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THE MODULAR CLASS AND ITS QUANTIZATION – MINICOURSE

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Abstract. In many of the quantum algebras studied in last years modular automorphisms play a relevant role. Recently, in the context of deformation quantization, Dolgushev showed how to relate the van den Bergh automorphism, carrying informations on duality between Hochschild homology and cohomology, with the Poisson modular class of the semiclassical limit. We will introduce his results and exhibit some interesting examples were trivial and non trivial modular automorphisms can be expected.

1. Introduction

Under the name *quantum geometry* one usually refers to a wide variety of techniques used to relate properties of non commutative algebras with geometric issues. One of the most widely known such a topic is the orbit method in representation theory, relating irreducible representations of the universal enveloping algebra to coadjoint orbits of the corresponding Lie algebra.

One peculiar tract of many classes of non commutative algebras studied in relation with quantization is the appearance of a naturally defined one parameter group of transformations on the algebra, related to *modular* properties. Examples of such modular automorphism are Connes–Takesaki automorphisms of von Neumann algebras, KMS automorphisms in groupoid algebras and multiplicative unitaries in compact quantum groups. At a more algebraic level we have the Nakayama automorphism in Noetherian Hopf algebras and the van den Bergh automorphism of general smooth algebras.

When the non commutative algebra under consideration is a deformation quantization algebra, i.e. it is an associative deformation of a commutative algebra, the deformation is determined by its infinitesimal datum: a Poisson bracket. It is then natural to relate the modular automorphism of non commutative algebras to a geometrical invariant of the underlying Poisson structures.

The aim of this short course is to explain the property of this invariant, not accidentally called the *modular class* of a Poisson manifold, and of its quantization.

The modular automorphism is maybe the easiest defined invariant on a Poisson manifold, and certainly one of the few that can be explicitly computed in most examples. It is determined by a vector field (unique only up to Hamiltonian vector fields) whose flow produces a one parameter group of infinitesimal Poisson transformations. A Poisson manifold is said to be unimodular when this modular flow can be chosen to be trivial. This property can be seen as a *type II* property. In fact Poisson manifolds are naturally foliated by symplectic manifolds, though the foliation in general fails to have constant rank. In the case the symplectic foliation is regular and the Poisson structure unimodular there exists a transverse measure to the foliation. In the non regular case its full meaning is subtler and, maybe, still not exploited at its fullness.

Quite recently Dolgushev showed how these modularity issues are not just related by analogy but indeed connected one to the other, at least in the context of Kontsevich formal deformation quantization. As we think that many consequences will follow from such results, and many more could be obtained in specific examples, our plan here is to clarify the general framework surrounding Dolgushev's theorems. In the first section we will review the basic ingredients of Poisson geometry. In the second one we will describe Dolgushev's theorems. We will also briefly touch upon the way in which modular properties should reflect on non commutative geometry *à la Connes*, when applied to deformation quantization algebras. Lastly we will describe two specific classes of Poisson manifolds with different modular properties and we will briefly comment on their quantizations (in which symmetry could replace formality in trying to prove analogues of Dolgushev's theorems).

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2. Poisson geometry

In this section we will review some basic results of the general theory of Poisson manifolds. Everything is quite standard and can be found in [**13, 31**] (and references therein). Let M be a smooth manifold. We will denote with $\mathfrak{X}(M)$ the space of vector fields on M and with $\mathfrak{X}^p(M)$ its p^{th}-wedge power, the space of multivector fields. We will denote with $\Omega^q M$ the space of q-differential forms. Given a p-multivector P and a p-form ω we will denote $\langle \omega, P\rangle$ the smooth function obtained by pairing them. More generally we will use $\imath_P\omega$ to denote the contraction of a q-form ω with a p-multivector (the result being a $q-p$ form) and, dually, $\imath_\omega P$ will stand for the contraction of the multivector with a form.

2.1. General theory.

DEFINITION 2.1. Let M be a manifold and let $\pi \in \Gamma(\wedge^2 TM)$ be a bivector. We will say that (M,π) is a Poisson manifold if the bracket on $\mathcal{C}^\infty(M)$ defined by

$$\{f,g\} = \langle \pi, df \wedge dg \rangle$$

is a Lie bracket.

Let us remark that the bivector requirement for π implies the Leibniz identity

$$\{fg,h\} = f\{g,h\} + g\{f,h\} \qquad \forall f,g,h, \in \mathcal{C}^\infty(M).$$

The condition that the bracket (2.1) is a Lie bracket can be expressed in compact form once the Schouten–Nijenhuis bracket between multivectors is introduced. Such bracket is just the extension of the usual Lie bracket of vector fields to a graded derivation of the whole Grassmann algebra of multivector fields $\mathfrak{X}^\bullet(M)$, which means that

$$[X, Y\wedge Z] = [X,Y]\wedge Z + Y \wedge [X,Z] \qquad \forall X,Y,Z \in \mathfrak{X}(M).$$

Using such bracket (2.1) is easily summarized as:

$$[\pi,\pi] = 0\,,$$

since

$$\langle [\pi,\pi], df \wedge dg \wedge dh \rangle = \{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\}\,.$$

For any smooth function $f \in \mathcal{C}^\infty(M)$, the vector field

$$X_f(g) = \{f,g\} = \langle \imath_{df}\pi, dg \rangle \qquad \forall g \in \mathcal{C}^\infty(M)$$

is called a *Hamiltonian* vector field of Hamiltonian function f. Using more generally 1-forms, rather than exact ones, one can define the so-called *sharp map*:

$$\sharp_\pi : \Omega^1 M \to \mathfrak{X}(M); \quad \alpha \mapsto i_\alpha \pi\,.$$

In particular, on exact forms, $\sharp_\pi(df) = X_f$. Since any 1-form is locally exact, vector fields in the image of the sharp map are called *locally Hamiltonian* vector fields. It is not difficult to show that, from an algebraic point of view, locally Hamiltonian vector fields are exactly those derivations of $\mathcal{C}^\infty(M)$, which are at the same derivations of the Poisson bracket (2.1):

$$X\{f,g\} = \{Xf,g\} + \{f,Xg\} \qquad \forall f,g \in \mathcal{C}^\infty(M).$$

Hamiltonian vector fields span an involutive non regular[1] distribution which is integrable. Any Poisson manifold is therefore equipped with a canonical non regular foliation. Two points on each leaf may be connected by a chain of locally defined integral paths of Hamiltonian fields. Furthermore each leaf carries a well defined symplectic form induced by the Poisson tensor. These data together constitutes what is called the *symplectic foliation* of (M, π).

Through the sharp map it is also possible to define a Lie bracket between 1-forms:

$$[\alpha, \beta] = L_{\sharp_\pi(\alpha)}\beta - L_{\sharp_\pi(\beta)}(\alpha) - d\pi(\alpha, \beta) \qquad \forall \alpha, \beta \in \Omega^1 M \,.$$

This bracket is relevant in that it shows that the cotangent bundle of any Poisson manifold T^*M, endowed with this bracket between its sections and the sharp map is an example of what is called a *Lie algebroid*. Much of what follows holds true for more general Lie algebroids.

2.2. Cohomology and homology.

On the vector spaces of multivector fields on a Poisson manifold (M, π) we can define, by mean of the Schouten–Nijenhuis bracket, the following degree 1 operator:

$$d_\pi : \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M); \quad d_\pi(P) = [\pi, P] \,.$$

From the graded Jacobi identity for the Schouten bracket one easily gets that $d_\pi^2 = 0$. The cohomology of the complex $(\mathfrak{X}^k(M), d_\pi)$ (first introduced in [**25**]) is called the (Lichnerowicz)–Poisson cohomology of M and will be denoted as $H_\pi^k(M)$.

