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ITERATIVE METHODS FOR HYPERBOLIC DIFFERENTIAL FUNCTIONAL EQUATIONS

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Abstract. We deal with the Darboux problem for the hyperbolic partial functional-differential equation

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = f\left(x, y, u_{(x,y)}, \frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y)\right) \quad \text{a.e. in } [0,a]\times[0,b],$$

$$u(x,y) = \psi(x,y) \quad \text{on } [-a_0,a]\times[-b_0,b]\backslash(0,a]\times(0,b],$$

where the function $u_{(x,y)} : [-a_0,0]\times[-b_0,0] \to \mathbb{R}^k$ is defined by $u_{(x,y)}(s,t) = u(s+x, t+y)$ for $(s,t) \in [-a_0,0]\times[-b_0,0]$. We study the existence and uniqueness of Carathéodory solutions of this problem by means of the iterative methods.

1. Introduction. Put $I = [0,a]\times[0,b]$, $D = [-a_0,0]\times[-b_0,0]$, $I^* = [-a_0,a]\times[-b_0,b]$, $I_0 = \overline{I^*\backslash I}$. We always assume that $a,b > 0$ and $a_0, b_0 \in \mathbb{R}_+$, where $\mathbb{R}_+ = [0,+\infty)$. The inequality $x < y$ in $\mathbb{R}^k$ means that $x_i < y_i$ for each $i \in \{1,\dots,k\}$. Similarly for "$\geq$","$>$" and "$\leq$". The function $f : I \times C(D,\mathbb{R}^k)\times\mathbb{R}^k\times\mathbb{R}^k \to \mathbb{R}^k$ of the variables (x,y,ω,μ,ν) is said to be nondecreasing with respect to the functional argument ω if the inequality $\omega_1 \leq \omega_2$ implies that $f(x,y,\omega_1,\mu,\nu) \leq f(x,y,\omega_2,\mu,\nu)$. Here $\omega_1 \leq \omega_2$ means that $\omega_1(s,t) \leq \omega_2(s,t)$ for all $(s,t) \in D$. Furthermore, $\langle u_1\rangle \leq \langle u_2\rangle$ means that $u_1 \leq u_2$, $\partial u_1/\partial x \leq \partial u_2/\partial x$ and $\partial u_1/\partial y \leq \partial u_2/\partial y$. In this paper we shall discuss Carathéodory solutions of the following Darboux problem:

$$(1) \qquad \begin{cases} \frac{\partial^2 u}{\partial x\partial y}(x,y) = f\left(x,y,u_{(x,y)}, \frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y)\right) & \text{a.e. in } I, \\ u(x,y) = \psi(x,y) & \text{on } I_0, \end{cases}$$

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where $f : I \times C(D, \mathbb{R}^k) \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ and $\psi : I_0 \to \mathbb{R}^k$ are given functions. We define $u_{(x,y)} : D \to \mathbb{R}^k$ by the formula $u_{(x,y)}(s,t) = u(s+x, t+y)$ for $(s,t) \in D$. In order to define this solutions we need an appropriately definition of an absolutely continuous function of two variables. For this purpose, we first introduce suitable notation. Given a rectangle $J = [a_1, a_2] \times [b_1, b_2]$ contained in I and $u : I \to \mathbb{R}$, let

$$\Delta_J(u) = u(a_1, b_1) - u(a_2, b_1) - u(a_1, b_2) + u(a_2, b_2).$$

A rectangle is called a subrectangle of I if its sides are parallel to the coordinate axes. Let m denote the Lebesgue measure on $\mathbb{R}^2$. We say that $u : I \to \mathbb{R}$ is absolutely continuous if the following two conditions are satisfied:

(a) Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{J \in \mathcal{J}} |\Delta_J(u)| < \epsilon$$

whenever $\mathcal{J}$ is a finite collection of pairwise non-overlapping subrectangles of I with

$$\sum_{J \in \mathcal{J}} m(J) < \delta.$$

(b) The marginal functions $u(\cdot, b)$ and $u(a, \cdot)$ are absolutely continuous functions of a single variable on $[0, a]$ and $[0, b]$, respectively.

We denote by

(a) $C(I, \mathbb{R}^k)$ the space of continuous functions from I into $\mathbb{R}^k$ with the usual supremum norm.
(b) $AC(I, \mathbb{R}^k)$ the space of absolutely continuous functions from I into $\mathbb{R}^k$.
(c) $C_x(I, \mathbb{R}^k)$ the space of functions ν of the variables (x, y) defined on I, continuous in $x \in [0, a]$ for almost all $y \in [0, b]$ and measurable in $y \in [0, b]$ for all $x \in [0, a]$ and such that

$$||\nu||_x = \int_0^b \max_{x \in [0,a]} |\nu(x, y)| dy < \infty.$$

(d) $C_y(I, \mathbb{R}^k)$ the space of functions μ of the variables (x, y) defined on I, continuous in $y \in [0, b]$ for almost all $x \in [0, a]$ and measurable in $x \in [0, a]$ for all $y \in [0, b]$ and such that

$$||\mu||_y = \int_0^a \max_{y \in [0,b]} |\mu(x, y)| dx < \infty.$$

(e) $L^1(I, \mathbb{R})$ the space of Lebesgue integrable functions from I into $\mathbb{R}$.

In [**1**] we can find that the following statements are equivalent:

(a) $u \in AC(I, \mathbb{R})$.

(b) There exist $g \in AC([0,a], \mathbb{R})$, $h \in AC([0,b], \mathbb{R})$ and $L \in L^1(I, \mathbb{R})$ such that

$$u(x,y) = g(x) + h(y) + \int_0^x \int_0^y L(s,t)dsdt.$$

Note that if $u \in AC(I, \mathbb{R})$ then $\partial u/\partial x$, $\partial u/\partial y$ and $\partial^2 u/\partial x \partial y$ exist almost everywhere on I. Furthermore, $\partial u/\partial x \in C_y(I, \mathbb{R})$ and $\partial u/\partial y \in C_x(I, \mathbb{R})$. Now, we are able to define the solution of problem (1). Namely, by the solution of this problem we mean a function $u : I^* \to \mathbb{R}^k$ continuous on I^* and absolutely continuous on I which satisfies the differential equation almost everywhere on I and the initial condition everywhere on I_0. Let $|\cdot|$ denote the maximum norm in $\mathbb{R}^k$. Moreover, $||w||_0$ denotes the usual supremum norm of $w \in C(D, \mathbb{R}^k)$. As in [**7**] we can verify that $||\cdot||_x$, $||\cdot||_y$ are norms and $(C_x(I, \mathbb{R}^k), ||\cdot||_x)$, $(C_y(I, \mathbb{R}^k), ||\cdot||_y)$ are Banach spaces. Moreover, for $(x,y) \in I$, we define

$$||\omega||^{(x,y)} = \max_{\substack{s \in [0,x] \\ t \in [0,y]}} |\omega(s,t)|,$$

$$||\nu||_x^{(x,y)} = \int_0^y \max_{s \in [0,x]} |\nu(s,t)|dt,$$

$$||\mu||_y^{(x,y)} = \int_0^x \max_{t \in [0,y]} |\mu(s,t)|ds.$$

Section 2 is devoted to the study of existence and uniqueness of solutions to problem (1) by means of the monotone iterative method. In [**9**] we considered a simpler Darboux problem, where f was independent of $\partial u/\partial x$ and $\partial u/\partial y$. Similar problem, but with classical solutions of this problem has been studied by Brzychczy and Janus in [**2, 3**] and by Lakshmikantham in [**11**]. Section 3 is dedicated to the Newton method for problem (1), where f is independent of $\partial u/\partial x$ and $\partial u/\partial y$. In this method the convergence that we get is quadratic. The Newton method for hyperbolic equations have been studied by Człapiński [**5, 6**]. Moreover, this method has been applied by Czernous [**4**] to the first order partial differential equations.

