UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLVII
2009

WEAK FUNDAMENTAL SOLUTION OF THE FIRST ORDER EVOLUTION EQUATION

by Mariusz Jużyniec

Abstract. The purpose of this paper is to present some theorems on existence, uniqueness, continuity and differentiability with respect to a parameter h of a weak solution of the evolution equation $\dot{u}(t) = A(h,t)u(t)+f(h,t)$ in case when operators $A(h,t)$ have domains depending on a parameter h.

Introduction. We consider the abstract first-order initial value problem

$$\frac{d}{dt}u(t) = A(h,t)u(t) + f(h,t) \quad \text{for} \quad t \in [0,T], \tag{1}$$

$$u(0) = x_h^0. \tag{2}$$

It is known that under some assumption on the family of the operators $A(h,t)$ and the function f, the problem (1)–(2) has the unique classical solution given by

$$u(h,t) = U(h,t,0)x_h^0 + \int_0^t U(h,t,s)f(h,s)ds, \tag{3}$$

where, for each $h \in \Omega$, U is the fundamental solution for the problem (1)–(2). In this paper we investigate the continuity and differentiability of the mapping

$$\Omega \times [0,T] \ni (h,t) \longrightarrow u(h,t) \in X,$$

where $u(h,\cdot)$ is a suitable defined weak solution of the problem (1)–(2).

1. Preliminaries. Now we consider a family $\{A(t)\}_{t\in[0,T]} \subset C(X)$ of densely defined operators. Assume that the domains $D(A(t)^*) = D^*$ are independent of $t \in [0,T]$ and suppose that $\forall t \in [0,T] : \ 0 \in \rho(A(t))$. By Theorem 5, paper [**6**], for any $t,s \in [0,T] : \ \overline{A^{-1}(t)A(s)} \in Aut(X)$.

Let

$$B(t,s) := \overline{A^{-1}(t)A(s)}.$$

THEOREM 1. *Let the family $\{A(t)\}_{t\in[0,T]}$ be strongly continuously differentiable and suppose that for each $t \in [0,T]$:*

(a) $A(t) \in C(X)$ and $\overline{D(A(t))} = X$,
(b) $0 \in \rho(A(t))$,
(c) a mapping $[0,T] \ni s \to B(s,t) \in Aut(X)$ is continuous in $s = t$,

then

(i) operators $A^{-1}(t)A^{'}(s)$ are bounded,
(ii) there exists $K \geq 0$ that

$$\|A^{-1}(t)A^{'}(s)\| \leq K \text{ for } t,s \in [0,T].$$

PROOF. By Theorem 7, paper [**6**], the family $\{A^{*}(t)\}_{t\in[0,T]}$ is w^{*}-differentiable. It easy to see that

$$[A^{'}(t)]^{*} = [A^{*}(t)]^{'}.$$

It follows that $[A^{-1}(t)A^{'}(s)]^{*} = [A^{'}(s)]^{*}[A^{-1}(t)]^{*}$. This operator is closed and with domain X^{*}, therefore the operator $A^{-1}(t)A^{'}(s)$ is bounded.

To prove (ii), first note that for any $x \in X$ and $v \in X^{*}$

$$\left\langle \overline{A^{-1}(0)A^{'}(s)}x, v \right\rangle = \left\langle \frac{\partial}{\partial s}\overline{A^{-1}(0)A(s)}x, v \right\rangle,$$

where $\frac{\partial}{\partial s}\overline{A^{-1}(0)A(s)}$ is a weak derivative of the family $\{\overline{A^{-1}(0)A(s)}\}$.

Indeed, for $x \in D$

$$\begin{aligned}\left\langle \frac{\partial}{\partial s}\overline{A^{-1}(0)A(s)}x, v \right\rangle &= \lim_{h\to 0}\left\langle \frac{A^{-1}(0)A(s+h)x - A^{-1}(0)A(s)x}{h}, v \right\rangle \\ &= \lim_{h\to 0}\left\langle \frac{A(s+h)x - A(s)x}{h}, \left(A^{-1}(0)\right)^{*} v \right\rangle = \left\langle A^{'}(s)x, \left(A^{-1}(0)\right)^{*} v \right\rangle \\ &= \left\langle A^{-1}(0)A^{'}(s)x, v \right\rangle\end{aligned}$$

and by density of D in X and Theorem 7, paper [**6**], it holds for each $x \in X$.

By Theorem 7 (iv), paper [**6**], the mapping $A^{-1}(0)A^{'}(\cdot)$ is weakly continuous, so it is uniformly bounded, i.e.

$$\|A^{-1}(0)A^{'}(s)\| \leq M \text{ for } s \in [0,T].$$

This implies, by Theorem 5 (i), paper [**6**], that

$$\|A^{-1}(t)A^{'}(s)\| \leq \|A^{-1}(t)A(0)\|\|A^{-1}(0)A^{'}(s)\| \leq K < \infty.$$

This ends proof. $\square$

2. Existence and uniqueness of the weak solution. In this section we consider a family of operators $\{A(t)\} \subset C(X)$, $t \in [0,T]$, where for every $t \in [0,T]: \ D(A(t)) = D$, $\overline{D} = X$ and $0 \in \varrho(A(t))$.

We investigate the Cauchy problem

$$\frac{du}{dt} = A(t)u, \qquad u(s) = x, \quad 0 \leq s \leq t \leq T, \tag{4}$$

where $x \in X$.