This cohomology has a (well founded) reputation for being quite hard to compute. This is due to the fact that it depends both on the topology of the foliation and on the variation of the symplectic form from leaf to leaf. Some interesting examples on linear spaces were explicitly computed recently (see [**1, 28, 29**]).

Let us try to understand it a little bit better by having, as usual, a closer look at the meaning of low-dimensional cohomology groups. If we take a 0-cochain $f \in \mathcal{C}^\infty(M)$, then $d_\pi(f) = X_f$. The 0-th cohomology group can be therefore described as the set of those functions such that $X_f = 0$. Such functions are called *Casimir functions* on the Poisson manifold and are constant along the leaves of the symplectic foliation.

[1]By non regular we man that its local dimension is non globally constant but it is only a lower semicontinuous function on M. Integrability of the distribution does not follow from the Frobenius theorem, though involutivity holds, but requires the subtler Stefan–Sussmann theorem (see [**6, 13, 31**] for more details).

Examples.

- Let $M = \mathbb{R}^{2n+1}$, with coordinates $p_i, q_i, t,\ i = 1, \dots n$ and with the canonical Poisson brackets having as only nontrivial commutators:

$$\{p_i, q_j\} = \delta_{i,j}\,.$$

  Then $H^0(M)$ is isomorphic to $\mathcal{C}^\infty(\mathbb{R})$, identified with functions on M depending only on t. In this case the symplectic leaves coincide with the level sets of the Casimir function t.
- Let $\pi = (x^2+y^2)\partial_x \wedge \partial_y$ on $\mathbb{R}^2$. Then a non constant function $f \in H^0_\pi(M)$ if and only if $(x^2+y^2)f \equiv 0$. This implies $f = 0$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ and, by continuity, $f \equiv 0$. Thus $H^0(M) \simeq \mathbb{R}$ and the Poisson manifold has no nonconstant Casimirs. Remark that this may happen even if the foliation is non trivial (hence leaves do no always coincide with level sets of Casimirs).

Let's now take a 1-cochain $X \in \mathfrak{X}^1(M)$. Then $d_\pi X = 0$ if and only if $X \in Der(\mathcal{C}^\infty(M), \{,\})$ is also a derivation of the Poisson bracket. Such a vector field is called a *Poisson vector field* and, as said, is locally Hamiltonian. Therefore $H^1_\pi(M)$ is the space of Poisson vector fields modulo Hamiltonian vector fields. A rephrasing of this statement is to say that the first Poisson cohomology group is the space of Poisson derivations modulo inner Poisson derivations. Our main invariant, the modular class, sit inside this cohomology groups and we will not need the higher order ones. A way of distinguishing locally Hamiltonian vector fields from Hamiltonian ones is the following: while the flow of Hamiltonian vector fields preserve, by definition, the symplectic foliation, the flow of locally Hamiltonian vector fields can move points from one leaf to another.

In the special case of a symplectic manifold M the sharp map is an isomorphism of vector spaces. At the level of 1-chains the sharp map sends closed 1-forms isomorphically onto Poisson vector fields and exact 1-forms isomorphically onto Hamiltonian vector fields. Its linear graded extension to the whole Grassmann algebras of differential forms and multivector fields is an isomorphism of cochain complexes, since it commutes with coboundary operators, and thus $H^k_\pi(M) \simeq H^k_{\mathrm{deR}}(M)$.

On Poisson manifolds it is possible also to give an homology theory, using differential forms, which is to a certain extent dual to the previous one. In fact, this duality problem together with how it is reflected in quantization is the core of these notes. On the space of smooth k-forms $\Omega^k M$ we can define a homology operator ∂_π as the graded commutator

$$\partial_\pi : \Omega^k M \to \Omega^{k-1} M; \quad \partial_\pi = [d, i_\pi]\,.$$

It can be checked (the computation being not so trivially easy) that $\partial_\pi^2 = 0$. The corresponding homology (introduced by Brylinski in [**5**]) will be called the *Poisson homology* of M and denoted $H_k^\pi(M)$.

Let us consider, as before, the simplest low-dimensional case. Let $\alpha = fdg$ be a generic 1-form on M. Then:

$$\partial_\pi(fdg) = di_\pi(fdg) - i_\pi(df \wedge dg) = -\{f, g\}\,.$$

Therefore

$$H_0^\pi(M) = \mathcal{C}^\infty(M)/\{\mathcal{C}^\infty(M), \mathcal{C}^\infty(M)\}\,.$$

2.3. The modular class.

Let, from now on, M be orientable of dimension n (this hypothesis is not, strictly speaking, necessary, but will simplify things a little bit). Let Ω be a volume form on M. Take any $f \in \mathcal{C}^\infty(M)$. Due to dimension reasons, the Lie derivative of the volume form Ω in the direction of the Hamiltonian vector field X_f is proportional to Ω itself, thus there exists a smooth function $\phi_\Omega(f) \in \mathcal{C}^\infty(M)$ such that:

$$L_{X_f}\Omega = \phi_\Omega(f)\Omega\,.$$

We have the following facts (which follows through a straightforward check from definitions):

1. the map $\phi_\Omega : f \mapsto \phi_\Omega(f)$ is a derivation of $\mathcal{C}^\infty(M)$, thus a vector field;
2. the map ϕ_Ω is a derivation of $\{-,-\}$, thus a Poisson vector field;
3. let $\Omega' = a\Omega$ be another volume form on M, $a \in \mathcal{C}^\infty(M)$, $a(x) \neq 0$, $\forall x \in M$; then
$$\phi_{\Omega'} = \phi_\Omega + X_{-log|a|}\,.$$

These three facts together imply that the vector field ϕ_Ω defines a Poisson cohomology class $[\phi_\Omega] \in H_\pi^1(M)$ which does not depend on M.[2] This class is called the *Poisson modular class* and, as everyone may have guessed by now, is the main character entering our story.

A Poisson manifold (M, π) such that $[\phi_\Omega] = 0$ will be called *unimodular*.

1. Let M be symplectic of dimension $2m$; then M is unimodular (use the symplectic volume $\Omega = \frac{\omega^m}{m!}$).
2. Let $M = \mathfrak{g}^*$ be the dual of a Lie algebra with the linear Poisson structure. In this case the modular class coincides with the Lie algebra cohomology class defined by the adjoint character $X \mapsto tr(ad_X)$. Therefore M is unimodular as a Poisson manifold if and only if $\mathfrak{g}$ is unimodular as a Lie algebra.

[2]In the algebraic setting which will be mentioned in the next section some additional care is needed; the appearance of a logarithm in the above formula forces to work with the subtler notion of log-Hamiltonian vector fields.

3. Let M be regularly foliated by symplectic manifolds. Then one can prove that there exists an injection $H^1(M) \hookrightarrow H^1_\pi(M)$ sending the Reeb class of the foliation to the Poisson modular class. In this case, therefore, Poisson unimodularity is equivalent to vanishing of the Reeb class.

Let us consider a fixed (M, π) and the volume form Ω. Then the following chain of equalities holds:

$$\begin{aligned} \int_M \{f, g\}\Omega &= \int_M \left(L_{X_f} g\right)\Omega \\ &= \int_M L_{X_f}(g\Omega) - g L_{X_f}\Omega \\ &= \int_M \left(d\imath_{X_f} g\Omega\right) + \imath_{X_f} d\,(g\Omega) - g L_{X_f}\Omega \\ &= -\int_M g\phi_\Omega(f)\Omega \qquad \forall f, g \in \mathcal{C}^\infty(M)\,. \end{aligned}$$

The equality between the first and last line of this chain is called *Poisson KMS condition*. Considering the integral w.r. to Ω as a trace tr_Ω on the associative algebra $\mathcal{C}^\infty(M)(M)$ the above condition states that in general such trace fails to be also a trace at the Poisson algebra level, since

$$tr_\Omega(\{f, g\}) = -tr_\Omega(\phi_\Omega(f)g)$$

and the modular class measures this failure.