2. Monotone iterative technique. Let $l_i \in L^1(I, \mathbb{R}_+)$ for $i = \{1,2,3\}$, c_1, c_2, $c_3 \in \mathbb{R}_+$. Define $r : I \to \mathbb{R}$ by the formula

$$r(x,y) = c_1 e^{12H(x,y)}, \tag{2}$$

where

$$H(x,y) = \sum_{i=1}^{3} \int_0^x \int_0^y l_i(s,t)dsdt + c_3 x + c_2 y.$$

Now, we prove some useful lemma.

Lemma 1. *The function r satisfy inequality*

$$\int_0^x \int_0^y l_1(s,t)r(s,t)dsdt \le \frac{1}{12} r(x,y) \ \textit{for } (x,y) \in I.$$

Proof. We integrate by parts the left-hand side of (3) obtain

$$12 \int_0^x \int_0^y \int_0^t l_1(s,z)dz \Big(\sum_{i=1}^{3} \int_0^s l_i(z,t)dz + c_2 \Big) r(s,t)dtds \tag{3}$$

$$= \int_0^x \int_0^y l_1(s,z)dz \ r(s,y)ds - \int_0^x \int_0^y l_1(s,t)r(s,t)dtds.$$

From (3) and the fact that l_2,$l_3 \in L^1(I, \mathbb{R}_+)$, $c_3 \in \mathbb{R}_+$ we get

$$\int_0^x \int_0^y l_1(s,t)r(s,t)dtds \le \int_0^x \int_0^y l_1(s,z)dz \ r(s,y)ds$$

$$\le \int_0^x \Big(\sum_{i=1}^{3} \int_0^y l_i(s,z)dz + c_3 \Big) r(s,y)ds = \frac{1}{12}(r(x,y) - r(0,y)) \le \frac{1}{12} r(x,y).$$

□

Remark 1. An easy computation shows that the function r satisfy inequalities

$$\int_0^x \Big(c_3 + \int_0^y l_3(s,t)dt \Big) r(s,y)ds \le \frac{1}{12} r(x,y),$$

$$\int_0^y \Big(c_2 + \int_0^x l_2(s,t)ds \Big) r(x,t)dt \le \frac{1}{12} r(x,y),$$

for $(x,y) \in I$.

In this section we shall discuss Carathéodory solutions for the following Darboux problem:

$$\begin{cases} \frac{\partial^2 u}{\partial x \partial y}(x,y) = f[u](x,y) + g[u](x,y) & \text{a.e. in } I, \\ u(x,y) = \psi(x,y) & \text{on } I_0, \end{cases} \tag{4}$$

where $f[u]$, $g[u]$ are defined by

$$f[u](x,y) = f\Big(x, y, u_{(x,y)}, \frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y) \Big),$$

$$g[u](x,y) = g\Big(x, y, u_{(x,y)}, \frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y) \Big).$$

Moreover, f,$g : I \times C(D, \mathbb{R}^k) \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ and $\psi : I_0 \to \mathbb{R}^k$ are given functions.

Assumption 1. *Suppose that $v^0, w^0 \in C(I^*, \mathbb{R}^k) \cap AC(I, \mathbb{R}^k)$, $\langle v^0 \rangle \leq \langle w^0 \rangle$ on I and v^0, w^0 are coupled lower and upper solution of (4), that is,*

$$\begin{cases} \frac{\partial^2 v^0}{\partial x \partial y}(x,y) \leq f[v^0](x,y) + g[w^0](x,y) & a.e.\ in\ I, \\ v^0(x,y) \leq \psi(x,y) & on\ I_0, \\ \frac{\partial v^0}{\partial x}(x,0) \leq \frac{\partial \psi}{\partial x}(x,0),\ \frac{\partial v^0}{\partial y}(0,y) \leq \frac{\partial \psi}{\partial y}(0,y) & for\ x \in [0,a],\ y \in [0,b], \end{cases}$$

$$\begin{cases} \frac{\partial^2 w^0}{\partial x \partial y}(x,y) \geq f[w^0](x,y) + g[v^0](x,y) & a.e.\ in\ I, \\ w^0(x,y) \geq \psi(x,y) & on\ I_0, \\ \frac{\partial w^0}{\partial x}(x,0) \geq \frac{\partial \psi}{\partial x}(x,0),\ \frac{\partial w^0}{\partial y}(0,y) \geq \frac{\partial \psi}{\partial y}(0,y) & for\ x \in [0,a],\ y \in [0,b]. \end{cases}$$

Assumption 2. *Suppose that functions $f, g : I \times C(D, \mathbb{R}^k) \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ of the variables (x, y, ω, μ, ν) are such that*

(A1) *$f, g(\cdot, \cdot, \omega, \mu, \nu) : I \to \mathbb{R}^k$ are measurable for all fixed $(\omega, \mu, \nu) \in (C(D, \mathbb{R}^k), \mathbb{R}^k, \mathbb{R}^k)$.*

(A2) *There are functions $l_i \in L^1(I, \mathbb{R}_+)$ for $i \in \{1, 2, 3\}$ and constants c_2, $c_3 \in \mathbb{R}_+$ such that*

$$\begin{aligned} |f(x,y,\tilde{\omega},\tilde{\mu},\tilde{\nu}) - f(x,y,\omega,\mu,\nu)| &\leq l_1(x,y)||\tilde{\omega} - \omega||_0 \\ &+ \left(c_2 + \int_0^x l_2(s,y)ds\right)|\tilde{\mu} - \mu| + \left(c_3 + \int_0^y l_3(x,t)dt\right)|\tilde{\nu} - \nu|, \\ |g(x,y,\tilde{\omega},\tilde{\mu},\tilde{\nu}) - g(x,y,\omega,\mu,\nu)| &\leq l_1(x,y)||\tilde{\omega} - \omega||_0 \\ &+ \left(c_2 + \int_0^x l_2(s,y)ds\right)|\tilde{\mu} - \mu| + \left(c_3 + \int_0^y l_3(x,t)dt\right)|\tilde{\nu} - \nu|, \end{aligned}$$

for $(x,y) \in I$ ω, $\tilde{\omega} \in C(D, \mathbb{R}^k)$, μ, $\tilde{\mu}$, ν, $\tilde{\nu} \in \mathbb{R}^k$ and $v^0_{(x,y)} \leq \omega$, $\tilde{\omega} \leq w^0_{(x,y)}$, $\frac{\partial v^0}{\partial x}(x,y) \leq \mu$, $\tilde{\mu} \leq \frac{\partial w^0}{\partial x}(x,y)$, $\frac{\partial v^0}{\partial y}(x,y) \leq \nu$, $\tilde{\nu} \leq \frac{\partial w^0}{\partial y}(x,y)$.