Definition 1. An operator valued function

$$U : \Delta_T := \{(t,s) : 0 \leq s \leq t \leq T\} \ni (t,s) \longrightarrow U(t,s) \in B(X)$$

is called the fundamental solution of the problem (4) if

(i) the family $\{U(t,s)\}$ is strongly continuous with respect to $(t,s) \in \Delta_T$,
(ii) for each $(t,s) \in \triangle_T : \|U(t,s)\| \leq Me^{\beta(t-s)}$,
(iii) for $0 \leq s \leq r \leq t \leq T : U(t,t) = I, \quad U(t,r)U(r,s) = U(t,s)$,
(iv) for each $x \in D : U(t,s)x \in D$,
(v) for each $x \in D$ and $(t,s) \in \Delta_T$:

$$\frac{\partial}{\partial t}U(t,s)x = A(t)U(t,s)x,$$

$$\frac{\partial}{\partial s}U(t,s)x = -U(t,s)A(s)x,$$

(vi) the mappings $\Delta_T \ni (t,s) \to \frac{\partial}{\partial t}U(t,s)$ and $\Delta_T \ni (t,s) \to \frac{\partial}{\partial s}U(t,s)$ are strongly continuous on D.

In [**8**] is proved that

Theorem 2. *If the family $\{A(t)\} \subset B(X)$ is strongly continuous, then there exists exactly one fundamental solution of the problem* (4).

Definition 2. If there exists a sequence of bounded operators $A_n(t)$, $t \in [0,T]$, such that

$$\forall n \in \mathbb{N} : \quad \text{a function } \ t \longrightarrow A_n(t) \text{ is strongly continuous}$$

and

$$\forall x \in X : \ \lim_{n\to\infty} \sup_{0 \leq t \leq T} \|[A(t) - A_n(t)]A^{-1}(t)x\| = 0$$

and the fundamental solutions of the problems

$$\frac{du}{dt}(t) = A_n(t)u(t), \quad u(s) = x \tag{5}$$

are uniformly bounded, i.e.

$$\|U_n(t,s)\| \leq M, \tag{6}$$

where M does not depend on $n \in \mathbb{N}$ and $(t,s) \in \Delta_T$, then we say that the family $\{A(t)\},\ t \in [0,T]$, is stably approximated by the sequence $\{A_n(t)\}$.

Definition 3. A family $\{A(t)\}_{t\in[0,T]}$ is called stable if there are constants $M \geq 1$ and $\beta \geq 0$ such that

$$\forall t \in [0,T]: \quad (\beta,\infty) \subset \varrho(A(t)) \tag{7}$$

and

$$\Big\| \prod_{j=1}^{n} R(\lambda, A(t_j)) \Big\| \leq M(\lambda-\beta)^{-n} \quad \text{for } \lambda > \beta \tag{8}$$

and for every finite sequence $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq T, \quad n \in \mathbb{N}$.

Now we give sufficient conditions for the family $\{A(t)\}_{t\in[0,T]}$ to be stably approximated.

Theorem 3. (*see* [**2**]) *Assume that*

(*i*) *the family* $\{A(t)\}_{t\in[0,T]}$ *is stable,*
(*ii*) *for each* $t \in [0,T]$, *the domain* $D(A(t)) = D$ *does not depend on* t,
(*iii*) $\forall x \in D$, *the mapping* $[0,T] \ni t \longrightarrow A(t)x \in X$ *is of class* C^1,
(*iv*) *for each* $t \in [0,T] : 0 \in \varrho(A(t))$.

Then the family $\{A(t)\}_{t\in[0,T]}$ *is stable approximated by the sequence* $\{A_n(t)\}$ *defined by*

$$A_n(t) := -nA(t)R(n, A(t)). \tag{9}$$

The sequence $(U_n(t,s))$ *of the fundamental solutions corresponding to* $\{A_n(t)\}$ *is strongly and uniformly convergent to* $U(t,s)$ *in* Δ_T.

Assumption $\mathrm{A_t}$. *Now suppose that*

(*i*) $\forall t \in [0,T]: \quad A(t) \subset C(X)$ *with* $D(A(t)) = D$ *and* $\overline{D} = X$,
(*ii*) $\forall t \in [0,T]: \quad D(A^*(t)) = D^*$,
(*iii*) $\exists M \geq 1,\ \beta \geq 0\ \forall t \in [0,T]: \quad A(t) \in G(M,\beta)$,
(*iv*) $\forall t \in [0,T]: \quad 0 \in \varrho(A(t))$,
(*v*) $\forall s \in [0,T]: \quad [0,T] \ni t \to \overline{A^{-1}(s)A(t)} \in Aut(X)$ *is continuous in* $t = s$,
(*vi*) $\forall s \in [0,T]$ *the family* $\{\overline{A^{-1}(s)A(t)}\}_{t\in[0,T]}$ *has weakly continuous weak derivative.*

Example. Let A ba a generator of a strongly continuous semigroup and let $0 \in \rho(A)$. Suppose that $\forall t \in [0,T]: \quad \Phi_t \in Aut(X) \cap Aut(D(A))$. If a mapping $t \to \Phi_t$ is suitable regular, then a family $\{A_t\}_{t\in[0,T]},\ A_t := \Phi_t \circ A$, satisfies Assumption $\mathrm{A_t}$.