In case (M, π) is a unimodular Poisson manifold the volume form Ω can be chosen in such a way that $\phi_\Omega = 0$. Then from the Poisson KMS condition we get:

$$\int_M \{f, g\}\Omega = 0 \qquad \forall f, g \in \mathcal{C}^\infty(M).$$

This means that there is a choice of volume form such that $\int_M \Omega = tr_\Omega$ is a *Poisson trace* on the Poisson algebra $\mathcal{C}^\infty(M)$. The existence of a Poisson trace is a non trivial fact (and examples of Poisson manifolds having no Poisson traces are easily produced). Let us remark that in general the space of Poisson traces can be seen as dual to the 0-dimensional Poisson homology $H_0^\pi(M)$.

Another point of view on the same property is the following. The Poisson boundary of a volume form can be expressed through the simpler formula

$$\partial_\pi\Omega = -\imath_{\phi_\Omega}\Omega$$

therefore (M, π) is Poisson unimodular if and only if there exists a volume form Ω such that $\partial_\pi\Omega = 0$. This means that such volume form defines a non trivial cycle for the higher Poisson homology and therefore implies $H_n^\pi(M) \neq 0$.

2.4. Duality.

To express correctly the duality between Poisson cohomology and homology we need their version *with coefficients*. Coefficients can be thought either, algebraically, as Poisson modules or, more geometrically, as being a vector bundle with a flat "connection." The right idea of connection in this context is that of contravariant connection

DEFINITION 2.2. Let $E \to M$ be a vector bundle on the Poisson manifold M. A flat contravariant connection on M is a linear map

$$D : \Omega^1 M \otimes \Gamma(E) \to \Gamma(E); \quad (\alpha, s) \mapsto D_\alpha s$$

such that:

1. $D_\alpha(fs) = f D_\alpha s + (\sharp_\pi(\alpha) f)s$, $\forall \alpha \in \Omega^1 M, s \in \Gamma(E), f \in \mathcal{C}^\infty(M)$;
2. $D_{f\alpha} s = f D_\alpha s$, $\forall \alpha \in \Omega^1 M, s \in \Gamma(E), f \in \mathcal{C}^\infty(M)$;
3. $[D_\alpha, D_\beta] = D[\alpha, \beta]_\pi$, $\forall \alpha, \beta \in \Omega^1 M$ (flatness condition).

As we were mentioning this notion admits other interpretations. We can say that the space of sections has a $\mathcal{C}^\infty(M)$-Poisson module structure[3] given by $f \cdot s = D_{df} s$, or that the Poisson Lie algebroid $(T^*M, [.], \sharp_\pi)$ has a Lie algebroid representation on E.

We will just need the easiest possible example: let $L \to M$ be a trivial line bundle on M. Any Poisson vector field $X \in \mathfrak{X}(M)$ defines a flat contravariant connection on L as:

$$D_{fdg} h = f D_{dg} h = f\,[\{g, h\} + (Xg)h]$$

and, in fact, any flat contravariant connection on L arises in this way, considering the Poisson vector field given by $Xg = D_{dg}1$.

EXAMPLES.

1. Let M be a symplectic manifold and let $E \to M$ be a flat vector bundle over M with flat connection ∇. Then $D_{df} = \nabla_{\sharp_\pi(df)} = \nabla_{X_f}$ defines a flat contravariant connection on E.
2. In case M is Poisson and orientable we can consider the canonical line bundle $\wedge^n T^*M$, together with the trivialization defined by the choice of a volume form Ω. This determines a Poisson vector field ϕ_Ω. The corresponding flat contravariant connection on $\wedge^n T^*M$ is called the *canonical Poisson line bundle*. In this language saying that a Poisson manifold is unimodular is tantamount to saying that its canonical Poisson line bundle is Poisson trivial.

[3]The defining properties of a Poisson module can then be derived just by rewriting 1.-2.-3. in this notation.

The key point now is that both Poisson homology and Poisson cohomology can be defined with coefficients in any Poisson vector bundle.[4] We will not give the exact definition here, which will take us too far away from our purposes. We just remark that in case in which M is symplectic the flat contravariant connection is just an ordinary flat connection and the Poisson cohomology with coefficients turns out to be exactly the de Rham cohomology with coefficients in a flat bundle (see [**3**] for more on the subject).

In the easiest possible case, that of trivial line bundles $L \to M$ carrying a flat contravariant connection defined by a Poisson vector field X, the *twisted* Poisson cohomology with coefficients in L is given by the explicit coboundary operator

$$d_{\pi,L} P = [\pi, P] + X \wedge P \qquad \forall P \in \mathfrak{X}^p(M)\,.$$

Dually, the Poisson homology with coefficients in L is given by the boundary operator

$$\partial_{\pi,L}\omega = \partial_\pi \omega - \imath_X \omega \qquad \forall \omega \in \Omega^p M\,.$$

We are now ready to state our duality result ([**14**] is the first paper in which a simpler version of this result was obtained; for more on the subject look at [**13**] and references therein; complete results expressed in the language of Lie algebroids can be found in [**19**]).

THEOREM 2.3. *Let (M,π) be an orientable Poisson manifold and let E be a Poisson vector bundle on M. The following Poincaré duality between Poisson homology and cohomology holds true*

$$H^k_\pi(M; E \otimes \wedge^n T^* M) \simeq H^\pi_{n-k}(M; E)\,.$$

In particular, if (M,π) is unimodular:

$$H^k_\pi(M) \simeq H^\pi_{n-k}(M)\,.$$

3. Quantum modular class

In this section we will explain results quite recently obtained by Dolgushev in [**11**], settling down the problem of quantization of the modular class in the formal case.

3.1. Deformation quantization.

Let (M,π_0) be an orientable Poisson manifold.[5] Let $\mathcal{A}_\hbar$ be a deformation quantization of this manifold. This basically means that $\mathcal{A}_\hbar$ is a topologically free $\mathbb{C}[[\hbar]]$-associative algebra (topologically free means Hausdorff and complete

[4]In the Lie algebroid setting we are dealing with Lie algebroid (co)homology with coefficients in a representation, a natural generalization of what is usually done for Lie algebras.

[5]The setting in which Dolgushev's results are obtained is that of a smooth affine variety over $\mathbb{C}$ with trivial canonical bundle.

in the $\hbar$-adic topology and torsion free with respect to $\hbar$; this is equivalent to the existence of a vector space isomorphism $\mathcal{A}_\hbar \simeq A[[\hbar]])$ such that, furthermore:

$$\frac{\mathcal{A}_\hbar}{\hbar \mathcal{A}_\hbar} \simeq \mathcal{F}(M); \qquad \frac{[f_1, f_2]}{\hbar} \mod \hbar = \{f_1, f_2\}\,.$$

Here by $\mathcal{F}(M)$ we will mean an algebra of functions on the manifold such that its characters allow to reconstruct the point set M. It can be, depending on the context, the algebra of regular function on a smooth affine variety, the algebra of smooth functions on a manifold and so on.