Assumption 3. *Suppose that the function $\psi : I_0 \to \mathbb{R}^k$ is such that $\psi \in C(I_0, \mathbb{R}^k)$, $\psi(\cdot, 0) \in AC([0,a], \mathbb{R}^k)$ and $\psi(0, \cdot) \in AC([0,b], \mathbb{R}^k)$.*

In the paper [**8**] we develop the theory of linear and nonlinear inequalities for the problem (4). In this paper we shall not use this theorems. We only use of following simple remark.

Remark 2. Suppose that u is a function continuous on I^* and absolute continuous on I. Furthermore,

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) \geq 0 \text{ a.e. in } I,$$

$$u(x,y) \geq 0 \text{ on } I_0,\ \frac{\partial u}{\partial x}(x,0) \geq 0 \text{ on } [0,a],\ \frac{\partial u}{\partial y}(0,y) \geq 0 \text{ on } [0,b].$$

Then

$$u(x,y) \geq 0, \quad \frac{\partial u}{\partial x}(x,y) \geq 0, \quad \frac{\partial u}{\partial y}(x,y) \geq 0 \text{ on } I.$$

Now, we give a constructive method for obtaining the solutions of the problem (4). Firstly, we note that problem (4) is equivalent to a problem:

$$(5) \qquad \begin{cases} u(x,y) = \psi(x,0) + \psi(0,y) - \psi(0,0) \\ \quad + \int_0^x \int_0^y \big(f[u](s,t) + g[u](s,t)\big)dsdt & \text{in } I, \\ u(x,y) = \psi(x,y) & \text{on } I_0. \end{cases}$$

Differentiating (5) with respect to x or y we respectively get

$$(6) \quad \begin{cases} \frac{\partial u}{\partial x}(x,y) = \frac{\partial \psi}{\partial x}(x,0) \\ \quad + \int_0^y \big(f[u](x,t) + g[u](x,t)\big)dt & \text{for all } y \in [0,b] \text{ and a.e. } x \in [0,a], \\ u(x,y) = \psi(x,y) & \text{on } I_0, \end{cases}$$

$$(7) \quad \begin{cases} \frac{\partial u}{\partial y}(x,y) = \frac{\partial \psi}{\partial y}(0,y) \\ \quad + \int_0^x \big(f[u](s,y) + g[u](s,y)\big)ds & \text{for all } x \in [0,a] \text{ and a.e. } y \in [0,b], \\ u(x,y) = \psi(x,y) & \text{on } I_0. \end{cases}$$

Now, we define the sequences $\{v^n\}$ and $\{w^n\}$ by

$$(8) \qquad \begin{cases} \frac{\partial^2 v^{n+1}}{\partial x \partial y}(x,y) = f[v^n](x,y) + g[w^n](x,y) & \text{a.e. in } I, \\ v^{n+1}(x,y) = \psi(x,y) & \text{on } I_0, \end{cases}$$

and

$$(9) \qquad \begin{cases} \frac{\partial^2 w^{n+1}}{\partial x \partial y}(x,y) = f[w^n](x,y) + g[v^n](x,y) & \text{a.e. in } I, \\ w^{n+1}(x,y) = \psi(x,y) & \text{on } I_0, \end{cases}$$

for $n = 0, 1, 2, \ldots$.

THEOREM 1. *Suppose that Assumptions 1–3 are satisfied. f is nondecreasing and g is nonincreasing in ω, μ and ν for $(x,y) \in I$ and $v^0_{(x,y)} \leq \omega \leq w^0_{(x,y)}$, $\frac{\partial v^0}{\partial x}(x,y) \leq \mu \leq \frac{\partial w^0}{\partial x}(x,y)$, $\frac{\partial v^0}{\partial y}(x,y) \leq \nu \leq \frac{\partial w^0}{\partial y}(x,y)$. Moreover the sequences $\{v^n\}$, $\{w^n\}$ are defined by (8) and (9), respectively. Then $v^n \to u$ and $w^n \to u$ in $C(I, \mathbb{R}^k)$, where u is the unique solution of the problem (4) such that $\langle v^0 \rangle \leq \langle u \rangle \leq \langle w^0 \rangle$ on I.*

PROOF. It is easily seen that v^n, $w^n \in AC(I, \mathbb{R}^k)$ for all n are the unique solutions of (8), (9), respectively. We shall show that

$$(10) \qquad \langle v^0 \rangle \leq \langle v^1 \rangle \leq \langle w^1 \rangle \leq \langle w^0 \rangle \text{ on } I.$$

Put $p = v^1 - v^0$, then

$$\frac{\partial^2 p}{\partial x \partial y}(x,y) \geq f[v^0](x,y) + g[w^0](x,y) - f[v^0](x,y) - g[w^0](x,y) = 0 \text{ a.e. in } I.$$

$$p(x,y) \geq \psi(x,y) - \psi(x,y) = 0 \text{ on } I_0.$$

Similarly, we get

$$\frac{\partial p}{\partial x}(x,0) \geq 0 \text{ on } [0,a], \quad \frac{\partial p}{\partial y}(0,y) \geq 0 \text{ on } [0,b].$$

From the above and Remark 2 we get that $\langle p \rangle \geq 0$ on I and this implies $\langle v^0 \rangle \leq \langle v^1 \rangle$ on I. Analogously we get $\langle w^1 \rangle \leq \langle w^0 \rangle$ on I.
Now, put $p = w^1 - v^1$. From monotone of f and g in ω, μ, ν and the assumption $\langle v^0 \rangle \leq \langle w^0 \rangle$ we get

$$\frac{\partial^2 p}{\partial x \partial y}(x,y) \geq f[w^0](x,y) + g[v^0](x,y) - f[v^0](x,y) - g[w^0](x,y) \geq 0 \text{ a.e. in } I.$$

Furthermore

$$p(x,y) = 0 \text{ on } I_0, \quad \frac{\partial p}{\partial x}(x,0) = 0 \text{ on } [0,a], \quad \frac{\partial p}{\partial y}(0,y) = 0 \text{ on } [0,b].$$

Therefore from Remark 2 we get that $\langle p \rangle \geq 0$ on I and this implies $\langle v^1 \rangle \leq \langle w^1 \rangle$ on I, and the proof of (10) is complete.
We shall show that if for some $n > 0$,

$$\langle v^{n-1} \rangle \leq \langle v^n \rangle \leq \langle w^n \rangle \leq \langle w^{n-1} \rangle \tag{11}$$

then

$$\langle v^n \rangle \leq \langle v^{n+1} \rangle \leq \langle w^{n+1} \rangle \leq \langle w^n \rangle. \tag{12}$$

Let $p = v^{n+1} - v^n$ then it is easily seen that $\frac{\partial^2 p}{\partial x \partial y}(x,y) \geq 0$ on I, $p(x,y) = 0$ on I_0, $\frac{\partial p}{\partial x}(x,0) = 0$ on $[0,a]$, $\frac{\partial p}{\partial y}(0,y) = 0$ on $[0,b]$. From this and Remark 2 we get that $\langle v^n \rangle \leq \langle v^{n+1} \rangle$ on I. Similar consideration apply to $\langle w^{n+1} \rangle \leq \langle w^n \rangle$, $\langle v^{n+1} \rangle \leq \langle w^{n+1} \rangle$ on I, which completes the proof of the implication (11)$\Rightarrow$(12).
Hence by induction we get for all $n = 1, 2, \ldots$

$$\langle v^0 \rangle \leq \langle v^1 \rangle \leq \cdots \leq \langle v^n \rangle \leq \langle w^n \rangle \leq \cdots \leq \langle w^1 \rangle \leq \langle w^0 \rangle \text{ on } I. \tag{13}$$