THEOREM 4. *If the family $\{A(t)\}$ is stable and strongly continuously differentiable and satisfies Assumption* $\mathrm{A_t}$, *then the fundamental solution of the problem* (4) *has following properties:*

(i) $\forall x \in X\ \forall s \in [0,T)\ \forall t \in [s,T]\ \forall v \in D^*$:

$$\frac{\partial}{\partial t}\langle U(t,s)x, v\rangle = \langle U(t,s)x, A^*(t)v\rangle\,, \tag{10}$$

(ii) $U^*(t,s)(D^*) \subset D^*$,

(iii) $\forall x \in X\ \forall t \in (0,T]\ \forall s \in [0,t]\ \forall v \in D^*$:

$$\frac{\partial}{\partial s}\langle U(t,s)x, v\rangle = -\langle x, A^*(s)U^*(t,s)v\rangle\,. \tag{11}$$

(iv) $\Delta_T \ni (t,s) \to \frac{\partial}{\partial t}\langle U(t,s)x, v\rangle$ *and* $\Delta_T \ni (t,s) \to \frac{\partial}{\partial s}\langle U(t,s)x, v\rangle$ *are continuous.*

PROOF. Equation (10) holds for $x \in D$. By w^*-differentiability of the function $A^*(\cdot)v$, $\|A^*(\cdot)v\|$ is uniformly bounded. Now, in view of the Banach–Steinhaus Theorem, (10) holds for each $x \in X$.

To prove *(ii)* we show that

$$\forall x \in D: \quad |\langle A(s)x, U^*(t,s)v\rangle| \le C\|x\|,$$

where a constant C does not depend on x. Since

$$\begin{aligned} |\langle A(s)x, U_n^*(t,s)v\rangle| &= |\langle A^{-1}(t)U_n(t,s)A(s)x, A^*(t)v\rangle| \\ &\le \|A^*(t)v\|\|A^{-1}(t)U_n(t,s)A(s)x\|, \end{aligned} \tag{12}$$

where $U_n(t,s)$ is the fundamental solution of the problem (5) with $A_n(t) = -nA(t)R(n,A(t))$, it is enough, by Theorem 5, paper [**6**], to show that the sequence $(W_n(t,s))$, where

$$W_n(t,s) := A^{-1}(t)U_n(t,s)A(s), \quad D(W_n(t,s)) = D \tag{13}$$

has following properties:

a) $\forall n\ \ \forall t,s \in \triangle_T$, operators $W_n(t,s)$ are densely defined and bounded,

b) the family $(\overline{W_n(t,s)})$ is uniformly bounded, i.e.

$$\exists K > 0\ \ \forall n \in \mathbb{N}\ \ \forall (t,s) \in \triangle_T: \ \ \|\overline{W_n(t,s)}\| \le K.$$

Functions $t \longrightarrow A^{-1}(t)$ and $(t,s) \longrightarrow U_n(t,s)$ are strongly continuously differentiable, so by (13) the function $t \longrightarrow W_n(t,s)x$ is of class C^1 and

$$\begin{aligned} \frac{\partial W_n(t,s)}{\partial t}x &= -A^{-1}(t)A'(t)A^{-1}(t)U_n(t,s)A(s)x + A^{-1}(t)A_n(t)U_n(t,s)A(s)x \\ &= [-A^{-1}(t)A'(t) + A_n(t)]A^{-1}(t)U_n(t,s)A(s)x. \end{aligned}$$

So

$$\frac{\partial W_n(t,s)}{\partial t}x = A_n(t)W_n(t,s)x - A^{-1}(t)A^{'}(t)W_n(t,s)x. \tag{14}$$

It follows from (14) that

$$W_n(t,s)x = U_n(t,s)x - \int_s^t U_n(t,r)A^{-1}(r)A^{'}(r)W_n(r,s)xdr, \tag{15}$$

where

$$W_n(s,s)x = U_n(s,s)x = x. \tag{16}$$

Let

$$W_n^{(0)}(t,s)x := U_n(t,s)x, \quad \text{for } \ x \in X \tag{17}$$

and

$$W_n^{(k)}(t,s)x := -\int_s^t U_n(t,r)A^{-1}(r)A^{'}(r)W_n^{(k-1)}(r,s)xdr \quad \text{for } x \in D. \tag{18}$$

Then one can verify, by induction, that $W_n^{(k)}(t,s)(D) \subset D$, and the operators $W_n^{(k)}(t,s)$ can be extended, by continuity, to bounded, everywhere defined operators. From (16) it follows that

$$\|\overline{W_n^{(0)}(t,s)}x\| = \|U_n(t,s)x\| \le M\|x\|. \tag{19}$$

By (9), (18) and (19)

$$\|\overline{W_n^{(k)}(t,s)}\| \le M^{k+1}K^k\frac{(t-s)^k}{k!}, \quad k = 0,1,2,\ldots \tag{20}$$

The estimates (20) imply that the series $\overline{W_n(t,s)} := \sum_{k=0}^{\infty} \overline{W_n^{(k)}(t,s)}$ converges uniformly, in the uniform operator topology, for $(t,s) \in \Delta_T$. As a consequence $\overline{W_n(t,s)}$ is uniformly continuous in $B(X)$ for $(t,s) \in \Delta_T$. The continuity of $\overline{W_n^{(k)}(t,s)}$, $n \in \mathbb{N}$ and (20) imply that one can interchange the summation and integration in

$$\overline{W_n(t,s)} = \sum_{k=0}^{\infty}\overline{W_n^{(k)}(t,s)} = U_n(t,s) - \sum_{k=1}^{\infty}\int_s^t U_n(t,r)A^{-1}(r)A^{'}(r)\overline{W_n^{(k-1)}}(r,s)dr$$

and thus see that $\overline{W_n(t,s)}$ is a solution of the integral equation (15). Moreover there exists constant K_1 such that

$$\forall (t,s) \in \Delta_T \ \ \forall n \in \mathbb{N}: \quad \|\overline{W_n(t,s)}\| \le K_1.$$

So

$$|\langle A(s)x, U_n^*(t,s)v\rangle| \le K_1\|A^*(t)v\|\|x\|$$

and passing with n to ∞ we have

$$|\langle A(s)x, U^*(t,s)v\rangle| \leq K_1 \|A^*(t)v\| \|x\|, \ \ \forall x \in D.$$

This implies that

$$U^*(t,s)v \in D(A^*(s)) = D^*.$$

Moreover

$$\sup\{\|A^*(s)U^*(t,s)v\| \ : \ (t,s) \in \triangle_T\} \leq K_2, \tag{21}$$

where $K_2 := K_1 \cdot \sup\{\|A^*(t)v\| \ : \ t \in [0,T]\} < \infty$.

