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THE ŁOJASIEWICZ EXPONENT OF NONDEGENERATE SINGULARITIES

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Abstract. In the article we give some estimations of the Łojasiewicz exponent of nondegenerate singularities in terms of their Newton diagrams. The results are stronger than Fukui inequality [**3**] in the case when Newton diagram contains exceptional faces. It is also a multidimensional generalization of the Lenarcik theorem [**5**].

1. Introduction. Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be a holomorphic function in an open neighborhood of $0 \in \mathbb{C}^n$ and $\sum_{\nu \in \mathbb{N}^n} a_\nu z^\nu$ be the Taylor expansion of f at 0. We define $\Gamma_+(f) := \operatorname{conv}\{\nu + \mathbb{R}^n_+ : a_\nu \neq 0\} \subset \mathbb{R}^n$ and call it *the Newton diagram* of f. Let $u \in \mathbb{R}^n_+ \setminus \{0\}$. Put $l(u, \Gamma_+(f)) := \inf\{< u, v >: v \in \Gamma_+(f)\}$ and $\Delta(u, \Gamma_+(f)) := \{v \in \Gamma_+(f) :< u, v >= l(u, \Gamma_+(f))\}$. We say that $S \subset \mathbb{R}^n$ is a *face* of $\Gamma_+(f)$, if $S = \Delta(u, \Gamma_+(f))$ for some $u \in \mathbb{R}^n_+ \setminus \{0\}$. The vector u is called *the primitive vector* of S. It is easy to see that S is a closed and convex set and $S \subset \operatorname{Fr}(\Gamma_+(f))$, where $\operatorname{Fr}(A)$ denotes the boundary of A. One can prove that a face $S \subset \Gamma_+(f)$ is compact if and only if all coordinates of its primitive vector u are positive. We call the family of all compact faces of $\Gamma_+(f)$ the *Newton boundary* of f and denote by $\Gamma(f)$. We denote by $\Gamma^k(f)$ the set of all compact k-dimensional faces of $\Gamma(f)$, $k = 0, \dots, n-1$. For every compact face $S \in \Gamma(f)$ we define quasihomogeneous polynomial $f_S := \sum_{\nu \in S} a_\nu z^\nu$. We say that f is *nondegenerate on the face* $S \in \Gamma(f)$, if the system of equations $\frac{\partial f_S}{\partial z_1} = \ldots = \frac{\partial f_S}{\partial z_n} = 0$ has no solution in $(\mathbb{C}^*)^n$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We say that f is *nondegenerate in the Kouchnirenko's sense* if it is nondegenerate on each face $\Gamma(f)$. We say that f is a *singularity* if f is a nonzero holomorphic

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function in some open neighborhood of the origin and $f(0) = 0$, $\nabla f(0) = 0$, where $\nabla f = (f'_{z_1}, \dots, f'_{z_n})$. We say that f is *an isolated singularity* if f is a singularity, which has an isolated critical point in the origin, i.e. additionally $\nabla f(z) \neq 0$ for $z \neq 0$.

Let $i \in \{1, \dots, n\}$, $n \geq 2$.

DEFINITION 1.1. We say that $S \in \Gamma^1(f) \subset \mathbb{R}^n$ is *an exceptional segment with respect to the axis Ox_i* if S is a segment lying in the plane Ox_ix_j for some $j \in \{1, \dots, n\}$, $j \neq i$, whose one end lies on Ox_i axis and second one is at distance 1 to Ox_i axis (see [**5**]).

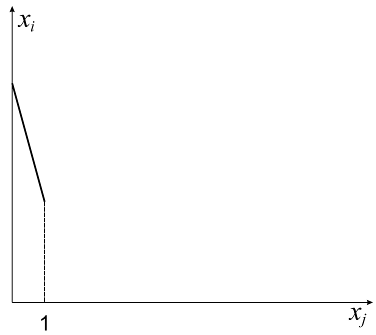

FIGURE 1. An exceptional segment with respect to the axis Ox_i

DEFINITION 1.2. We say that $S \in \Gamma^{n-1}(f) \subset \mathbb{R}^n$ is *an exceptional face with respect to the axis Ox_i* if its intersection with every plane Ox_ix_j, $j \in \{1, \dots, n\}$, $j \neq i$, is an exceptional segment with respect to axis Ox_i.