Now we show that

$$||w^n - v^n||^{(x,y)} \leq \frac{1}{2^n} r(x,y), \tag{14}$$

$$\left|\left| \frac{\partial w^n}{\partial x} - \frac{\partial v^n}{\partial x} \right|\right|_y^{(x,y)} \leq \frac{1}{2^n} r(x,y), \tag{15}$$

$$\left\| \frac{\partial w^n}{\partial y} - \frac{\partial v^n}{\partial y} \right\|_x^{(x,y)} \le \frac{1}{2^n} r(x,y) \tag{16}$$

for $n = 1,2,\dots$, $(x,y) \in I$ and r is defined by (2), where

$$c_1 = \max\left\{ \max_{(x,y)\in I^*} \left|w^0(x,y) - v^0(x,y)\right|, \left\| \frac{\partial w^0}{\partial x} - \frac{\partial v^0}{\partial x} \right\|_y^{(a,b)}, \left\| \frac{\partial w^0}{\partial y} - \frac{\partial v^0}{\partial y} \right\|_x^{(a,b)} \right\}.$$

Estimates (14)–(16) will be proved by induction on n.
In order to show that estimates (14)–(16) hold for $n = 1$, we note that

$$\begin{aligned}
&\int_0^x \int_0^y |f(s,t,\tilde{\omega}_{(s,t)}, \tilde{\mu}(s,t), \tilde{\nu}(s,t)) - f(s,t,\omega_{(s,t)}, \mu(s,t), \nu(s,t))| dsdt \\
&\le \int_0^x \int_0^y \left[l_1(s,t) ||\tilde{\omega}_{(s,t)} - \omega_{(s,t)}||_0 + \left(c_2 + \int_0^s l_2(z,t)dz\right) |\tilde{\mu}(s,t) - \mu(s,t)| \right. \\
&\quad \left. + \left(c_3 + \int_0^t l_3(s,z)dz\right) |\tilde{\nu}(s,t) - \nu(s,t)| \right] dsdt \\
&\le \int_0^x \int_0^y l_1(s,t) ||\tilde{\omega}_{(s,t)} - \omega_{(s,t)}||_0 dsdt \\
&\quad + \int_0^y \left[\left(c_2 + \int_0^x l_2(s,t)ds\right) \int_0^x |\tilde{\mu}(s,t) - \mu(s,t)| ds \right] dt \\
&\quad + \int_0^x \left[\left(c_3 + \int_0^y l_3(s,t)dt\right) \int_0^y |\tilde{\nu}(s,t) - \nu(s,t)| dt \right] ds \\
&\le \int_0^x \int_0^y l_1(s,t) ||\tilde{\omega}_{(s,t)} - \omega_{(s,t)}||_0 dsdt + \int_0^y \left[\left(c_2 + \int_0^x l_2(s,t)ds\right) ||\tilde{\mu} - \mu||_y^{(x,t)} \right] dt \\
&\quad + \int_0^x \left[\left(c_3 + \int_0^y l_3(s,t)dt\right) ||\tilde{\nu} - \nu||_x^{(s,y)} \right] ds,
\end{aligned} \tag{17}$$

where (ω,μ,ν), $(\tilde{\omega},\tilde{\mu},\tilde{\nu}) \in C(I^*, \mathbb{R}^k) \times C_y(I, \mathbb{R}^k) \times C_x(I, \mathbb{R}^k)$. Similar arguments to those above show that

$$\begin{aligned}
&\int_0^x \int_0^y |g(s,t,\tilde{\omega}_{(s,t)}, \tilde{\mu}(s,t), \tilde{\nu}(s,t)) - g(s,t,\omega_{(s,t)}, \mu(s,t), \nu(s,t))| dsdt \\
&\le \int_0^x \int_0^y l_1(s,t) ||\tilde{\omega}_{(s,t)} - \omega_{(s,t)}||_0 dsdt + \int_0^y \left[\left(c_2 + \int_0^x l_2(s,t)ds\right) ||\tilde{\mu} - \mu||_y^{(x,t)} \right] dt \\
&\quad + \int_0^x \left[\left(c_3 + \int_0^y l_3(s,t)dt\right) ||\tilde{\nu} - \nu||_x^{(s,y)} \right] ds.
\end{aligned}$$

By the above, (5)–(7) and the inequality $x \leq e^x$, it is easily seen that

$$\begin{aligned}||w^1 - v^1||^{(x,y)} &\leq 2\int_0^x \int_0^y l_1(s,t)||w^0_{(s,t)} - v^0_{(s,t)}||_0 dsdt \\ &+ 2\int_0^y \left[\left(c_2 + \int_0^x l_2(s,t)ds\right)\left|\left|\frac{\partial w^0}{\partial x} - \frac{\partial v^0}{\partial x}\right|\right|_y^{(x,t)}\right]dt \\ &+ 2\int_0^x \left[\left(c_3 + \int_0^y l_3(s,t)dt\right)\left|\left|\frac{\partial w^0}{\partial y} - \frac{\partial v^0}{\partial y}\right|\right|_x^{(s,y)}\right]ds \\ &\leq 2c_1\Big(\int_0^x \int_0^y l_1(s,t)dsdt + c_2 y + \int_0^x \int_0^y l_2(s,t)dsdt \\ &+ c_3 x + \int_0^x \int_0^y l_3(s,t)dsdt\Big) \leq \frac{1}{2}r(x,y),\end{aligned}$$

$$\begin{aligned}\left|\left|\frac{\partial w^1}{\partial x} - \frac{\partial v^1}{\partial x}\right|\right|_y^{(x,y)} &\leq 2\int_0^x \int_0^y l_1(s,t)||w^0_{(s,t)} - v^0_{(s,t)}||_0 dsdt \\ &+ 2\int_0^y \left[\left(c_2 + \int_0^x l_2(s,t)ds\right)\left|\left|\frac{\partial w^0}{\partial x} - \frac{\partial v^0}{\partial x}\right|\right|_y^{(x,t)}\right]dt \\ &+ 2\int_0^x \left[\left(c_3 + \int_0^y l_3(s,t)dt\right)\left|\left|\frac{\partial w^0}{\partial y} - \frac{\partial v^0}{\partial y}\right|\right|_x^{(s,y)}\right]ds \leq \frac{1}{2}r(x,y),\end{aligned}$$

$$\begin{aligned}\left|\left|\frac{\partial w^1}{\partial y} - \frac{\partial v^1}{\partial y}\right|\right|_x^{(x,y)} &\leq 2\int_0^x \int_0^y l_1(s,t)||w^0_{(s,t)} - v^0_{(s,t)}||_0 dsdt \\ &+ 2\int_0^y \left[\left(c_2 + \int_0^x l_2(s,t)ds\right)\left|\left|\frac{\partial w^0}{\partial x} - \frac{\partial v^0}{\partial x}\right|\right|_y^{(x,t)}\right]dt \\ &+ 2\int_0^x \left[\left(c_3 + \int_0^y l_3(s,t)dt\right)\left|\left|\frac{\partial w^0}{\partial y} - \frac{\partial v^0}{\partial y}\right|\right|_x^{(s,y)}\right]ds \leq \frac{1}{2}r(x,y),\end{aligned}$$