Equation (11) holds for $x \in D$ and by (21), it holds for $x \in X$. □

We now investigate the Cauchy problem

$$\frac{du}{dt} = A(t)u + f(t), \qquad u(0) = x \ \ 0 \leq t \leq T, \tag{22}$$

where $f \in L^1(0,T;X)$, $\{A(t)\}_{t\in[0,T]} \subset C(X)$ and $\forall t \in [0,T] : \ \ \overline{D(A(t))} = X$ and $D(A^*(t)) = D^*$.

Definition 4. A function $u \in C([0,T];X)$ is a weak solution of (22) if for each $v \in D^*$

(i) the function $[0,T] \ni t \longrightarrow \langle u(t), v\rangle \in \mathbb{R}$ is absolutely continuous and differentiable almost everywhere in $[0,T]$,
(ii) $\forall v \in D^* : \ \ \frac{d}{dt}\langle u(t), v\rangle = \langle u(t), A^*(t)v\rangle + \langle f(t), v\rangle$ *a.e. in* $[0,T]$,
(iii) $u(0) = x$.

Theorem 5. *If the family $\{A(t)\}_{t\in[0,T]}$ satisfies the assumptions of Theorem 4 and $f \in L^1(0,T;X)$, then for each $x \in X$ there exists exactly one weak solution of the problem* (22) *and it is given by*

$$u(t) = U(t,0)x + \int_0^t U(t,s)f(s)ds \quad t \in [0,T], \tag{23}$$

where $\{U(t,s)\}_{(t,s)\in\Delta_T}$ is the fundamental solution of the problem (4).

Proof. By Theorem 3, there exists the fundamental solution of the problem (4) and the function u, given by (23), is continuous.

Fix $v \in D^*$. First assume that $f \in C([0,T];X)$. By (10) and continuity of the function $\Delta_T \times X \ni (t,s,x) \to U(t,s)x \in X$ we see that

$$\frac{d}{dt}\int_0^t \langle U(t,s)f(s), v\rangle ds = \langle f(t), v\rangle + \int_0^t \langle U(t,s)f(s), A^*(t)v\rangle ds.$$

This implies that

$$\frac{d}{dt}\langle u(t), v\rangle = \langle u(t), A^*(t)v\rangle + \langle f(t), v\rangle. \tag{24}$$

Now suppose that f is Bochner integrable, i.e. $f \in L^1(0,T;X)$. Let ϕ_n be an approximate identity. Then $f * \phi_n$ is of class C^∞ and $f * \phi_n \to f$ in L^1 norm. Let a sequence $(f_n) \subset C([0,T];X)$ converg to f in $L^1(0,T;X)$ norm. Let

$$u_n(t) := U(t,0)x + \int_0^t U(t,s) f_n(s)ds \ \text{ for } \ t \in [0,T]. \tag{25}$$

Function u_n is a weak solution of the problem

$$\frac{d}{dt}\langle u_n(t), v\rangle = \langle u_n(t), A^*(t)v\rangle + \langle f_n(t), v\rangle, \quad u_n(0) = x$$

and by integrating this equation over $[0,t]$, we have

$$\langle u_n(t), v\rangle = \langle x, v\rangle + \int_0^t [\langle u_n(s), A^*(s)v\rangle + \langle f_n(s), v\rangle]ds. \tag{26}$$

From

$$\|u_n(t) - u(t)\| \leq Me^{\beta T} \int_0^T \|f_n(s) - f(s)\|ds$$

it follows that $u_n \to u$ in $C([0,T],X)$ norm. Thus, by the Lebesgue Theorem, u, given by (23), satisfies equation

$$\langle u(t), v\rangle = \langle x, v\rangle + \int_0^t [\langle u(s), A^*(s)v\rangle + \langle f(s), v\rangle]ds.$$

Existence of the weak solution is proved.

To prove uniqueness suppose that there exists $\overline{u}$ another weak solution of the problem (22) and set $w = u - \overline{u}$. Then the function w is a weak solution of the problem

$$\frac{d}{dt}w(t) = A(t)w(t), \quad w(0) = 0.$$

Fix $v \in D^*$ and $t \in (0,T]$. By Theorem 4,

$$\begin{aligned}\frac{d}{ds}\langle U(t,s)w(s), v\rangle &= \frac{d}{ds}\langle w(s), U^*(t,s)v\rangle \\ &= \langle w(s), A^*(s)U^*(t,s)v\rangle - \langle w(s), A^*(s)U^*(t,s)v\rangle = 0.\end{aligned}$$

From continuity of the function $\langle U(t,\cdot)w(\cdot), v\rangle$ it follows that

$$\langle U(t,s)w(s), v\rangle = \mathit{const}.$$

If $s_1, s_2 \in [0,t]$, then $\langle U(t,s_1)w(s_1), v\rangle = \langle U(t,s_2)w(s_2), v\rangle$ for each $v \in D^*$. By w^*-density of D^*, $U(t,s_1)w(s_1) = U(t,s_2)w(s_2)$ for $s_1, s_2 \in [0,t]$. Taking $s_1 = 0$ and $s_2 = t$ we get $w(t) = 0$. $\square$

3. Dependence of the weak solution on a parameter. Let Ω be a compact subset of $\mathbb{R}^m$. We shell consider the following initial value problem with a parameter $h \in \Omega$:

$$\frac{d}{dt}u(t) = A(h,t)u(t) + f_h(t), \quad u(0) = u_h^0, \tag{27}$$

where $A : \Omega \times [0,T] \ni (h,t) \longrightarrow A(h,t) \in C(X),\ u_h^0 \in X,\ \ f_h \in L^1(0,T;X)$.