It is easy to see that there is at most one exceptional face for each coordinate axis. Denote by E_f the set of exceptional faces in $\Gamma^{n-1}(f)$.

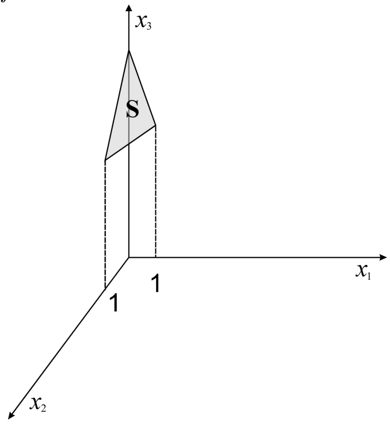

FIGURE 2. An exceptional face S with respect to axis Ox_3

Definition 1.3. We say that the Newton diagram of f is *convenient*, if it has nonempty intersection with every coordinate axis.

Definition 1.4. We say that the Newton diagram of f is *nearly convenient*, if its distance to every coordinate axis doesn't exceed 1.

For every $(n-1)$-dimensional compact face $S \in \Gamma(f)$ we shall denote by $x_1(S), \dots, x_n(S)$ coordinates of intersection of the hyperplane determined by face S with the coordinate axes. We define $m(S) := \max\{x_1(S), \dots, x_n(S)\}$. It is easy to see that

$$x_i(S) = l(u, \Gamma_+(f))/u_i,\ i = 1, \dots, n,$$

where u is a primitive vector of S. It is easy to check that the Newton diagram $\Gamma_+(f)$ of an isolated singularity f is nearly convenient. So, "nearly convenience" of the Newton diagram is a neccesary condition for f to be an isolated singularity. For a singularity f such that $\Gamma^{n-1}(f) \neq \emptyset$, we define

$$m_0(f) := \max_{S \in \Gamma^{n-1}(f)} m(S). \tag{1}$$

It is easy to see that in the case $\Gamma_+(f)$ is convenient $m_0(f)$ is equal to the maximum of the length from the origin to the points of the intersection of the Newton diagram and the union of all axes.

Remark 1.5. A definition of $m_0(f)$ for all singularities (even for $\Gamma^{n-1}(f) = \emptyset$) can be found in [**3**]. In the case $\Gamma^{n-1}(f) \neq \emptyset$ both definitions are equivalent.

Let $f = (f_1, \dots, f_n) : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$ be a holomorphic mapping having an isolated zero at the origin. We define a number

$$l_0(f) := \inf\{\alpha \in \mathbb{R}_+ : \exists_{C>0} \exists_{r>0} \forall_{\|z\|<r} \|f(z)\| \geq C\|z\|^\alpha\} \tag{2}$$

and call it *the Lojasiewicz exponent* of the mapping f. There are formulas and estimations of the number $l_0(f)$ under some nondegeneracy conditions of f (see [**1**]).

Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be an isolated singularity. We define a number $\text{Ł}_0(f) := l_0(\nabla f)$ and call it *the Lojasiewicz exponent of singularity* f. Now we give some important known properities of the Łojasiewicz exponent (see [**6**]):

(a) $\text{Ł}_0(f)$ is a rational number.
(b) $\text{Ł}_0(f) = \sup\{\frac{\operatorname{ord} \nabla f(z(t))}{\operatorname{ord} z(t)} : 0 \neq z(t) \in \mathbb{C}\{t\}^n,\ z(0) = 0\}$.
(c) The infimum in the definition of the Łojasiewicz exponent is attained for $\alpha = \text{Ł}_0(f)$.
(d) $s(f) = [\text{Ł}_0(f)] + 1$, where $s(f)$ is *the degree of* C^0*-sufficiency of* f [**2**].