so that estimates (14)–(16) hold for $n = 1$.
Now, we note that if $\omega = \tilde{\omega}$ on I_0 then by (17) we have

$$\begin{aligned}&\int_0^x \int_0^y |f(s,t,\tilde{\omega}_{(s,t)}, \tilde{\mu}(s,t), \tilde{\nu}(s,t)) - f(s,t,\omega_{(s,t)}, \mu(s,t), \nu(s,t))|dsdt \\ &\leq \int_0^x \int_0^y l_1(s,t)||\tilde{\omega} - \omega||^{(s,t)}dsdt + \int_0^y \left[\left(c_2 + \int_0^x l_2(s,t)ds\right)||\tilde{\mu} - \mu||_y^{(x,t)}\right]dt \\ &\quad + \int_0^x \left[\left(c_3 + \int_0^y l_3(s,t)dt\right)||\tilde{\nu} - \nu||_x^{(s,y)}\right]ds,\end{aligned}$$

$$\int_0^x \int_0^y |g(s,t,\tilde{\omega}_{(s,t)},\tilde{\mu}(s,t),\tilde{\nu}(s,t)) - g(s,t,\omega_{(s,t)},\mu(s,t),\nu(s,t))|dsdt$$
$$\leq \int_0^x \int_0^y l_1(s,t)||\tilde{\omega}_{(s,t)} - \omega_{(s,t)}||^{(s,t)}dsdt + \int_0^y \left[\left(c_2 + \int_0^x l_2(s,t)ds\right)||\tilde{\mu}-\mu||_y^{(x,t)}\right]dt$$
$$+ \int_0^x \left[\left(c_3 + \int_0^y l_3(s,t)dt\right)||\tilde{\nu}-\nu||_x^{(s,y)}\right]ds.$$

By the above and (5)–(7), it is easily seen that for $n \geq 1$ we have

$$\begin{aligned} ||w^{n+1} - v^{n+1}||^{(x,y)} &\leq 2\int_0^x \int_0^y l_1(s,t)||w^n - v^n||^{(s,t)}dsdt \\ &+ 2\int_0^y \left[\left(c_2 + \int_0^x l_2(s,t)ds\right)\left|\left|\frac{\partial w^n}{\partial x} - \frac{\partial v^n}{\partial x}\right|\right|_y^{(x,t)}\right]dt \\ &+ 2\int_0^x \left[\left(c_3 + \int_0^y l_3(s,t)dt\right)\left|\left|\frac{\partial w^n}{\partial y} - \frac{\partial v^n}{\partial y}\right|\right|_x^{(s,y)}\right]ds, \end{aligned} \tag{18}$$

$$\begin{aligned} \left|\left|\frac{\partial w^{n+1}}{\partial x} - \frac{\partial v^{n+1}}{\partial x}\right|\right|_y^{(x,y)} &\leq 2\int_0^x \int_0^y l_1(s,t)||w^n - v^n||^{(s,t)}dsdt \\ &+ 2\int_0^y \left[\left(c_2 + \int_0^x l_2(s,t)ds\right)\left|\left|\frac{\partial w^n}{\partial x} - \frac{\partial v^n}{\partial x}\right|\right|_y^{(x,t)}\right]dt \\ &+ 2\int_0^x \left[\left(c_3 + \int_0^y l_3(s,t)dt\right)\left|\left|\frac{\partial w^n}{\partial y} - \frac{\partial v^n}{\partial y}\right|\right|_x^{(s,y)}\right]ds, \end{aligned} \tag{19}$$

$$\begin{aligned} \left|\left|\frac{\partial w^{n+1}}{\partial y} - \frac{\partial v^{n+1}}{\partial y}\right|\right|_x^{(x,y)} &\leq 2\int_0^x \int_0^y l_1(s,t)||w^n - v^n||^{(s,t)}dsdt \\ &+ 2\int_0^y \left[\left(c_2 + \int_0^x l_2(s,t)ds\right)\left|\left|\frac{\partial w^n}{\partial x} - \frac{\partial v^n}{\partial x}\right|\right|_y^{(x,t)}\right]dt \\ &+ 2\int_0^x \left[\left(c_3 + \int_0^y l_3(s,t)dt\right)\left|\left|\frac{\partial w^n}{\partial y} - \frac{\partial v^n}{\partial y}\right|\right|_x^{(s,y)}\right]ds. \end{aligned} \tag{20}$$

Now we assume that estimates (14)–(16) hold for $n \geq 1$. Then by Lemma 1 and Remark 1 obtain

$$\int_0^x \int_0^y l_1(s,t)||w^n - v^n||^{(s,t)}dsdt \leq \frac{1}{12}\cdot\frac{1}{2^n}r(x,y),$$

$$\int_0^y \left[\left(c_2 + \int_0^x l_2(s,t)ds\right)\left|\left|\frac{\partial w^n}{\partial x} - \frac{\partial v^n}{\partial x}\right|\right|_y^{(x,t)}\right]dt \leq \frac{1}{12}\cdot\frac{1}{2^n}r(x,y),$$

$$\int_0^x \left[\left(c_3 + \int_0^y l_3(s,t)dt\right)\left|\left|\frac{\partial w^n}{\partial y} - \frac{\partial v^n}{\partial y}\right|\right|_x^{(s,y)}\right] ds \leq \frac{1}{12} \cdot \frac{1}{2^n} r(x,y).$$

By the above and estimates (18)–(20) we see

$$||w^{n+1} - v^{n+1}||^{(x,y)} \leq 6 \cdot \frac{1}{12} \cdot \frac{1}{2^n} r(x,y) = \frac{1}{2^{n+1}} r(x,y),$$

$$\left|\left|\frac{\partial w^{n+1}}{\partial x} - \frac{\partial v^{n+1}}{\partial x}\right|\right|_y^{(x,y)} \leq 6 \cdot \frac{1}{12} \cdot \frac{1}{2^n} r(x,y) = \frac{1}{2^{n+1}} r(x,y),$$

$$\left|\left|\frac{\partial w^{n+1}}{\partial y} - \frac{\partial v^{n+1}}{\partial y}\right|\right|_x^{(x,y)} \leq 6 \cdot \frac{1}{12} \cdot \frac{1}{2^n} r(x,y) = \frac{1}{2^{n+1}} r(x,y),$$

so that if estimates (14)–(16) hold for some $n \geq 1$ then estimates (14)–(16) hold for $n+1$. Hence, by virtue of the induction the proof of estimates (14)–(16) is complete. From this estimates and (13) we see that that sequences $\{(v^n, \frac{\partial v^n}{\partial x}, \frac{\partial v^n}{\partial y})\}$, $\{(w^n, \frac{\partial w^n}{\partial x}, \frac{\partial w^n}{\partial y})\}$ are convergent in $C(I,\mathbb{R}^k) \times C_y(I,\mathbb{R}^k) \times C_x(I,\mathbb{R}^k)$ and

$$\lim_{n\to\infty} \left(v^n, \frac{\partial v^n}{\partial x}, \frac{\partial v^n}{\partial y}\right) = \lim_{n\to\infty} \left(w^n, \frac{\partial w^n}{\partial x}, \frac{\partial w^n}{\partial y}\right).$$