The renowned Kontsevich formality theorem (see [**20**]) implies that for any Poisson manifold such a deformation quantization exists. Furthermore any deformation quantization is uniquely determined by the so called Kontsevich class, which is a formal Poisson bivector, i.e. $\pi \in \hbar\Gamma(\wedge^2 TX)[[\hbar]]$, such that $\pi = \hbar\pi_0 + O(\hbar^2)$ and $[\pi, \pi] = 0$. Most of what was said in the previous chapters about Poisson manifolds keeps being true for formal Poisson manifolds, where we will just allow formal power series of functions, vectors, multivectors, differential forms and so on.

Let us remark, in particular, that if a formal Poisson bivector is unimodular then also its 0-order term π_0, which is again a Poisson bivector, is unimodular. If X is a formal Poisson vector field for π, i.e. an element of $\Xi(M)[[\hbar]]$ which is a derivation of the formal Lie bracket defined by π, then its lower order term X_0 is a Poisson vector field for π_0.

Our aim here is to give an answer to the following questions:

1. Does the quantization of unimodular Poisson manifolds carry special features?
2. Can the modular vector field be lifted to a quantization?

To address such questions we will first address a seemingly unrelated algebraic duality.

3.2. The van den Bergh duality.

The van den Bergh duality theorem is a purely algebraic results which clarifies the extent to which Hochschild homology and cohomology of an associative algebra are dual.

We do not attempt to provide full definitions here (the standard reference being [**26**]); we just remark that Hochschild (co)homology is the natural homology theory arising in the category of unital associative algebras,[6] ruling, for example, associative algebra deformations. When the associative algebra is $\mathcal{C}^\infty(M)$ (and similar results hold true for other function algebras) the Hochschild homology (resp. cohomology) with coefficients in the algebra itself, seen as a bimodule by left and right multiplication, is isomorphic to $\Omega^* M$ (resp. to $\mathfrak{X}(M)$). This result is known as Hochschild–Kostant–Rosenberg theorem;

[6] They can be shortly defined, for example, in terms of *Ext* and *Tor* functors, which is not very helpful for someone unfamiliar with this language, see [**22**].

we will call *HKR map* the map on chains descending to this vector space isomorphism.

Let B be any complex associative algebra and B^{op} the opposite algebra $a \cdot^{op} b = b \cdot a$; let us denote with $B^e = B \otimes B^{op}$ the associative envelope of B. This space carries a natural B-bimodule structure, given by left and right multiplication in B. Let us define $\dim B$ as the projective dimension[7] of the algebra. The algebra B is said to be smooth whenever $\dim B < \infty$. With $HH^k(B; M)$ (resp. $HH_k(B; M)$) we will denote the Hochschild cohomology (resp. homology) with coefficients in a B-bimodule M ([**26**] for general definitions).

The next proposition appears, at first, as a very specialized assertion on this algebraic cohomology theory. It is our purpose to show how it is intimately related to the Poisson modular class when the algebra under investigation is a deformation quantization algebra. Let us recall that a B-bimodule M is called invertible if there exists another B-bimodule, denoted M^{-1}, such that:

$$M \otimes_B M^{-1} \simeq B; \qquad M^{-1} \otimes_B M \simeq B,$$

where $\simeq$ stands for B-bimodule isomoprhism.

Theorem 3.1 (van den Bergh). *Let B be a smooth algebra. Suppose that there exist $n \in \mathbb{N}$ and an invertible bimodule ω_B such that:*

$$HH^k(B, B^e) = \begin{cases} \omega_B & k = n \\ 0 & k \neq n \end{cases}$$

then

1. $n = \dim B$;
2. $HH_k(B, \omega_B \otimes_B M) \simeq HH^{n-k}(B, M)$ *for any B-bimodule M.*

In this case ω_B is called the *dualizing bimodule* of B.

What does this theorem tell us in the *classical case*, $B = \mathcal{C}^\infty(M)$? In that case an invertible projective B-bimodule is equivalent to[8] a line bundle L on M, such that there exists another line bundle for which $L \otimes L^{-1}$ and $L^{-1} \otimes L$ are the trivial line bundle. The bundle ω_B, in the orientable case, is nothing but the canonical line bundle (with non zero sections given by volume form) and the van den Bergh duality, in that case, is nothing but a different avatar of the usual Poincaré duality appears.

Let us now move to deformation quantization algebras. A special family of B-bimodules we need to consider in the non commutative case is given by algebra automorphisms $\nu \in Aut(B)$. Any such automorphisms defines two B-bimodule structure on B itself, to be denoted, respectively, by ${}_\nu B$ and B_ν given by:

[7]This means the length of a projective resolution of B in the category of finitely generated B-bimodules.

[8]Think at the projective bimodule as the space of sections of the bundle.

1. $(b \cdot_\nu b_1) \cdot_\nu b_2 = \nu(b) b_1 b_2$, $\forall b, b_1, b_2 \in B$;
2. $(b \odot_\nu b_1) \odot_\nu b_2 = b b_1 \nu(b_2)$, $\forall b, b_1, b_2 \in B$.

PROPOSITION 3.2. *Let $\mathcal{A}_\hbar$ be a deformation quantization of the Poisson manifold (M, π). Then $\mathcal{A}_\hbar$ satisfies the hypothesis of van den Bergh theorem. Furthermore* $\dim \mathcal{A}_\hbar = \dim M$ *and the dualizing bimodule $\omega_{\mathcal{A}_\hbar}$ is isomorphic to $\mathcal{A}_\hbar \nu$, where $\nu \in Aut(\mathcal{A}_\hbar)$, $\nu = \mathrm{Id} \mod \hbar$.*

Let us remark that the last equality means that the automorphism ν equals the identity at order 0, i.e. for any $a \in \mathcal{A}_\hbar$ we have that $(\nu(a) - a) \in \hbar \mathcal{A}_\hbar$.

Let us consider the trivial case of a manifold with zero Poisson structure and its obvious deformation quantization, which is nothing but its $\mathbb{C}[[\hbar]]$-linear extension. Then this proposition recovers the usual Poincaré duality with $\nu = Id$, i.e. the dualizing bimodule is a trivial rank 1 projective bimodule, the space of sections of a trivial line bundle.

In the general case this proposition, as mentioned in the introduction, states that any deformation quantization algebra comes equipped with a distinguished automorphism ν. To be precise, ν is not unique; it is uniquely defined only up to inner automorphisms of $\mathcal{A}_\hbar$ (i.e. conjugation by elements of $\mathcal{A}_\hbar$). Any such automorphism will be called a *modular automorphism* of $\mathcal{A}_\hbar$.

It is also worth mentioning that the statement above does not apply only to Kontsevich's deformation quantization but to any deformation quantization, in the loose sense specified at the beginning of this section.

We would like to relate this natural modular automorphism with the Poisson modular flow of the undeformed algebra. Since in the following we will need formality theorems (for the deformed algebra and for its Hochschild (co)chains) we will restrict ourselves to Kontsevich's deformation quantizations.

