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A RESULT ABOUT BLOWING UP HENSELIAN EXCELLENT REGULAR RINGS

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Abstract. We prove a theorem on analytic factorization when blowing up along a regularly embedded center a henselian excellent regular ring containing the rationals. In our next paper, it will be applied to the proof of the rank theorem for differentially algebraic relations or, more generally, for convergent Weierstrass systems of Hartogs type.

Let $(A, \mathfrak{m})$ be a regular local ring of dimension m, $u_1, \dots, u_m$ a system of parameters of A and $J = (u_1, \dots, u_k)$ with $1 \le k \le m$. It is well known that for the Rees algebra $\mathfrak{R}(J)$ we have the following isomorphism

$$A[T_1, \dots, T_k]/I \longrightarrow \mathfrak{R}(J) := \bigoplus_{n=0}^{\infty} J^n, \quad T_i \mapsto u_i,$$

where $T_1, \dots, T_k$ are indeterminates and I is the ideal generated by the elements $u_i T_j - u_j T_i$. This is true, in fact, for any commutative ring A provided that the sequence $u_1, \dots, u_k$ is regular (see e.g. [**4**]). This result says in geometric language that the blowing-up of A along the regularly embedded center determined by the ideal J, which is the projective spectrum of $\mathfrak{R}(J)$, has projective spaces as fibres, and implies that the normal cone of the ideal J, whose projectivization is the exceptional divisor of the blowing-up, is a vector bundle. In the local charts $T_i \neq 0$ the blowing-up is thus the spectrum of the ring

$$A[v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k]/I, \quad i = 1, \dots, k,$$

where $v_j = T_j/T_i$ and I is the ideal generated by the elements $u_j - u_i v_j$. Therefore, any blowing-up of the ring A along a smooth center can be determined in a local chart near $v = 0$ by the ring monomorphism:

$$A \longrightarrow A[v]/I, \quad v = (v_1, \dots, v_{k-1}),$$

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where I is the ideal generated by the elements $u_i - u_k v_i$, $i = 1, \dots, k-1$. Then the exceptional divisor of the blowing-up is given by the equation $u_k = 0$.

In the sequel, I shall denote, by abuse of notation, the ideals generated by the above elements in the rings under study. Passing to the completion $\widehat{A}$ of the ring A in the Krull topology, we get the blowing-up given in a local chart near $v = 0$ by the inclusion

$$\widehat{A} \longrightarrow \widehat{A}[v]/I.$$

It induces the local ring monomorphism

$$\widehat{A} \longrightarrow \widehat{A}[[v]]/I.$$

Denote by H and $\widetilde{H}$ the henselizations of the local rings

$$B := A[v]_{(u,v)}/I \quad \text{and} \quad \widetilde{B} := \widehat{A}[v]_{(u,v)}/I,$$

respectively. We may, of course, regard H and $\widetilde{H}$ as subrings of the integral closures of B and $\widetilde{B}$ in $\widehat{A}[[v]]/I$, respectively.

We now state an important observation about henselization. Let $(R, \mathfrak{m})$ be a local ring, $\widehat{R}$ its completion in the Krull topology and $\widehat{R}^{alg}$ the ring of all those elements from $\widehat{R}$ that are algebraic over R. It seems to be known that the henselization R^h of R coincides with that subring of $\widehat{R}$, $R^h = \widehat{R}^{alg}$, whenever the local ring R is excellent and normal. Below we present a proof of this fact, because we are not able to indicate an appropriate reference to the literature.

The henselization R^h is the inductive limit of the system of equiresidual étale local R-algebras S. The analytic criterion for a local equiresidual R-algebra S to be étale (cf. [**2**], Chap. III, § 5, Theorem 3) is that their completions in the Krull topology coincide: $\widehat{S} = \widehat{R}$. Therefore, we always have $R^h \subset \widehat{R}$. Hence and by a standard étale presentation of equiresidual étale local R-algebras (cf. [**7**], Chap. V, Theorem 1, and also [**5**]), we get the inclusion $R^h \subset \widehat{R}^{alg}$.

Suppose now that R is normal and excellent. Take an element $a \in \widehat{R}^{alg}$, put $S := R[a]$ and let S' be the integral closure of S. Since every excellent ring is a Nagata ring (cf. [**1**], Chap. 13, Theorem 78), S' is finite over S. If an excellent local ring is normal, so is its completion in the Krull topology. Consequently, the completion $\widehat{R}$ is normal, and thus we have embeddings

$$R \subset S \subset S' \subset \widehat{R}.$$

The contraction of the maximal ideal of $\widehat{R}$ determines unique prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of S and S', respectively. We thus get embeddings