We note that from (13) and the monotone character of f, g we have

$$f[v^0](x,y) + g[w^0](x,y) \leq f[v^n](x,y) + g[w^n](x,y) \leq f[w^0](x,y) + g[v^0](x,y),$$

$$f[v^0](x,y) + g[w^0](x,y) \leq f[w^n](x,y) + g[v^n](x,y) \leq f[w^0](x,y) + g[v^0](x,y),$$

where $n = 0, 1, \ldots$ Moreover, from assumption (A1) and (A2) we have that $f[v^0](x,y) + g[w^0](x,y) \in L^1(I,\mathbb{R}^k)$ and $f[w^0](x,y) + g[v^0](x,y) \in L^1(I,\mathbb{R}^k)$. From the above, assumption (A2) and Lebesgue theorem on dominated convergence we see that the function u defined by

$$u(x,y) = \lim_{n\to\infty} v^n(x,y) = \lim_{n\to\infty} w^n(x,y)$$

is the solution of problem (4) such that $\langle v^0\rangle \leq \langle u\rangle \leq \langle w^0\rangle$ on I. In order to prove that u is the unique solution of (4) such that $\langle v^0\rangle \leq \langle u\rangle \leq \langle w^0\rangle$ on I. Assume that $\tilde{u}$ is solution of (4) such that $\langle v^0\rangle \leq \langle u\rangle \leq \langle w^0\rangle$ on I. If for some $n \geq 0$ we have $v^n \leq \tilde{u} \leq w^n$ and $p = w^{n+1} - \tilde{u}$ then from monotone of f and g we get

$$\frac{\partial^2 p}{\partial x \partial y}(x,y) = f[w^n](x,y) + g[v^n](x,y) - f[\tilde{u}](x,y) - g[\tilde{u}](x,y) \geq 0.$$

Moreover, $p = 0$ on I_0 and $\partial p/\partial x(x,0) = \partial p/\partial y(0,y) = 0$ for $x \in [0,a]$ and $y \in [0,b]$. Therefore $w^{n+1} \leq \tilde{u}$ on I. Analogously, we get $v^{n+1} \geq \tilde{u}$ on I. Therefore u is the unique solution of (4) such that $\langle v^0\rangle \leq \langle u\rangle \leq \langle w^0\rangle$ on I. $\square$

3. The Newton method. Let $(X, ||\cdot||_X)$ be a Banach space and $S = \{u \in X : ||u - u_0||_X \leq \delta\}$, where $u_0 \in X$ and $\delta > 0$ are arbitrary. Let $\mathcal{F} : S \to X$ be a given operator such that $\mathcal{F}'(u)$ exists for $u \in S$. We consider the equation

$$\mathcal{F}(u) = 0 \tag{21}$$

and the Newton sequence $\{u^n\}$ defined by

$$u^0 = u_0 \quad \text{i} \quad u^{n+1} = u^n - [\mathcal{F}'(u^n)]^{-1}\mathcal{F}(u^n) \quad \text{for } n \geq 0. \tag{22}$$

We now state Kantorovich theorem and some lemmas that will be of use later.

THEOREM 2. *Suppose that* $\mathcal{F} : S \to X$ *and*

1°) *The Fréchet derivative* $\mathcal{F}'(u)$ *exists for* $u \in S$.

2°) *There exists* $A \in \mathbb{R}_+$ *such that*

$$||\mathcal{F}'(u) - \mathcal{F}'(\tilde{u})||_* \leq A||u - \tilde{u}||_X \quad \textit{for } u, \tilde{u} \in S.$$

3°) *The operator* $\mathcal{F}'(u_0)$ *has a inverse and there is* $B \in \mathbb{R}_+$ *such that* $||[\mathcal{F}'(u_0)]^{-1}||_* \leq B$.

4°) *For* u_0 *the estimate* $||[\mathcal{F}'(u_0)]^{-1}\mathcal{F}(u_0)||_X \leq \eta$ *holds.*

5°) *The constants* A, B, η *fulfil* $h = AB\eta \leq \frac{1}{2}$.

6°) *For* $\delta > 0$ *we have*

$$\frac{1 - \sqrt{1 - 2h}}{h}\eta \leq \delta.$$

Then

a) *There exists the solution of* (21).

b) *The Newton sequence* (22) *exists and there is* u^* *such that* $u^* = \lim_{n\to\infty} u^n$.

c) $\mathcal{F}(u^*) = 0$ *and the following estimate holds*

$$||u^n - u^*||_X \leq \frac{1}{2^{n-1}}(2h)^{2^n - 1}\eta, \textit{ for } n \geq 0.$$

The above theorem can be found in [**10**].

LEMMA 2. *Let* $A : X \to X$ *be the bounded operator such that* $||A||_* < 1$ *then* $E - A : X \to X$ *is a bijection,* $(E - A)^{-1}$ *is bounded and*

$$||(E - A)^{-1}||_* \leq \frac{1}{1 - ||A||_*},$$

where E *is identity mapping.*

LEMMA 3. *Let* $l \in L^1(I, \mathbb{R}_+)$. *Then*

$$\int_0^x \int_0^y l(s,t)e^{4H(s,t)}dsdt \leq \frac{1}{4}\left(e^{4H(x,y)} - 1\right) \quad \textit{for } (x,y) \in I,$$

where

$$H(x,y) = \int_0^x \int_0^y l(s,t)dsdt.$$

PROOF. We integrate by parts the left-hand side of (23) obtain

$$\begin{aligned}
&4\int_0^x \int_0^y \int_0^s l(z,t)dz \int_0^t l(s,z)dze^{4H(s,t)}dtds \\
&= \int_0^x \Big\{ \Big[\int_0^t l(s,z)dze^{4H(s,t)} \Big]_{t=0}^{t=y} - \int_0^y l(s,t)e^{4H(s,y)}dt \Big\} ds \\
&= \int_0^x \int_0^y l(s,z)dze^{4H(s,y)}ds - \int_0^x \int_0^y l(s,t)e^{4H(s,t)}dsdt.
\end{aligned} \tag{23}$$

From (23) and the fact that $l \in L^1(I, \mathbb{R}_+)$ we get

$$\int_0^x \int_0^y l(s,t)e^{4H(s,t)}dsdt \leq \int_0^x \int_0^y l(s,z)dze^{4H(s,y)}ds = \frac{1}{4}(e^{4H(x,y)} - 1).$$

□

We consider the problem

$$\begin{cases} \frac{\partial^2 u}{\partial x \partial y}(x,y) = f(x,y,u_{(x,y)}) & \text{a.e. in } I, \\ u(x,y) = \psi(x,y) & \text{on } I_0, \end{cases} \tag{24}$$

where $f : I \times C(D, \mathbb{R}) \to \mathbb{R}$. Problem (24) is equivalent to the equation

$$\mathcal{F}[u](x,y) = 0 \quad \text{for } (x,y) \in I^*, \tag{25}$$

where

$$\begin{aligned}
\mathcal{F}[u](x,y) &= u(x,y) - \psi(x,0) - \psi(0,y) + \psi(0,0) \\
&\quad - \int_0^x \int_0^y f(s,t,u_{(s,t)})dsdt \qquad \text{for } (x,y) \in I,
\end{aligned} \tag{26}$$

$$\mathcal{F}[u](x,y) = u(x,y) - \psi(x,y) \qquad \text{for } (x,y) \in I_0. \tag{27}$$