THEOREM 3.3 (Dolgushev). *The van den Bergh dualizing bimodule of $\mathcal{A}_\hbar$ is isomorphic to $\mathcal{A}_\hbar$ (thus ν is inner) iff π is unimodular.*

SKETCH OF PROOF. Let us first suppose that $\omega_B \simeq \mathcal{A}_\hbar$ as a bimodule. Then van den Bergh theorem implies that there is an isomorphism:

$$V : HH^0(\mathcal{A}_\hbar, \mathcal{A}_\hbar) \to HH_n(\mathcal{A}_\hbar, \mathcal{A}_\hbar)\,.$$

The formality theorems for chains and cochains (see [**12**]) guarantee the existence of isomorphisms

$$\mu_1 : HH^0(\mathcal{A}_\hbar, \mathcal{A}_\hbar) \to H^0_\pi(M); \qquad \mu_2 : HH_n(\mathcal{A}_\hbar, \mathcal{A}_\hbar) \to H^\pi_n(M)\,. \tag{3.1}$$

Let us now consider $[1] \in H^0_\pi(M)$. Then $\mu_2 \circ V \circ \mu_1^{-1}[1] = [\omega]$, where $\omega \in \Omega^n M[[\hbar]]$ is such that $\partial_\pi \omega = 0$; if we prove that ω is a volume form (i.e. nowhere vanishing) on M then we are done (see remark 2 after the definition of the modular class). Let $\omega = \omega_0 + O(\hbar)$. Dolgushev shows that ω is a volume form iff ω_0 is. To prove that ω_0 never vanishes it is enough to use the map V_0

(the $\hbar = 0$ term of the van den Bergh isomorphism) and Hochschild–Kostant–Rosenberg theorem.
Let us now start from π being unimodular. Then there exists a volume form $\omega \in \Omega^n M[[\hbar]]$such that $\partial_\pi \omega = 0$ and ω_0 is a volume form. Now van den Bergh theorem establishes an isomorphism

$$\tilde{V} : HH_n(\mathcal{A}_\hbar, \mathcal{A}_\hbar) \to HH^0(\mathcal{A}_\hbar, \nu^{-1}\mathcal{A}_\hbar).$$

Let us consider $\tilde{V}(\mu_2^{-1}[\omega])$. This is represented by an invertible $b \in \mathcal{A}_\hbar$ which is a twisted 0-cocycle, i.e.:

$$\nu^{-1}(a) \star b - b \star a = 0\,, \qquad \forall a \in \mathcal{A}_\hbar.$$

Thus $\nu^{-1}(a) = b \star a \star b^{-1}$, which means that ν is an inner automorphism and therefore $\mathcal{A}_\hbar \nu \simeq \mathcal{A}_\hbar$ as $\mathcal{A}_\hbar$-bimodules. □

In general an associative algebra such that the van den Bergh dualizing bimodule is isomorphic to the algebra itself is sometimes called a Calabi–Yau algebra [**16**]. In this language the theorem states that a deformation quantization algebra is Calabi–Yau if and only if the underlying Poisson bivector is unimodular.

We would like to remark here that the proof of this theorem does not really need formality theorems in their full form. What is really needed is the existence of the isomorphisms given in (3.1). In specific cases such isomorphisms can be obtained without relying on the ∞-algebra's approach (e.g. through spectral sequences, as we will see for quantum groups).

3.3. Quantization of Poisson vector fields.

Let $X \in \mathfrak{X}(M)[[\hbar]]$ be a formal vector field. How does this quantize? Certainly any derivation $D \in Der(\mathcal{A}_\hbar)$ has a lower order term D_0[9] which is a derivation on functions, thus a vector field on M. We would like to build a section of this map, at least for Poisson vector fields (at the formal level).

PROPOSITION 3.4. *There exists a $\mathbb{C}[[\hbar]]$-linear map*

$$\mathfrak{X}[[\hbar]] \cap \operatorname{Ker} d_\pi \to Der(\mathcal{A}_\hbar); X \mapsto D_X \tag{3.2}$$

such that:

1. $D_X = X \mod (\hbar)$;
2. $[D_X, D_Y] = D_{[X,Y]} + J$, *where J is an inner derivation of $\mathcal{A}_\hbar$.*

Let us remark that if $X \in \mathfrak{X}[[\hbar]] \cap \operatorname{Ker} d_\pi$ then its lower order term X_0 is a π_0-Poisson vector field. Let us also stress the point that this map is nothing but a lifting at the level of cycles of the isomorphism $\mu : H^1_\pi(M) \simeq HH^1(\mathcal{A}_\hbar, \mathcal{A}_\hbar)$ provided by formality for Hochschild chains. The existence of such map can be expressed by saying that all Poisson vector fields can be quantized as derivations of the quantum algebra. We do not know, apart

[9]Say $D \in \hbar^n Der(\mathcal{A}_\hbar)$ but $D \notin \hbar^{n-1} Der(\mathcal{A}_\hbar)$; then $D_0 = D/\hbar^n \mod \hbar$.

from a bunch of examples, results of this kind which do not rely on formality techniques.

Take now ϕ_Ω to be a modular vector field for (M, π). Recall that the Kontsevich class of the quantization is $\pi = \hbar\pi_0 + O(\hbar^2)$, thus $\phi_\Omega \in \hbar\Gamma(TM)[[\hbar]] \cap \operatorname{Ker} d_\pi$, i.e. $\phi_\Omega = 0 \mod \hbar$. Then the quantization of the modular vector field via (3.2) verifies $D_{\phi_\Omega} = 0 \mod \hbar$. This allows to define an automorphism of $\mathcal{A}_\hbar$ by formal exponentiation:

$$exp(D_{\phi_\Omega}) = \sum_{n\geq 0} \hbar^n \frac{D^n_{\phi_\Omega}}{n!} \,.$$

In the next section we will see that this automoprhism can be identified with the modular one.

3.4. Main theorem.

THEOREM 3.5 (Dolgushev). *Let $\mathcal{A}_\hbar$ be a deformation quantization of (M, π_0) and let π be a representative of the Kontsevich class, with modular vector field ϕ_Ω w.r.t. to a formal volume form $\Omega \in \Gamma(\wedge^n T^*M)[[\hbar]]$. Consider the corresponding derivation $D_{\phi_\Omega} \in Der(\mathcal{A}_\hbar)$. Then the modular automorphism of $\mathcal{A}_\hbar$ is*

$$\nu = exp(D_{\phi_\Omega})$$

modulo inner automorphisms.

In another language the same results can be expressed by saying that the semiclassical limit of the v.d. Bergh dualizing bimodule of $\mathcal{A}_\hbar$ is the canonical flat contravariant connection on M.

HINTS ON THE PROOF. Let $\sigma = exp(D_{\phi_\Omega})$. Define on $\mathcal{A}_\hbar[t, t^{-1}]$ the associative product

$$(at^n) \cdot (bt^m) = a \star \sigma^n(b) t^{n+m}$$

this product extend to a star product on $A[t, t^{-1}][[\hbar]]$ which is a deformation quantization of

$$\pi_0 + t\partial_t \wedge \phi_\Omega \,.$$

This last bivector is always Poisson and unimodular[10] on $M \times \mathbb{C}^*$. We can now apply Theorem 3.3 to this deformation quantization and conclude that its v.d. Bergh dualizing bimodule is isomorphic to the algebra itself.

At this point the missing (more technical) step is relating the Hochschild (co)homology $A[t, t^{-1}][[\hbar]]$ to the one of $\mathcal{A}_\hbar$. We will refrain from discussing it here. □

[10]This kind of Poisson bivectors are called also Poisson–Ore extensions; in particular this is called the standard unimodular extension in [**7**].

REMARK. What Dolgushev proves, in passing, is that an Ore extension of $\mathcal{A}_\hbar$ defined by an automorphism (thus: no twisted derivation) is the Kontsevich quantization of the corresponding Poisson–Ore extension $M \times \mathbb{C}^*$. It would be interesting to understand whether such a result still holds true for general Ore extensions admitting non trivial twisted derivations.