Lenarcik gave in [**5**] the formula for the Łojasiewicz exponent for singularities of two variables, nondegenerate in Kouchnirenko's sense, in terms of its Newton diagram.

Theorem 1.6 ([**5**]). *Let* $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ *be an isolated singularity and nondegenerate in Kouchnirenko's sense and* $\Gamma^1(f) \setminus E_f \neq \emptyset$*. Then*

$$\pounds_0(f) = \max_{S \in \Gamma^1(f) \setminus E_f} m(S) - 1. \tag{3}$$

Remark 1.7. In two-dimensional case one can prove that for isolated singularities such that $\Gamma^1(f) \setminus E_f = \emptyset$, i.e. $\Gamma^1(f)$ consist of only exceptional segments, we have $\pounds_0(f) = 1$.

In multidimensional case we have only an upper estimation, which was given by T. Fukui in 1991.

Theorem 1.8 ([**3**]). *Let* $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ *be an isolated singularity and nondegenerate in Kouchnirenko's sense. Then*

$$\pounds_0(f) \leq m_0(f) - 1. \tag{4}$$

We give now the main result, which is the improvement of the above theorem in the case $E_f \neq \emptyset$.

Theorem 1.9. *Let* $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 2$, *be an isolated singularity and nondegenerate in Kouchnirenko's sense and* $\Gamma^{n-1}(f) \setminus E_f \neq \emptyset$*. Then*

$$\pounds_0(f) \leq \max_{S \in \Gamma^{n-1}(f) \setminus E_f} m(S) - 1. \tag{5}$$

Remark 1.10. For $n=2$ if $\Gamma^{n-1}(f) \setminus E_f = \emptyset$ see Remark 1.7. One can prove that for singularities of n-variables, $n > 2$, if $E_f \neq \emptyset$, then $\Gamma^{n-1}(f) \setminus E_f \neq \emptyset$.

2. Proof of the main theorem. We give now some lemmas used in the proof of the main theorem.

Lemma 2.1. *Let* $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 3$, *be a holomorphic function in an open neighborhood of* $0 \in \mathbb{C}^n$ *and* $g(z_1, \ldots, z_k) := f(z_1, \ldots, z_k, 0, \ldots, 0) \neq 0$, $k \geq 2$. *Then*

$$\Gamma(g) = \{S \in \Gamma(f) : S \subset \{x_{k+1} = \ldots = x_n = 0\}\}. \tag{6}$$

Proof. "$\subset$". Let $S \in \Gamma(g)$, so $S = \Delta(u, \Gamma_+(g))$ for some $u \in (\mathbb{R}_+ \setminus \{0\})^k$. Of course $S \subset \Gamma_+(f) \cap \{x_{k+1} = \ldots = x_n = 0\}$. Set $u' = (u_1, \ldots, u_k, l(u, \Gamma_+(g)) + 1, \ldots, l(u, \Gamma_+(g)) + 1) \in \mathbb{R}^n$. We show that $S = \Delta(u', \Gamma_+(f))$. By definition of u' we have that $l(u', \Gamma_+(f))$ can be realised only for $v \in \Gamma_+(f) \cap \{x_{k+1} = \ldots = x_n = 0\}$. But it is easy to check that $\Gamma_+(f) \cap \{x_{k+1} = \ldots = x_n = 0\} = \Gamma_+(g)$. So we get $l(u', \Gamma_+(f)) = l(u, \Gamma_+(g))$ and $\Delta(u', \Gamma_+(f)) = \Delta(u, \Gamma_+(g))$. Reasumming $S = \Delta(u', \Gamma_+(f))$, it is in $\Gamma(f)$.
"$\subset$" Let $S \in \Gamma(f)$ i $S \subset \{x_{k+1} = \ldots = x_n = 0\}$. Then $S = \Delta(u, \Gamma_+(f))$ for some $u \in (\mathbb{R}_+ \setminus \{0\})^n$ and as we observed above $\Gamma_+(f) \cap \{x_{k+1} = \ldots = x_n = 0\} = \Gamma_+(g)$. So $l(u, \Gamma_+(f)) = l(u', \Gamma_+(g))$, where $u' = (u_1, \ldots, u_k)$. It follows that $\Delta(u', \Gamma_+(g)) = \Delta(u, \Gamma_+(f))$ and $S \in \Gamma(g)$. That concludes the proof. $\square$