We note that

$$(\mathcal{F}'[u]v)(x,y) = v(x,y) - \int_0^x \int_0^y \frac{\partial f}{\partial \omega}(s,t,u_{(s,t)})v_{(s,t)}dsdt \quad \text{for } (x,y) \in I, \tag{28}$$

$$(\mathcal{F}'[u]v)(x,y) = v(x,y) \qquad \text{for } (x,y) \in I_0. \tag{29}$$

We define the function $u_0 : I^* \to \mathbb{R}$ by

$$u_0(x,y) = \begin{cases} \psi(x,y) & \text{for } (x,y) \in I_0, \\ \psi(x,0) + \psi(0,y) - \psi(0,0) & \text{for } (x,y) \in I. \end{cases}$$

We note that if ψ satisfies Assumption 3 then u_0 is continuous on I^* and absolutely continuous on I. Obviously we can choose u_0 as an arbitrary continuous function on I^* such that $u_0(x,y) = \psi(x,y)$ for $(x,y) \in I_0$. The Newton sequence for (25) has the form

$$\begin{cases} u^0(x,y) = u_0(x,y) & \text{a.e. in } I^*, \\ u^{n+1}(x,y) = u^n(x,y) - [\mathcal{F}'(u^n)]^{-1}\mathcal{F}(u^n)(x,y) & \text{a.e. in } I^*, \end{cases} \tag{30}$$

hence

$$(\mathcal{F}'(u^n)u^{n+1})(x,y) = (\mathcal{F}'(u^n)u^n)(x,y) - \mathcal{F}(u^n)(x,y) \quad \text{a.e. in } I^*. \tag{31}$$

From (26) and (28) we get that (31) is equivalent to the equation

$$\begin{aligned} &u^{n+1}(x,y) - \int_0^x \int_0^y \frac{\partial f}{\partial \omega}(s,t,u^n_{(s,t)})u^{n+1}_{(s,t)}dsdt \\ &= u^n(x,y) - \int_0^x \int_0^y \frac{\partial f}{\partial \omega}(s,t,u^n_{(s,t)})u^n_{(s,t)}dsdt - u^n(x,y) \\ &\quad + \psi(x,0) + \psi(0,y) - \psi(0,0) + \int_0^x \int_0^y f(s,t,u^n_{(s,t)})dsdt, \end{aligned}$$

for $(x,y) \in I$. Hence

$$\begin{aligned} u^{n+1}(x,y) &= \psi(x,0) + \psi(0,y) - \psi(0,0) \\ &+ \int_0^x \int_0^y \left[f(s,t,u^n_{(s,t)}) + \frac{\partial f}{\partial \omega}(s,t,u^n_{(s,t)})(u^{n+1}_{(s,t)} - u^n_{(s,t)}) \right] dsdt. \end{aligned}$$

From (27) and (29) we get that (31) is equivalent to the equation

$$u^{n+1}(x,y) = \psi(x,y)$$

for $(x,y) \in I_0$. Hence $\{u^n\}$ defined by (30) has the form

$$\begin{cases} u^0(x,y) = u_0(x,y) & \text{a.e. in } I^*, \\ u^{n+1}(x,y) = \psi(x,0) + \psi(0,y) - \psi(0,0) & \\ \quad + \int_0^x \int_0^y \left[f(s,t,u^n_{(s,t)}) + \frac{\partial f}{\partial \omega}(s,t,u^n_{(s,t)})(u^{n+1}_{(s,t)} - u^n_{(s,t)}) \right] dsdt & \text{in } I, \\ u^{n+1}(x,y) = \psi(x,y) & \text{on } I_0. \end{cases} \tag{32}$$

Note that problem (32) is equivalent to the problem

$$\begin{cases} u^0(x,y) = u_0(x,y) & \text{a.e. in } I^*, \\ \frac{\partial^2 u^{n+1}}{\partial x \partial y}(x,y) = f(x,y,u^n_{(x,y)}) + \frac{\partial f}{\partial \omega}(x,y,u^n_{(x,y)})(u^{n+1}_{(x,y)} - u^n_{(x,y)}) & \text{a.e. in } I, \\ u^{n+1}(x,y) = \psi(x,y) & \text{on } I_0. \end{cases}$$

ASSUMPTION 4. *Suppose that the function $f : I \times C(D,\mathbb{R}) \to \mathbb{R}$ of the variables (x,y,ω) is such that*

A1) $f(\cdot\ ,\omega): I \to \mathbb{R}$ *is measurable for all* $\omega \in C(D, \mathbb{R})$ *and* $f(x, y, \cdot)\colon C(D, \mathbb{R}) \to \mathbb{R}$ *is continuous for a.e.* $(x, y) \in I$.

A2) *There is* $m \in L^1(I, \mathbb{R}_+)$ *such that*

$$|f(x, y, (u_0)_{(x,y)})| \leq m(x, y) \quad \textit{for a.e. } (x, y) \in I.$$

A3) *For a.e.* $(x, y) \in I$ *Frechet derivative* $\frac{\partial f}{\partial \omega}(x, y, \omega)$ *exists and is continuous linear operator for* $\omega \in C(D, \mathbb{R})$. *Moreover* $\frac{\partial f}{\partial \omega}(x, y, \cdot)$ *is continuous operator.*

A4) *There is a function* $l \in L^1(I, \mathbb{R}_+)$ *such that*

$$\left|\frac{\partial f}{\partial \omega}(x, y, \omega)u\right| \leq l(x, y)||u||_0, \tag{33}$$

$$\left|\left|\frac{\partial f}{\partial \omega}(x, y, \omega) - \frac{\partial f}{\partial \omega}(x, y, \tilde{\omega})\right|\right|_* \leq l(x, y)||\omega - \tilde{\omega}||_0 \tag{34}$$

for a.e. $(x, y) \in I$ *and* ω, $\tilde{\omega}$, $u \in C(D, \mathbb{R})$.

