a line may pass through a point p_1 just once, the dimension of L_3 is equal to -1.

For $d = 4$ we will point out a curve of degree four that omits point p_2. Non-speciality of the system $\mathcal{L}_4(2p_1, p_3, \ldots, p_8)$ can be done by induction.

Let the conic through p_3, p_4, p_5, p_6 and p_1 be the first component of a quartic curve. We introduce a point p_{10} which lies on the conic and is different from p_i, for $i = 2, \ldots, 9$. Let the conic passing through points p_7, p_8, p_9, p_1 and p_{10} be the second component of the desired curve. By Bézout's Theorem, non of these conics passes through p_2.

Therefore the systems $\mathcal{L}_d(2p_1, p_2, \ldots, p_9)$ are non-special for every d.

Now we consider the same configuration of points as in the example above. The difference is that we relabel points in such a way that the double point p_1 lies on a conic. We will see that this has an impact on the dimension of the system L_3.

Example 22. Let us compute the dimensions of systems $L_d = \mathcal{L}_d(2p_1, p_2, \ldots, p_9)$, where points $p_1, \ldots, p_8$ lie on an irreducible conic and a point p_9 lies out of this conic.

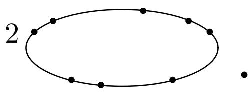

Figure 8. The considered configuration of points

As in the previous example, for $d = 1, 2$ the system of curves passing through given points is empty by Bézout's Theorem.

For $d = 3$, the conic is a component of the system. The second component must be a line through p_1 and p_9. Therefore the dimension of L_3 is equal to 0, while its expected dimension equals -1.

For $d = 4$, we will show a quartic curve such that it omits p_2. Non-speciality of $L_4(2p_1, p_3, \ldots, p_9)$ can be derived by induction. Take the conic through p_3, p_4, p_5, p_6 and p_1 to be the first component of the curve of degree four. Introduce point p_{10} which does not lie on this conic and is different from p_9. Let the conic passing through p_7, p_8, p_9, p_{10} and p_1 be the second component of the desired curve.

So one of the systems $\mathcal{L}_d(2p_1, p_2, \ldots, p_9)$, namely L_3, is special. Recall that in the previous example with the same configuration but different position of a double point all L_d's were non-special.

Example 23. Let us compute the dimensions of systems $L_d = \mathcal{L}_d(2p_1, p_2, \ldots, p_8)$, where points p_2, p_3, p_4 are collinear, points p_5, p_6, p_7 are also collinear, and a double point p_1 and a simple point p_8 lie out of these lines.

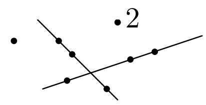

Figure 9. The considered configuration of points

By Bézout's Theorem the systems L_1 and L_2 are empty.

For $d = 3$ the expected dimension equals -1. Using Bézout's Theorem and Theorem 14 for the system with seven points, we obtain that through six points (two triples of collinear points) and one double point passes a uniquely determined cubic. By Bézout's Theorem it intersects a line, which passes through p_1 and p_8, three times. Therefore if p_8 lies on this cubic then the dimension of L_3 is equal to 0. Otherwise there does not exist a curve of degree three passing through all the points $p_1, \ldots, p_8$ so L_3 is empty in this case.

To know the dimensions for all d, we have to compute the dimension of L_4 in the first case. We will show a curve of degree four which omits p_2. The system $\mathcal{L}_4(2p_1, p_3, p_4, \ldots, p_8)$ is non-special, which can be shown by induction.

Let the conic passing through p_3, p_4, p_5, p_6 and p_1 be the first component of our curve. Now we consider lines through pairs of points p_1 and p_7, p_7 and p_8, p_1 and p_8. As the second component we take one of these lines which does not pass through p_2 – such a line exists because p_1, p_7 and p_8 are not collinear. The third component is a line passing through one remaining point among p_1, p_7, p_8, omitting p_2. So in this case $\dim L_4 = \operatorname{edim} L_4 = 4$.

Therefore the system $\mathcal{L}_d(2p_1, p_2, \ldots, p_8)$ is special iff $d = 3$ and all the points lie on an irreducible, singular cubic with a node in the double point.

Remark 24. As in the case of non-general position on a cubic, the fact that points lie on a nodal cubic is independent of position of the points on conics or lines. There exist points, such that no three are collinear and no six lie on a conic, which determine a nodal cubic, e.g. a double point $(0, 0)$ and simple points $(-1, 0), (1, 2), (1, -2), (2, 12), (2, -12), (3, 30), (3, -30), (4, 80)$.

4.4. Conclusions. As we saw in Example 23, if we consider systems with a double point, we have to extend a definition of configuration: apart from having information about position of points on conics, lines and non-general position on a cubic, we have to know whether or not the points lie on a singular cubic with a node in the double point.

After computing the dimensions of all the systems $\mathcal{L}_d(2p_1, p_2, \ldots, p_s)$ for $s \leqslant 9$, we may state a theorem similar to Theorem 18:

Theorem 25. *If $s \leq 9$ then for an arbitrary d and for an arbitrary configuration of points $p_1, \ldots, p_s$ the dimension of a system $\mathcal{L}_d(2p_1, p_2, \ldots, p_s)$ may be computed using only Bézout's Theorem and Theorem 14.*

A proof can be completed by analysing all the configurations, analogously to those presented in examples. The number of possibilities to consider is much bigger then in the proof of Theorem 18 because we have to distinguish the situations where a configuration of points is the same but the points are relabelled in such a way that a double point lies on a different curve (or out of curves given in the description of a configuration). For example for six points, where p_1 is a double point, we will have to consider such cases: six points on a conic; six points collinear; five points, among which is p_1, collinear; five simple points collinear; four points, among which is p_1, collinear and two remaining points on one line with p_1; four points, among which is p_1, collinear and two remaining points on a line which does not pass through p_1; four simple points collinear; three points, among which is p_1, collinear and three remaining points also collinear, and the lines do not intersect in p_1; three points, among which is p_1, collinear and two of the remaining points on one line with p_1; three points, among which is p_1, collinear and the remaining points in general position with no two collinear with p_1; three simple points collinear; six points in general position.

With a generalized definition of configuration, we obtain a corollary:

Corollary 26. *For $s \leq 9$ the dimension of a system $\mathcal{L}_d(2p_1, p_2, \ldots, p_s)$ is independent of coordinates of points $p_1, \ldots, p_s$, but depends only on configuration of points.*

4.5. Closing remarks. Methods presented in this paper are also useful while computing the dimensions of other systems with number of points less than or equal to nine. Note that we may have to extend the definition of configuration. In extending the definition -1-curves are involved (see [**4**]).

We have examined many cases to find a counterexample for which Bézout's Theorem and omitting of points do not work. Systems with up to nine points which appeared at first to be the counterexamples eventually were done. The problem which occurred was with the definition of configuration. It is known from [**4**] that there are infinitely many -1-curves which should be dealt with when considering configurations of nine multiple points.

We state a conjecture:

Conjecture 27. *Given a complete description of a configuration of up to nine multiple points we can always proceed with Bézout's Theorem and Theorem 14 to calculate the dimension of a system of curves of any degree d.*

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Institute of Mathematics
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland
e-mail: lucja.farnik@gmail.com