3.5. Applications to NC geometry.

In non commutative geometry *à la Connes* the key notion is that of a spectral triple, the axiomatic set of data that should generalize the concept of a spin manifold. The objects in a spectral triple $(\mathcal{A}, H, D)$ are a non commutative algebra $\mathcal{A}$, an Hilbert space H and an operator D; having in mind the triple $\mathcal{A} = \mathcal{C}^\infty(M)$, $H = L^2(M; S)$ the space of L^2-sections of the spin bundle S, D a Dirac operator on S. In [**18**] the author made an attempt to understand what can be said when, in the spectral triple, the algebra is a deformation quantization $\mathcal{A}_\hbar$. Spectral triples verify a long list of axioms which would take us too far away to describe here. One whose role seems, however, fundamental, concerns the existence of a noncommutative trace. In fact if M is a spin manifold, it is orientable and integration with respect to a volume form defines a linear map on $\mathcal{C}^\infty(M)_c$ which is trivially a trace. For this reason it is reasonable to expect that non commutative integration is given by a non commutative trace. This axiom is strictly connected to non triviality of a suitable Hochschild homology group, the so called *dimension axiom.*

PROPOSITION 3.6. *Let (M, π) be a Poisson manifold with a fixed volume form Ω and let $\mathcal{A}_\hbar$ be any deformation quantization of (M, π). If there exists a trace on $\mathcal{A}_\hbar$ such that*

$$\int_M f\Omega = \tau(\bar{f}) \mod \hbar\,,$$

where $f = \bar{f} \mod \hbar$, then $\phi_\Omega = 0$, i.e. (M, π) is unimodular.

This can be seen as a simplified version of Dolgushev theorem (3.3) holding true for any deformation quantization. It tells us that if we insist on using traces as a non commutative version of integration then we are forced to restrict ourselves to the quantization of unimodular Poisson manifolds.

On the other hand for spectral triples, as we said, indeed a trace is necessary. This simple proposition shows that it is unreasonable to expect a spectral triple if you start with a non unimodular Poisson manifolds.

From another point of view the orientability and dimension axioms of spectral triples together implies that $HH_n(\mathcal{A}_\hbar, \mathcal{A}_\hbar) \neq 0$; a non trivial cohomology class in this space can be seen as a quantized volume form. But then formality for chains[11] imply that $H_n^\pi(M) \neq 0$ with a volume form having non zero class. This is exactly, as we said, Poisson unimodularity in disguise.

Thus, whenever we are quantizing a non unimodular Poisson manifold, we should expect that the deformed algebra has smaller cohomological dimension than its semiclassical limit. This is exactly what happens, for example, with compact quantum groups, where a phenomenon known as *dimension drop* in Hochschild homology (see [**17**]) makes its appearance.

To recover the correct cohomological dimension in those cases the use of *twisted traces* was proposed. From the point of view of our notes this is no surprise since twisted Hochschild homology can be seen as a quantization of Poisson homology with coefficients in a line bundle, precisely the canonical Poisson line bundle. From this point of view the introduction of so-called *type III spectral triples* (see [**9**]) provides the most natural setting for standard quantum groups (and in general for quantization of non unimodular Poisson manifolds).

Unluckily, since standard quantum groups are not known to be ismorphic to the Kontsevich deformation quantization of underlying Poisson–Lie groups the previous argument is not a theorem. What one misses here is the isomorphism between Poisson homology and Hochschild homology of the deformed algebra. We will further comment on this point in the last section.

4. Group manifolds

In this section we will consider some special cases of Poisson manifolds carrying a large set of symmetries. This will result in a wide class of examples of unimodular and non unimodular Poisson manifolds on which the modular flow is pretty well understood.

4.1. Poisson–Lie groups.

The aim of this last lecture is to describe in more detail what happens when the Poisson manifold is, in fact, a Lie group and all the structures considered up to now are compatible with the multiplication. Let, from now on, then G be a Lie group and $\mathfrak{g}$ its Lie algebra. Let us denote by l_g (resp r_g) the left translation diffeomorphism $h \mapsto gh$) (resp. the right translation diffeomorphism $h \mapsto hg$). We will denote with $l_{g,*}$ and $r_{g,*}$ the corresponding tangent maps between vector spaces, and also their square wedge powers.

[11] The already invoked formality for Hochschild chains is a statement about the existence of a specific quasi isomorphism certain of L_∞-modules. Rather than trying to explain it, even vaguely, which would bring us quite far from this discussion we recall its main corollary: Hochschild (co)homology of Kontsevich's deformation quantization $\mathcal{A}_\hbar$ is given by formal power series in the Poisson (co)homology of the Kontsevich's representative π (a formal Poisson bivector).

DEFINITION 4.1. Let G be a Lie group. A Poisson structure π on G is said to be multiplicative (and (G,π) is then called a Poisson–Lie group) if

$$\pi(gh) = l_{g,*}\pi(h) + r_{h,*}\pi(g) \qquad \forall h, g \in G\,.$$

As a Poisson manifold a non trivial Poisson–Lie group is always non regular; as an exercise the reader can easily show that $\pi(e) = 0$. Let us now consider the following maps:

$$\eta : g \mapsto l_{g^{-1},*}\pi(g); \qquad \eta : G \to \wedge^2\mathfrak{g}$$
$$\delta : X \mapsto \frac{d}{dt}\eta(exp(tX))\big|_{t=0}; \qquad \delta : \mathfrak{g} \to \wedge^2\mathfrak{g}$$

The two properties of being multiplicative and of being Poisson for π can be rewritten as corresponding properties of the linear map δ. They respectively turn out to be the fact that δ is a 1-cocycle in Lie algebra cohomology with values in $\wedge^2\mathfrak{g}$, and that its transpose map ${}^t\delta : \wedge^2\mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket on $\mathfrak{g}^*$.

A Lie algebra endowed with a map δ with such properties is called a *Lie bialgebra*.

The Lie algebra $\mathfrak{g}^*$ integrates to a unique connected simply connected Lie group G^*. This group turns out to be Poisson–Lie as well (this notion is self-dual) and will be called the *Poisson dual* of G.

Let us consider the case in which G is compact. Say that $G_{\mathbb{C}}$ is the connected, simply connected, semisimple Lie group that integrates the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$, so that G is its (unique up to inner automorphisms of $G_{\mathbb{C}}$) compact real form. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, such that $\mathfrak{h} \cap \mathfrak{g} = \mathfrak{t}$ is the Lie algebra of the maximal torus of G, and choose a basis $\alpha_1, ..., \alpha_n \in \mathfrak{h}^*$ of simple roots. Let $X_i^{\pm}$ and H_i $(i = 1, \ldots, n)$ be the Chevalley generators corresponding to the simple roots. Denote with X_α and H_α the Chevalley generators corresponding to generic roots.

The *standard* Poisson–Lie group structure on the compact group G is determined by the (coboundary) Lie bialgebra $\delta(X) = \mathrm{ad}_X\, r$, where

$$r = \sum_{\alpha>0} X_{-\alpha} \wedge X_\alpha \in \wedge^2\mathfrak{g}\,.$$

This is the Poisson structure underlying compact quantum groups; its symplectic foliation resembles more a stratification (it is highly singular); leaves are connected to the Bruhat cells.

4.2. Poisson homogeneous spaces and θ-manifolds.

Let (G,π) be a Poisson–Lie group and let M be a G-homogeneous space endowed with a Poisson bivector π_M. Denote with $\phi : G \times M \to M$ the action and consider the orbit maps:

$$\phi_g : x \mapsto \phi(g,x); \qquad \phi_x : g \mapsto \phi(g,x),$$

where $g \in G$ and $x \in M$. As before with $\phi_{g,*}$ and $\phi_{x,*}$ we will denote the tangent maps and their wedge powers. The Poisson manifold (M, π_M) is called a Poisson homogeneous space if

$$\pi_M(\phi(g,x)) = \phi_{g,*}\pi_M(x) + \phi_{x,*}\pi_G(g) \qquad \forall g \in G, x \in M .$$

If $\pi_G = 0$ then π_M is called a G-invariant Poisson structure.