Remark 3. From the mean value theorem and (33) we get that

$$||f(x, y, \omega) - f(x, y, \tilde{\omega})|| \leq l(x, y)||\omega - \tilde{\omega}||_0 \qquad \text{for a.e. } (x, y) \in I \text{ and } \omega, \tilde{\omega}, u \in C(D, \mathbb{R}).$$

By the above, it follows that if $\omega^n \to \omega$ in $C(I, \mathbb{R})$ then

$$f(x, y, \omega^n) \to f(x, y, \omega) \qquad \text{for a.e. } (x, y) \in I.$$

Theorem 3. *Suppose that assumptions 3 and 4 are satisfied. Moreover*

$$\sinh(2H(a, b)) \int_0^a \int_0^b m(x, y) dx dy \leq \frac{1}{2}, \quad \textit{where} \quad H(x, y) = \int_0^x \int_0^y l(s, t) ds dt. \tag{35}$$

Then

a) *There exists the solution of* (25) *for* $(x, y) \in I^*$.

b) *The Newton sequence* (32) *exists and there is* u^* *such that* $u^* = \lim_{n\to\infty} u^n$ *on* I^*.

c) $(\mathcal{F}(u^*))(x, y) = 0$ *and the following estimate holds*

$$||u^n - u^*|| \leq \frac{C}{2^{n-1}}(2h)^{2^n - 1}\eta \quad \textit{for } n \geq 0,$$

where

$$C = e^{2H(a,b)}, \quad \eta = \int_0^a \int_0^b m(x, y) dx dy, \quad h = \sinh(2H(a, b))\eta. \tag{36}$$

Proof. We consider the Banach space $(C(I^*,\mathbb{R}), ||\cdot||_1)$, where

$$||u||_1 = \max_{(x,y)\in I} ||u||^{(x,y)} e^{-2H(x,y)}.$$

Note that

$$||u||^{(a,b)} e^{-2H(a,b)} \le ||u||_1, \quad \text{so} \quad ||u|| \le C||u||_1, \tag{37}$$

where C is defined by (36). Now, we will show that assumptions of theorem Kantorovich are satisfied. From (A3) we get that the Fréchet derivative of the operator $\mathcal{F}$ exists. Moreover it follows from (28) and (29) that

$$(\mathcal{F}'[u]v)(x,y) = ((E - A[u])v)(x,y),$$

where E is identity mapping and

$$\begin{aligned} (A[u]v)(x,y) &= \int_0^x \int_0^y \frac{\partial f}{\partial \omega}(s,t,u_{(s,t)}) v_{(s,t)} ds dt \qquad &&\text{for } (x,y)\in I, \\ (A[u]v)(x,y) &= 0 &&\text{for } (x,y)\in I_0. \end{aligned} \tag{38}$$

Let us estimate the norm of A. For $(x,y)\in I$ we get

$$\begin{aligned} |(A[u]v)(x,y)| &\le \int_0^x \int_0^y \left| \frac{\partial f}{\partial \omega}(s,t,u_{(s,t)}) v_{(s,t)} e^{-2H(s,t)} e^{2H(s,t)} \right| ds dt \\ &\le \int_0^x \int_0^y l(s,t) ||v||^{(s,t)} e^{-2H(s,t)} e^{2H(s,t)} ds dt \\ &\le ||v||_1 \int_0^x \int_0^y l(s,t) e^{2H(s,t)} ds dt \le \frac{1}{2} ||v||_1 e^{2H(x,y)}. \end{aligned}$$

The right side of the above inequality is nondecreasing in x and y. Therefore

$$||A[u]v||_1 \le \frac{1}{2}||v||_1.$$

Thus $||A[u]||_* \le \frac{1}{2} < 1$ and consequently from (38) and Lemma 2 we see that $[\mathcal{F}']^{-1}$ exists with $||[\mathcal{F}']^{-1}||_* \le 2$. From (34) and Lemma 3 we get that

$$\begin{aligned} |(F'[u] - F'[\tilde{u}])v(x,y)| &\le \int_0^x \int_0^y \left| \left(\frac{\partial f}{\partial \omega}(s,t,u_{(s,t)}) - \frac{\partial f}{\partial \omega}(s,t,\tilde{u}_{(s,t)}) \right) v(s,t) \right| ds dt \\ &\le \int_0^x \int_0^y l(s,t) ||u_{(s,t)} - \tilde{u}_{(s,t)}||_0 ||v||^{(s,t)} ds dt \\ &= \int_0^x \int_0^y l(s,t) ||u_{(s,t)} - \tilde{u}_{(s,t)}||_0 e^{-2H(s,t)} ||v||^{(s,t)} e^{-2H(s,t)} e^{4H(s,t)} ds dt \\ &\le ||u-\tilde{u}||_1 ||v||_1 \int_0^x \int_0^y l(s,t) e^{4H(s,t)} ds dt \le \frac{1}{4} ||u-\tilde{u}||_1 ||v||_1 (e^{4H(x,y)} - 1). \end{aligned}$$

Therefore

$$|(F'[u]v - F'[\tilde{u}]v)(x,y)|e^{-2H(x,y)} \leq \frac{1}{2}||u - \tilde{u}||_1||v||_1 \sinh(2H(x,y)).$$

Whence

$$||(F'[u] - F'[\tilde{u}])v||_1 \leq \frac{\sinh(2H(x,y))}{2}||u - \tilde{u}||_1||v||_1,$$

that is

$$||F'[u] - F'[\tilde{u}]||_* \leq \frac{\sinh(2H(a,b))}{2}||u - \tilde{u}||_1.$$

Let $\eta > 0$ be a constant defined by (36). Then

$$||u^1 - u^0||_1 \leq \int_0^a \int_0^b |f(x,y,(u_0)_{(x,y)})|dxdy \leq \eta. \tag{39}$$

From (35) it follows that

$$h = \sinh(2H(a,b))\eta \leq \frac{1}{2}.$$

Moreover from (39) we get that

$$||[\mathcal{F}'[u^0]]^{-1}\mathcal{F}[u^0]||_1 = ||u^1 - u^0||_1 \leq \eta.$$

Let

$$\delta = \frac{1 - \sqrt{1-2h}}{h}\eta.$$

Then from Theorem 2 and inequality (37) we get the assertion of our theorem.

□

References

1. Berkson E., Gillespie T. A., *Absolutely continuous functions of two variables and well-bounded operators*, J. London Math. Soc., Vol. **30**, No. **2** (1984), 305–321.
2. Brzychczy S., Janus J., *Monotone iterative methods for nonlinear integro-differential hyperbolic equations*, Univ. Iagell. Acta Math., **37** (1999), 245–261.
3. Brzychczy S., Janus J., *Monotone iterative methods for nonlinear hyperbolic differential-functional equations*, Univ. Iagell. Acta Math., **38** (2000), 141–152.
4. Czernous W., *On the Chaplyghin method for generalized solutions of partial differential functional equations*, Univ. Iagell. Acta Math., **43** (2005), 125–141.
5. Człapiński T., *Iterative methods for Darboux problem for partial functional differential equations*, J. of Inequal. & Appl., **4** (1998), 141–161.
6. Człapiński T., *Hyperbolic Functional Differential Equations*, Gdańsk, 1999.
7. Deimling K., *A Carathéodory theory for systems of integral equations*, Ann. Mat. Pura Appl., **86** (1970), 217–260.
8. Karpowicz A., *Carathéodory solutions of hyperbolic functional differential inequalities with first order derivatives*, Ann. Pol. Math., **94** (2008), 53–78.
9. Karpowicz A., *Monotone methods for hyperbolic differential-functional equations*, to appear.

10. Krylov V. I., Bobkov V. V., Monastyrnyi P. I., *Vychislitelnye metody vysshey matematiki*, Vysshaya Shkola, Minsk, 1972.
11. Lakshmikantham V., Köksal S., *Monotone Flows and Rapid Convergence for Nonlinear Partial Differential Equations*, Taylor & Francis, London, 2003.
12. Pandit S. G., *Monotone methods for systems of nonlinear hyperbolic problems in two independet variables*, Nonlinear Anal., Theory, Meth. & Appl., Vol. **30**, No. **5** (1997), 2735–2742.

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