DEFINITION 5. A family $\{A(h,t)\}_{(h,t)\in\Omega\times[0,T]}$ is said to be uniformly stable with respect to $h \in \Omega$, when there exist constants M and β such that for any $h \in \Omega$ the family $\{A(h,t)\}_{t\in[0,T]}$ is stable with constants M and β.

In this section we adopt the following:

ASSUMPTION $\mathrm{A_{h,t}}$.

(i) $\forall t \in [0,T]\ \forall h \in \Omega : \ \ D(A(h,t)) = D_h$,

(ii) $\forall t \in [0,T]\ \forall h \in \Omega : \ \ D(A^*(h,t)) = D^*$,

(iii) $\forall t \in [0,T]\ \forall h \in \Omega : \ \ 0 \in \varrho(A(h,t))$,

(iv) $\forall h,k \in \Omega\ \forall s \in [0,T]$, *the mapping*

$$[0,T] \ni t \longrightarrow \overline{A^{-1}(k,s)A(h,t)} \in Aut(X)$$

is continuous in $t = s$,

(v) $\forall h,k \in \Omega\ \forall s \in [0,t]$, *the family* $\left\{\overline{A^{-1}(h,s)A(h,t)}\right\}_{t\in[0,T]}$ *has weakly continuous weak derivative,*

(vi) $\forall k \in \Omega\ \forall t \in [0,T]$, *a mapping*

$$\Omega \ni h \longrightarrow \overline{A^{-1}(k,t)A(h,t)} \in Aut(X)$$

is continuous, uniformly in $t \in [0,T]$.

One can easily verify (see Theorems 5 and 6, paper [**6**]) that this operators are well-defined.

COROLLARY 1. *From Assumption* $\mathrm{A_{h,t}}$ *it follows that mappings*

- $[0,T] \ni t \longrightarrow \overline{A^{-1}(k,s)A(h,t)} \in Aut(X)$,
- $[0,T] \ni t \longrightarrow \overline{A^{-1}(h,t)A(k,s)} \in Aut(X)$,
- $[0,T] \ni t \longrightarrow \overline{A^{-1}(k,t)A(h,t)} \in Aut(X)$,

are continuous.

PROOF. One can easily verify that $[0,T] \ni t \longrightarrow \overline{A^{-1}(k,s)A(h,t)} \in Aut(X)$ is continuous.

It is known that if a mapping $[0,T] \ni t \longrightarrow B(t) \in Aut(X)$ is continuous and $\forall t \in [0,T] : 0 \in \varrho(B(t))$, then the mapping $[0,T] \ni t \longrightarrow B^{-1}(t) \in Aut(X)$ is continuous. Continuity of the mapping $[0,T] \ni t \longrightarrow \overline{A^{-1}(h,t)A(k,s)} \in Aut(X)$ follows from the above. $\square$

THEOREM 6. *If the family* $\{A(h,t)\}_{(h,t)\in\Omega\times[0,T]}$ *is uniformly stable with respect to* $h\in\Omega$, *satisfies Assumption* $\mathrm{A_{h,t}}$ *and* $\forall h\in\Omega\ \forall x\in D_h : [0,T]\ni t\longrightarrow A(h,t)x\in X$ *is of class* C^1, *then there exists for each* $h\in\Omega$, *one fundamental solution* $\{U(h,t,s)\}_{(t,s)\in\Delta_T}\subset B(X)$. *Moreover for each* $v\in D^*$ *and* $x\in X$

(i) $\dfrac{\partial}{\partial t}\langle U(h,t,s)x,v\rangle=\langle U(h,t,s)x,A^*(h,t)v\rangle$,

(ii) $U^*(h,t,s)(D^*)\subset D^*$,

(iii) $\dfrac{\partial}{\partial s}\langle U(h,t,s)x,v\rangle=-\langle x,A^*(h,s)U^*(h,t,s)v\rangle$,

(iv) $\lim_{h\to h_0}\langle U(h,t,s)x,v\rangle=\langle U(h_0,t,s)x,v\rangle$ *uniformly in* $(t,s,x)\in\triangle_T\times K$, *where* K *is a compact subset of* X.

PROOF. (i), (ii) and (iii) follow from Theorems 3 and 4.

Fix $x\in X$ and $v\in D^*$.

$$\frac{d}{d\tau}\langle U(h,\tau,s)x,U^*(h_0,t,\tau)v\rangle = \langle U(h,\tau,s)x,[A^*(h,\tau)-A^*(h_0,\tau)]U^*(h_0,t,\tau)v\rangle. \tag{28}$$

By integrating (28) over $[s,t]$ we have

$$\langle U(h,t,s)x-U(h_0,t,s)x,v\rangle = \int_s^t\langle U(h,\tau,s)x,[A^*(h,\tau)-A^*(h_0,\tau)]U^*(h_0,t,\tau)v\rangle\,d\tau, \tag{29}$$

so

$$\langle U(h,t,s)x-U(h_0,t,s)x,v\rangle$$
$$=\int_s^t\left\langle U(h,\tau,s)x,\left\{\left[\overline{A^{-1}(h_0,\tau)A(h,\tau)}\right]^*-I^*\right\}A^*(h_0,\tau)U^*(h_0,t,\tau)v\right\rangle d\tau.$$

By (21), Assumption $\mathrm{A_{h,t}}$ and the Lebesgue Theorem

$$\lim_{h\to h_0}\langle U(h,t,s)x,v\rangle=\langle U(h_0,t,s)x,v\rangle \tag{30}$$

uniformly in $(t,s)\in\triangle_T$.