Examples of Poisson homogeneous spaces are given by quotients by Poisson–Lie subgroups (i.e. subgroups such that the Poisson bivector is tangent to them). However, such case is far from exhaustive.

Let π_G^r now be simply a right invariant Poisson structure on G (i.e. $l_{g,*}\pi_G^r = \pi_G^r$, $\forall g \in G$). An easy way to produce such π_G^r is to start with an element $r \in \wedge^2\mathfrak{g}$ satisfying the classical Yang–Baxter equation $[r, r] = 0$, and to consider the corresponding right invariant tensor $\pi_G^r(g) = r_{g,*}(r)$. Let P be a G-manifold, with a G-invariant Poisson structure π_P;[12] and consider $G \times P$ as a product Poisson manifold with Poisson bivector $\pi_G \oplus \pi_P$. The natural diagonal G action $g \cdot (h, x) = (hg^{-1}, gx)$ preserves this Poisson structure and therefore it induces a Poisson structure on the quotient $(G \times P)/G \simeq P$, which is just the sum of π_P with the image of r via the wedge square of the infinitesimal G-action.

In the case in which $G \simeq \mathbb{T}^n$ and π_G^r has maximal rank, one has that π_G^r is determined by the choice of an antisymmetric matrix θ. The corresponding projected Poisson structure π_θ on P, where $\pi_P = 0$, is called a *Poisson θ-manifold*.

Let us now consider the specific example in which $P = G$, and the left action is simply given by left multiplication. Fix any element $r \in \wedge^2\mathfrak{g}$ satisfying CYBE, let $\pi_G^r(g) = r_{g,*}r$ and take as left invariant Poisson structure just $\pi_P(g) = l_{g,*}r$. Then the projected Poisson structure is the sum of this two bivectors and turns out to be a Poisson–Lie structure. This also works if $r \in \wedge^2\mathfrak{h}$, where $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. In the special case in which G is compact and $\mathfrak{h}$ is the Lie algebra of a maximal torus, the CYBE equation turns out to be trivially satisfied. With this procedure any antisymmetric matrix $\theta \in \wedge^2\mathfrak{h}$ provides a Poisson–Lie structure π_θ on G which is always different from the standard structure $\pi_{\rm std}$. It can be proved that any Poisson–Lie structure on a compact G is a linear combination $\pi_{\rm std} + \lambda\pi_\theta$, $\lambda \in \mathbb{R}$ (the proof relies on the celebrated Belavin Drinfel'd theorem: see [**21**]).

4.3. Poisson modular class computations.

Let G be a Poisson–Lie group. Let Ω be a right invariant volume form on the group. Let x_0 be the modular character of $\mathfrak{g}^*$, i.e. $x_0(\xi) = tr\, \mathrm{ad}_\xi^*$ and let ξ_0 be the modular character of the Lie algebra $\mathfrak{g}$. Let us denote with x_0^L (resp. x_0^R) the left (resp. right) invariant vector field on G such that its value at the identity is x_0. Similarly for ξ_0^L and ξ_0^R (which are read as right invariant

[12] It is always possible to chose $\pi_P = 0$.

1-forms on G). Then Evens–Lu–Weinstein ([**14**]) proved, with a rather direct computation, that the modular class with respect to a right invariant volume form is:

$$\phi_\Omega = \frac{1}{2}\left(x_0^L + x_0^R - \sharp_\pi(\xi_0^r)\right) . \tag{4.1}$$

A similar formula can be proven for Poisson homogeneous spaces, in full generality [**27**]; however, since explaining it here would require some (not so relevant) additional complications we will stick to the simpler case in which the homogeneous space is the quotient of a Poisson–Lie subgroup. In that case, once a G-invariant volume form on the quotient is chosen, the correspoinding modular vector field on G/H is simply the projection of the one on G.

I would like to use this computation to describe to large classes of Poisson manifolds "with symmetries," one of them being unimodular and the other one being not. The unimodular ones will be Poisson θ-manifolds, the non unimodular will be standard Poisson–Lie groups and their homogeneous spaces. In the last section I will comment about their quantizations.

Let us now consider the case of Poisson θ-manifolds. Let $\rho : \mathfrak{t}^n \to M$ be an infinitesimal action of a torus on a manifold, and let $\theta \in \wedge^2\mathfrak{t}$, so that $\pi_\theta(x) = \rho^{\wedge 2}(\theta)(x)$. Now $\rho^{\wedge 2}(\theta) = \sum_{i<j} \theta_{i,j}\xi_i \wedge \xi_j$, with $\theta_{i,j} \in \mathbb{R}$. Let Ω be a left invariant volume form on M. If we prove that $\partial_{\pi_\theta}\Omega = \sum_{i<j} \theta_{ij} d i_{\xi_i\wedge\xi_j}\Omega$ is zero then we've proven that θ-manifolds are unimodular. Now take into account that $\mathfrak{t}^n$ is commutative. Therefore $[\xi_i, \xi_j] = 0$, $\forall i, j$. Hence:

$$0 = i_{[\xi_i,\xi_j]}\Omega = (L_{\xi_i} i_{\xi_j} - i_{\xi_j} L_{\xi_i})\Omega = -d i_{\xi_i\wedge\xi_j}\Omega.$$

In the case of Poisson–Lie θ-groups one can perform a completely analogous computation.[13]

Thus also Poisson–Lie θ-groups are unimodular.

Let us move to the standard compact case.

PROPOSITION 4.2. *Let (G, π_{std}) be a standard compact Poisson–Lie group and Ω be an invariant volume form. Let us denote with H_ρ the Cartan element corresponding to the semisum ρ of all positive roots. This element can be identified to an element in $\mathfrak{t}^*$ via the Killing form, and therefore to an invariant vector field on G. Then*

$$\phi_\Omega = 2iH_\rho$$

and (G, π_{std}) is non unimodular.

[13]The same proof holds true whenever M carries an invariant G-form, regardless of abelianity and compactness of the group G in our special case. We stuck to the special case of torus actions as it was the one raising some interest, in last years, in the context of quantization issues (e.g. [**8, 30**]). On the other hand $\mathbb{R}^n$-actions may well provide nonunimodular Poisson θ-manifolds. A family of interesting such manifolds is currently under investigation in [**2**].

PROOF. The computation of the modular vector field is simply an application of formula (4.1), once the explicit form of the Poisson–Lie bivector is given. Remark that being in the G-compact case, the Lie algebra $\mathfrak{g}$ is unimodular and therefore its modular characters provides no contribution in formula (4.1), i.e.:

$$\phi_\Omega = -\frac{1}{2}\sharp_\pi(\xi_0^l)\,.$$

Knowing that $\phi_\Omega \neq 0$ for a single volume form is, of course, not enough to check unimodularity. We need to know here whether its cohomology class is zero. The point here is that even without computing the full Poisson cohomology of G (which is not known[14] we can conclude that $[\phi_\Omega] \neq 0$. In fact ϕ_Ω is tangent to the maximal torus T having as Lie algebra $\mathfrak{t}$ the imaginary part of Cartan root vectors. The restriction of its flow on this torus is not trivial. On the other hand $\pi_{\mathrm{std}} = 0$ when restricted to this torus. Hence the flow of ϕ_Ω does not take place on symplectic leaves but rather moves one 0-leaf to another. This is possible (basically by definition) exactly if ϕ_Ω is a Poisson derivation which is not Hamiltonian, therefore $[\phi_\Omega] \neq 0$ in $H^1_\pi(G)$. □

4.4. Quantization.

Let us consider first Poisson θ-manifolds and groups. Much is known about their quantizations, approached from many possible points of view. Just think of the easiest example: invariant symplectic structures on the 2-torus, which quantize to the quantum torus. Deformation quantization (also in the strict sense), non commutative algebra by generators and relations, C^*-algebraic closure, spectral triples, even the connection with geometric quantization via symplectic groupoids were developed. No non trivial modular flow appears in such quantizations. Non commutative traces and spectral triples can be explicitly built up (see [**8**]) and the Hochschild dimension coincide with the classical one. This is in perfect agreement with Dolgushev's results, though we do not know, as usual, whether such quantizations are equivalent to Kontsevich's one. It is tempting to say that an analogue of Theorems 3.3 and 3.5 should hold true in the context of strict deformation quantization; they certainly hold for this class of algebras.