$$R \subset S_{\mathfrak{p}} \subset S'_{\mathfrak{q}} \subset \widehat{R},$$

and the localizations $S_{\mathfrak{p}}$, $S'_{\mathfrak{q}}$ are, of course, local equiresidual R-algebras. Moreover, the local rings R and $S'_{\mathfrak{q}}$ are of the same dimension, because the dimension formula holds between R and S', as between universally catenary rings. The embedding $\varphi : S'_{\mathfrak{q}} \longrightarrow \widehat{R}$ induces a surjective homomorphism $\widehat{\varphi} : \widehat{S'_{\mathfrak{q}}} \longrightarrow \widehat{R}$. Again, the completion $\widehat{S'_{\mathfrak{q}}}$ is normal, and *a fortiori* a domain. Therefore the homomorphism $\widehat{\varphi}$ is injective, and thus an isomorphism. Hence and, as before, by the analytic criterion for a local equiresidual algebra to be étale, $S'_{\mathfrak{q}}$ is an equiresidual étale local R-algebra, and consequently, $a \in R^h$. This proves the converse inclusion $\widehat{R}^{alg} \subset R^h$.

The main purpose of this paper is to establish the following theorem about analytic factorization in a henselian excellent regular ring blown up along a regularly embedded center.

THEOREM. *Keep the foregoing notation and assume that A is a henselian regular excellent ring containing the rationals. Let $f \in \widehat{A}$ be an irreducible element and $g \in \widetilde{B}$ be the transform of f under the blowing-up under study, i.e. $f = g \cdot u_k^p$, where $p \in \mathbb{N}$ is the largest power of u_k that factors from f. If an irreducible factor h of g in the completion $\widehat{A}[[v]]/I$ of $\widetilde{B}$ lies in the henselization H, then g lies in B up to an invertible factor from $\widetilde{B}$.*

REMARKS. 1) Since A and thus all the rings under study are excellent, it is well-known that the irreducible factors of g in the completion $\widehat{A}[[v]]/I$ of $\widetilde{B}$ can always be taken from the henselization $\widetilde{B}$.

2) As established by Rotthaus [**8, 9**], a local ring A containing the rationals has the Artin approximation property iff it is henselian and excellent.

3) Consider a polynomial $P(T) \in R[T_1, \dots, T_n]$ with coefficients from a local ring $(R, \mathfrak{m})$. The coefficients of a factor of the polynomial $P(T)$ over R or $\widehat{R}$ satisfy a certain system of polynomial equations with coefficients from R; in fact, the coefficients of $P(T)$ are involved in this system. Therefore, if the ring R has the Artin approximation property, the polynomial $P(T)$ is irreducible over R iff it is so over $\widehat{R}$.

For a proof of the theorem, observe first that g is irreducible in the ring $\widetilde{B}$, and the element h is algebraic over the ring B. Let $P(T) \in B[T]$ be its minimal polynomial. Actually, according to the above remarks, $P(T)$ coincides with the minimal polynomial of h over the ring $\widetilde{B}$.

Let $b \in B$ be the constant term of the polynomial $P(T)$. It is clear that $b \in h \cdot H \subset h \cdot \widetilde{H}$. Let

$$g = h_1 \cdot \ldots \cdot h_s \quad \text{with} \quad h_1 = h,\ h_2, \dots, h_s \in \widetilde{H}$$

be a decomposition of g into irreducible factors over $\widetilde{H}$. Clearly, g is irreducible in $\widetilde{B}$, and thus the ideal $\mathfrak{p} := g \cdot \widetilde{B}$ is prime. Then

$$\mathfrak{p}\widetilde{H} = \bigcap_{i=1}^{s} \mathfrak{q}_i \quad \text{with} \quad \mathfrak{q}_i := h_i \cdot \widetilde{H}.$$

Since the henselization $\widetilde{H}$ is flat over $\widetilde{B}$ (cf. [**7**], Chap. VIII, Theorem 3 or [**3**], Chap. VII, Theorem (43.8)), the going-down theorem holds for the embedding $\widetilde{B} \subset \widetilde{H}$, and thus we get (cf. [**1**], Chap. 2, Section (5.B)):

$$\mathfrak{q}_i \cap \widetilde{B} = \mathfrak{p} \quad \text{for all} \quad i = 1, \ldots, s;$$

in particular, $h \cdot \widetilde{H} \cap \widetilde{B} = g \cdot \widetilde{B}$ whence $b \in g \cdot \widetilde{B}$. Again, since A has the Artin approximation property, it follows from Remark 3 that any two decompositions of an element from B into irreducible factors over B and $\widetilde{B}$, respectively, are associate. Consequently, $g \cdot \beta \in B$ for an invertible element β from $\widetilde{B}$, as asserted.

The above result will be applied in our paper [**6**] to the proof of the rank theorem for differentially algebraic relations or, more generally, for convergent Weierstrass systems of Hartogs type.

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