Let $B_h(t,s):X\longrightarrow\mathbb{R},\ h\in\Omega,\ (t,s)\in\triangle_T$ be a family of linear functionals given by

$$B_h(t,s)x:=\langle U(h,t,s)x,v\rangle\quad\text{for } x\in X$$

for fixed $v\in D^*$.

This family is uniformly bounded: $\|B_h(t,s)\|\le\|v\|\|U(h,t,s)\|\le Me^{\beta T}\|v\|$ and by (30)

$$\lim_{h\to h_0}\langle U(h,t,s)x,v\rangle=\lim_{h\to h_0}B_h(t,s)x=B_{h_0}(t,s)x=\langle U(h_0,t,s)x,v\rangle$$

uniformly in $(t,s,x)\in\triangle_T\times K$. □

Theorem 7. *Let X be a reflexive Banach space and suppose that the family $\{A(h,t)\}_{(h,t)\in\Omega\times[0,T]}$ satisfies the assumptions of Theorem 6, then $\forall v \in X^*$:*

$$\lim_{h\to h_0} \langle U(h,t,s)x, v\rangle = \langle U(h_0,t,s)x, v\rangle$$

uniformly in $(t,s,x) \in \triangle_T \times K$, where K is a compact subset of X.

Proof. By reflexivity of X, D^* is dense in X^*, so by Theorem 6 the assertion follows. □

Theorem 8. *If the family $\{A(h,t)\}_{(h,t)\in\Omega\times[0,T]}$ satisfies the assumptions of Theorem 7 and mappings $\Omega \ni h \longrightarrow u_h^0 \in X$, $h \longrightarrow f_h \in L^1(0,T;X)$ are continuous, then for each $h \in \Omega$ there exists exactly one weak solution of the problem (27) given by*

$$u_h(t) = U(h,t,0)u_h^0 + \int_0^t U(h,t,s)f_h(s)ds \tag{31}$$

and for any $v \in D^*$

$$\lim_{h\to h_0} \langle u_h(t), v\rangle = \langle u_{h_0}(t), v\rangle,$$

uniformly in $t \in [0,T]$.

Proof. By Theorem 5, function given by (31) is a weak solution of a problem (27) and

$$u_h(t) - u_{h_0}(t) = [U(h,t,0) - U(h_0,t,0)]u_h^0 + U(h_0,t,0)[u_h^0 - u_{h_0}^0]$$
$$+ \int_0^t [U(h,t,s) - U(h_0,t,s)]f_{h_0}(s)ds + \int_0^t U(h,t,s)[f_h(s) - f_{h_0}(s)]ds.$$

Fix $v \in D^*$. By Theorem 6 (iv), for $K := \{u_h^0\,;\, h \in \Omega\}$:

$$\lim_{h\to h_0} \langle [U(h,t,0) - U(h_0,t,0)]u_h^0, v\rangle = 0,$$

uniformly in $t \in [0,T]$.

From $\|U(h,t,s)\| \le Me^{\beta T}$ it follows that $\lim_{h\to h_0} U(h_0,t,0)[u_h^0 - u_{h_0}^0] = 0$, uniformly in $t \in [0,T]$ and moreover by continuity of the mapping $h \to f_h$,

$$\lim_{h\to h_0} \int_0^t U(h,t,s)[f_h(s) - f_{h_0}(s)]ds = 0,$$

uniformly in $t \in [0,T]$.

To complete the proof we note that there exists a sequence $(\varphi_n) \subset C([0,T]; X)$ such that $\lim_{n\to\infty} \varphi_n = f_{h_0}$ in L^1-norm. For fixed $\varepsilon > 0$ there exists such $n_0 \in \mathbb{N}$ that

$$\|f_{h_0} - \varphi_{n_0}\|_{L^1} \le \varepsilon(4Me^{\beta T}\|v\|)^{-1}. \tag{32}$$

From compactness of $K_1 := \{\varphi_{n_0}(s); s \in [0,T]\}$, the inequality $\|U(h,t,s)\| \leq Me^{\beta T}$, (32) and Theorem 6 (iv) there exists $\delta > 0$ for which

$$\left| \left\langle \int_0^t [U(h,t,s) - U(h_0,t,s)] f_{h_0}(s) ds, v \right\rangle \right| \leq 2Me^{\beta T} \|v\| \int_0^t \|f_{h_0}(s) - \varphi_{n_0}(s)\| ds$$

$$+ \int_0^t |\langle [U(h,t,s) - U(h_0,t,s)] \varphi_{n_0}(s), v \rangle| ds < \varepsilon$$

for $|h - h_0| < \delta$ and $t \in [0,T]$. The assertion is proved. □

From Theorems 7 and 8 it follows that

THEOREM 9. *If the assumptions of Theorem 8 hold in a reflexive Banach space X, then for any $v \in X^*$:*

$$\lim_{h \to h_0} \langle u_h(t), v \rangle = \langle u_{h_0}(t), v \rangle,$$

uniformly in $t \in [0,T]$.