Let us move, now, to standard quantum groups. In this case quantization has been studied mainly from the algebraic point of view. It is well known since the work of Woronowicz on Haar's measures that, as we may expect from Poisson geometry, non trivial modular automorphisms appear. To prove the relation between the classical and semiclassical pictures, what we are tempted to call *exact quantization properties*, one should try to use the rich symmetries of this case.

[14]But upcoming new results on it have been recently announced in [**27**].

Using spectral sequences Feng–Tsygan ([**15**]) showed, in fact, that for the the standard compact Poisson–Lie group G the Hochschild homology of, respectively, the quantized algebra of formal functions $\mathbb{C}^h(G)$ and of quantized regular functions $\mathbb{C}_h[G]$ are given by:

$$\begin{aligned} HH_n(\mathbb{C}^h(G)) &= \Omega^n_f(H) \otimes \mathbb{C}((\hbar)), \\ HH_n(\mathbb{C}_h[G]) &= \Omega^n(N(H)) \otimes \mathbb{C}((\hbar)), \end{aligned}$$

where $\Omega^n_f(H)$ represents the space of formal differential n-forms of degree n on the Cartan subgroup H and $\Omega^n(N(H))$ the space of regular differential n-forms on the normalizer of H. It would be interesting to understand whether an extension of these results to cohomology (eventually with coefficients) could allow to prove directly an analogue of Dolgushev's results for standard quantizations. This, in principle, should allow to recover by different means (and maybe to obtain explicit formulas for it) the Nakayama automorphism studied in [**4**] from a purely algebraic point of view.

Feng–Tsygan results were extended to a specific subclass of compact Poisson homogeneous spaces by [**10**]. Since quite explicit results on its quantization were recently given in [**23**], one could consider this as another interesting test case.

References

1. Ammar M., Poncin N., *Formal Poisson cohomology of twisted r-matrix Poisson structures,* Isr. J. Math., **165** (2008), 381–411.
2. Bonechi F., Ciccoli N., Staffolani N., Tarlini M., *Poisson Theta-manifolds*, in preparation.
3. Bott R., Tu J. L., *Differential forms in algebraic topology*, Springer, Graduate Text in Math., **82**, Springer Verlag, New York, 1982.
4. Brown K. A., Zhang J. J., *Dualizing complexes and twisted Hochschild (co)homology for Noetherian Hopf algebras*, J. Algebra, **320** (2008), 1814–1850.
5. Brylinski J. L., *A differential complex for Poisson manifolds*, J. Diff. Geom., **28** (1988), 93–114.
6. Ciccoli N., *From Poisson to quantum geometry*, Lecture Notes, available at `http://toknotes.mimuw.edu.pl/`.
7. Ciccoli N., *Poisson Ore extensions*, Seminari di Geometria dell'Universitá di Bologna 2007/2008, S. Coen ed., at press.
8. Connes A., Dubois–Violette M., *Non commutative finite dimensional manifolds I, Spherical manifolds and related examples*, Comm. Math. Phys., **230** (2002), 539–579.
9. Connes A., Moscovici H., *Type III and spectral triples*, S. Albeverio (ed.) et al., *Traces in number theory, geometry and quantum fields*, Vieweg, Wiesbaden, Aspects of Mathematics, **E38** (2008), 57–71.
10. Dahl E., Enriquez B., *Homologie cyclique et de Hochschild de certaines èspaces homogènes quantiques*, K-theory, **6** (1992), 499–517.
11. Dolgushev V., *The van den Bergh duality and the modular symmetry of a Poisson variety*, Selecta Math., **14** (2009), 199–228.
12. Dolgushev V., *A formality theorem for Hochschild chains*, Adv. Math., **200** (2006), 51–101.

13. Dufour J. P., Zung N. T., *Poisson structures and their normal forms*, Birhäuser Progress in Math., **242**, Basel 2005.
14. Evens S., Lu J.–H., Weinstein A., *Transverse measures, the modular class and cohomology pairing for Lie algebroids*, Quart. J. Math. Oxford, **50** (1999), 417–436.
15. Feng P., Tsygan B., *Hochschild and cyclic homology of quantum groups*, Comm. Math. Phys., **140** (1990), 481–521.
16. Ginzburg V., *Calabi–Yau algebras*, available at `arXive math.QA/0612139`.
17. Hadfield T., Krähmer U., *On the Hochschild homology of quantum $SL(N)$*, C.R.Acad. Sci. Paris, **343** (2006), 9–13.
18. Hawkins E., *Noncommutative rigidity*, Commun. Math. Phys., **246** (2004), 211–235.
19. Huebschmann J., *Duality for Lie–Rinehart algebras and the modular class*, J. Reine Angew. Math., **510** (1999), 103–159.
20. Kontsevich M., *Deformation quantization of Poisson manifolds*, Lett. Math. Phys., **66** (2003), 157–216.
21. Korogodskii V., Soibelman J., *The algebra of functions on quantum groups I*, Math. Surv. and Monographs 56, American Mathematical Society 1998.
22. Krähmer U., *Poincaré duality in Hochschild (co)homology*, in: S. Caenepeel et al. (eds.), Proceedings *New techniques in Hopf algebras and graded ring theory*, Brussels, 2007.
23. Krähmer U., *The Hochschild cohomology ring of the standard Podleś quantum sphere*, Arab J. Math., at press.
24. Launois S., Richard L., *Twisted Poincaré duality for some quadratic Poisson algebras*, Lett. Math. Phys., **79** (2007), 161–174.
25. Lichnerowicz A., *Les variétés de Poisson et leurs algebrés de Lie associées*, J. Diff. Geom., **12** (1977), 253–300.
26. Loday J. L., *Cyclic homology*, Grundlehren der Mathematische Wissenschaften, **301**, Springer Verlag, Berlin 1998.
27. Lu J.–H., *A note on Poisson homogeneous spaces*, in: G. Dito (ed.) et al., *Poisson geometry in mathematics and physics. Proceedings of the international conference*, Tokyo, Japan, June 5–9, 2006. Providence, RI: American Mathematical Society (AMS), Contemporary Mathematics, **450** (2008), 173–198.
28. Picherau A., *Poisson (co)homology and isolated singularities*, J. Algebra, **99** (2006), 747–777.
29. Roger C., Vanhaecke P., *Poisson cohomology of the affine plane*, J. Algebra, **251** (2002), 448–460.
30. Rieffel M., *Deformation quantization for actions of $\mathbb{R}^d$*, Mem. Amer. Math. Soc., **106**, 1993.
31. Vaisman I., *The geometry of Poisson manifolds*, Birhäuser Progress in Math., 118, Basel 1994.

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