We will now present theorems on differentiability of the weak solution with respect to a parameter h. To do this assume

ASSUMPTION $\mathrm{Z_h}$. *For each $v \in D^*$ a mapping*

$$\Omega \ni h \longrightarrow A^*(h,t)v \in X^*$$

is of class C^1 and $\dfrac{\partial}{\partial h}(A^*(h,t)v)$ is continuous with respect to $(h,t) \in \Omega \times [0,T]$.

It is not difficult to show that

COROLLARY 2. *For each $t \in [0,T]$ and $k \in \Omega$ the family $\left\{\overline{A^{-1}(k,t)A(h,t)}\right\}$ is weakly differentiable with respect to h and $\forall x \in X$, $\forall v \in X^*$:*

$$\frac{\partial}{\partial h} \left\langle \overline{A^{-1}(k,t)A(h,t)} x, v \right\rangle = \left\langle x, \frac{\partial}{\partial h} \left[A^*(h,t) \left(A^*(k,t)\right)^{-1} v \right] \right\rangle.$$

Moreover above partial derivative is a continuous function with respect to (h,t).

COROLLARY 3. *There exists a constant M_1 independent of h and s such that*

$$\left\| \frac{\overline{A^{-1}(k,s)A(h,s)} - I}{h-k} \right\| \leq M_1. \tag{33}$$

PROOF. By Assumption $\mathrm{Z_h}$ and Corollary 2, for each $v \in X^*$ we obtain

$$\lim_{h \to k} \left\langle \frac{\overline{A^{-1}(k,s)A(h,s)} - I}{h-k} x, v \right\rangle = \left\langle x, \frac{\partial}{\partial h} \left(\overline{A^{-1}(k,s)A(h,s)} \right)^*_{|h=k} v \right\rangle, \tag{34}$$

uniformly in $s \in [0,T]$. Setting

$$f(h,s) := \left\langle \overline{A^{-1}(k,s)A(h,s)}x, v \right\rangle ,$$

for fixed $k \in \Omega$, we will show that

$$\frac{f(k+\triangle h,s)-f(k,s)}{\triangle h} \to f_h^{'}(k,s), \quad \triangle h \to 0, \tag{35}$$

uniformly in s.

The mapping $(h,t) \longrightarrow f_h^{'}(h,s)$ is uniformly continuous, so for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(|h_1 - h_2| < \delta \quad \wedge \quad |s_1 - s_2| < \delta) \Rightarrow (|f_h^{'}(h_2,s_2) - f_h^{'}(h_1,s_1)| < \varepsilon). \tag{36}$$

By the Lagrange Theorem there exists $\theta \in (0,1)$ such that

$$f_h^{'}(k+\theta\triangle h,s) = \frac{f(k+\triangle h,s)-f(k,s)}{\triangle h}. \tag{37}$$

Setting $s_1 = s_2 = s$, $h_1 = k$ and $h_2 = k + \theta\triangle h$ for $|\triangle h| < \delta$ we have, by (36) and (37),

$$\left| \frac{f(k+\triangle h,s)-f(k,s)}{\triangle h} - f_h^{'}(k,s) \right| < \varepsilon.$$

Next, let us consider the mapping $B_x(h,s) \in B(X^*,\mathbb{R})$ defined by

$$B_x(h,s)v := \left\langle \frac{\overline{A^{-1}(k,s)A(h,s)} - I}{h-k} x, v \right\rangle .$$

The family $H := \{B_x(h,s) \in X^{**}\,;\,(h,s) \in \Omega \times [0,T]\}$ satisfies the assumptions of the Banach–Steinhaus Theorem, so

$$\|B_x(h,s)\| = \left\| \frac{\overline{A^{-1}(k,s)A(h,s)} - I}{h-k} x \right\| \leq M(x,k), \tag{38}$$

with a constant M independent of h and s. By inequality (38) and the Banach–Steinhaus Theorem we have

$$\left\| \frac{\overline{A^{-1}(k,s)A(h,s)} - I}{h-k} \right\| \leq M_1(k), \tag{39}$$

where a constant M_1 is independent of h and s. □

Let X be a reflexive Banach space.

THEOREM 10. *If the family $\{A(h,t)\}_{(h,t)\in\Omega\times[0,T]}$ satisfies the assumptions of Theorem 6 and Assumption* $\mathrm{Z_h}$, *mappings $\Omega \ni h \longrightarrow u_h^0 \in X$ and $f : \Omega \ni h \longrightarrow f_h \in L^1(0,T;X)$ are of class C^1, then for each $v \in D^*$ a mapping*

$$\Omega \times [0,T] \ni (h,t) \longrightarrow \langle u_h(t), v\rangle \in \mathbb{R}$$

is differentiable with respect to a parameter h and

$$\frac{\partial}{\partial h}\langle u_h(t),v\rangle_{|h=h_0} = \left\langle U(h_0,t,0)(u^0_{h_0})',v\right\rangle + \int_0^t \langle U(h_0,t,s)f'_{h_0}(s),v\rangle ds$$
$$+\int_0^t \left\langle u_{h_0}(s), \frac{\partial}{\partial h}A^*(h,s)_{|h=h_0}U^*(h_0,t,s)v\right\rangle ds.$$

PROOF. Let u_h be a weak solution of the problem (27). This function is continuous and almost everywhere weakly differentiable. Fix $v\in D^*$. A function $s\longrightarrow U^*(h_0,t,s)v$ is w^*-differentiable, so

$$\begin{aligned}\frac{d}{ds}\langle u_h(s),U^*(h_0,t,s)v\rangle &= \langle u_h(s),A^*(h,s)U^*(h_0,t,s)v\rangle\\ &+\langle f(h,s),U^*(h_0,t,s)v\rangle - \langle u_h(s),A^*(h,s)U^*(h_0,t,s)v\rangle.\end{aligned}\tag{40}$$

Integrating (40) over $[0,t]$ and applying this formula to $\left\langle \frac{u_h(t)-u_{h_0}(t)}{h-h_0},v\right\rangle$ we get

$$\begin{aligned}\left\langle \frac{u_h(t)-u_{h_0}(t)}{h-h_0},v\right\rangle &= \left\langle U(h_0,t,0)\frac{u^0_h-u^0_{h_0}}{h-h_0},v\right\rangle\\ &+\int_0^t\left\langle U(h_0,t,s)\frac{f(h,s)-f(h_0,s)}{h-h_0},v\right\rangle ds\\ &+\int_0^t\left\langle u_h(s),\frac{A^*(h,s)-A^*(h_0,s)}{h-h_0}U^*(h_0,t,s)v\right\rangle ds \quad h\neq h_0.\end{aligned}\tag{41}$$

Denote

$$z_h(t) := U(h_0,t,0)\frac{u^0_h-u^0_{h_0}}{h-h_0}+\int_0^t U(h_0,t,s)\frac{f_h(s)-f_{h_0}(s)}{h-h_0}ds.$$

The function z_h is a weak solution of the Cauchy problem

$$\begin{cases}\frac{d}{dt}z_h(t)=A(h_0,t)z_h(t)+F(h,t)\\ z_h(0)=z^0_h,\end{cases}$$

where

$$F(h,t)=\begin{cases}\frac{f(h,t)-f(h_0,t)}{h-h_0} & \text{for } h\neq h_0\\ f'(h_0,t) & \text{for } h=h_0\end{cases}$$

and

$$z^0_h=\begin{cases}\frac{u_h-u_{h_0}}{h-h_0} & \text{for } h\neq h_0\\ (u^0_{h_0})' & \text{for } h=h_0.\end{cases}$$

By Theorem 8,

$$\lim_{h\to h_0}\langle z_h(t),v\rangle = \langle z_{h_0}(t),v\rangle$$

uniformly in $t \in [0,T]$, where

$$z_{h_0}(t) = U(h_0,t,0)(u^0_{h_0})' + \int_0^t U(h_0,t,s)f'_{h_0}(s)ds.$$

Now we consider third term of the right side of the formula (41).

$$\int_0^t \left\langle u_h(s), \frac{A^*(h,s) - A^*(h_0,s)}{h-h_0} U^*(h_0,t,s)v \right\rangle ds$$
$$= \int_0^t \left\langle u_h(s) - u_{h_0}(s), \frac{A^*(h,s) - A^*(h_0,s)}{h-h_0} U^*(h_0,t,s)v \right\rangle ds$$
$$+ \int_0^t \left\langle u_{h_0}(s), \left(\frac{\overline{A^{-1}(h_0,s)A(h,s)} - I}{h-h_0} \right)^* A^*(h_0,s)U^*(h_0,t,s)v \right\rangle ds.$$

For any $(t,s) \in \triangle_T$

$$\lim_{h\to h_0} \left\langle u_h(s) - u_{h_0}(s), \frac{A^*(h,s) - A^*(h_0,s)}{h-h_0} U^*(h_0,t,s)v \right\rangle = 0.$$

Indeed, the mapping $\Omega \ni h \to u_h(s) \in X$ is weakly continuous and the mapping $\Omega \ni h \to A^*(h,s)w \in X^*$ is differentiable for $w \in D^*$.

Moreover,

$$\|u_h(s)\| \le C \quad \text{for } (h,s) \in \Omega \times [0,T].$$

By the Assumption $\mathrm{Z_h}$, (39), (21) and the Lebesgue Theorem we get

$$\int_0^t \left\langle u_h(s), \frac{A^*(h,s) - A^*(h_0,s)}{h-h_0} U^*(h_0,t,s)v \right\rangle ds$$
$$\longrightarrow \int_0^t \left\langle u_{h_0}(s), \frac{\partial}{\partial h} A^*(h,s)_{|h=h_0} U^*(h_0,t,s)v \right\rangle ds,$$

when $h \to h_0$. This ends proof. □

References

1. Ball J. M., *Strongly continuous semigroups, weak solutions, and the variation of constant formula*, Proc. Amer. Math. Soc., **63(2)** (1977), 370–373.
2. Bochenek J., Winiarska T., *Evolution equations with parameter in the hyperbolic case*, Ann. Polon. Math., **64** (1996), 47–60.
3. Hill E., Philips R. S., *Functional Analysis and Semi-Groups*, 1985.
4. Jużyniec M., *Weak solutions of evolution equations with parameter*, Univ. Iagel. Acta Math., **35** (1997), 189–204.
5. Jużyniec M., *Dependence on parameter of weak solution of the first order differential equation*, Univ. Iagel. Acta Math., **37** (1999), 237–243.
6. Jużyniec M., *Dependence of a weak solution of the first order differential equation on a parameter*, Univ. Iagel. Acta Math., **46** (2008), 35–46.
7. Kato T. *Perturbation theory for linear operators*, Springer-Verlag, 1966.

8. Krein S. G., *Linear differential equations in Banach spaces*, Transl. Math. Monographs, Vol. **29**, Amer. Math. Soc., 1971.
9. Pazy A., *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, 1983.
10. Winiarska T., Winiarski T., *Regularity of domains of parameterized families of closed linear operators*, Ann. Polon. Math., **80** (2003), 231–241.

Received July 19, 2007

Institute of Mathematics
Cracow University of Technology
Warszawska 24
31-155 Kraków, Poland
e-mail: juzyniec@usk.pk.edu.pl