Lemma 2.2. *Let* $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 2$, *be a singularity and* $\nabla f \circ \Phi = 0$ *for some* $\Phi = (\phi_1, \ldots, \phi_n) \in \mathbb{C}\{t\}^n$, $\Phi(0) = 0$, $\phi_i \neq 0$, $i = 1, \ldots, n$. *Then there is* $S \in \Gamma(f)$ *such that* $\nabla f_S \circ \operatorname{in} \Phi = 0$, *so* f *is degenerate on face* S.

Proof. Denote $d_i := \operatorname{ord} \phi_i > 0$, $w := (d_1, \ldots, d_n)$ and set $S := \Delta(w, \Gamma_+(f))$. Since all coordinates of w are positive, S is a compact face of $\Gamma_+(f)$. Expand f according to the weights system w we have $f = \operatorname{in}_w f + \ldots$. We have the following cases:
(a) $S \not\subset \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i = 0\}$, $i = 1, \ldots, n$. Then for all $i \in \{1, \ldots, n\}$ we can find a monomial in $\operatorname{in}_w f$ in which the variable z_i appears. Hence $(\operatorname{in}_w f)'_{z_i} = \operatorname{in}_w f'_{z_i}$, $i = 1, \ldots, n$. By assumption $f'_{z_i} \circ \Phi = 0$, $i = 1, \ldots, n$, so

$$(7) \qquad 0 = \operatorname{in}_w f'_{z_i} \circ \operatorname{in} \Phi = (\operatorname{in}_w f)'_{z_i} \circ \operatorname{in} \Phi = (f_S)'_{z_i} \circ \operatorname{in} \Phi, \quad i = 1, \ldots, n.$$

It follows that $\nabla f_S \circ \operatorname{in} \Phi = 0$, hence f is degenerate on face S.
(b) $S \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_{i_0} = 0\}$ for some $i_0 \in \{1, \ldots, n\}$. Set

$$(8) \qquad J := \{j \in \{1, \ldots, n\} : S \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_j = 0\}\}.$$

Then for $i \notin J$ we can find a monomial in $\operatorname{in}_w f$ in which the variable z_i appears. Hence for $i \notin J$ we get $(\operatorname{in}_w f)'_{z_i} = \operatorname{in}_w f'_{z_i}$. So we have

$$(9) \qquad 0 = \operatorname{in}_w f'_{z_i} \circ \operatorname{in} \Phi = (\operatorname{in}_w f)'_{z_i} \circ \operatorname{in} \Phi = (f_S)'_{z_i} \circ \operatorname{in} \Phi.$$

On the other hand $(f_S)'_{z_i} \circ \operatorname{in} \Phi = 0$, for $i \in J$.

Reasumming $\nabla f_S \circ \operatorname{in} \Phi = 0$, thus f is degenerate on S. □

Lemma 2.3 ([**8**, Lemma 1.4]). *Let* $f = (f_1, \ldots, f_n)$, $g = (g_1, \ldots, g_n)$: $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *be holomorphic maps in an open neighborhood of zero and let* f *has an isolated zero at the origin. If* $\operatorname{ord}(g - f) > l_0(f)$, *then* g *has an isolated zero and* $l_0(g) = l_0(f)$.

We can go to the proof of the main theorem.

Proof of Theorem 1.9. Note in the beginning that if $E_f = \emptyset$, then our theorem is an immediate consequence of Theorem 1.8. So it can be assumed that $E_f \neq \emptyset$. Without lost of generality we can suppose that $E_f = \{S_1, \ldots, S_k\}$, for some $k \in \{1, \ldots, n\}$ and S_i is exceptional face with respect to axis Ox_i, $i = 1, ..., k$. First note that $m(S_i) = x_i(S_i)$, $i = 1, \ldots, k$. Set

$$(10) \qquad J := \{j \in \{1, \ldots, k\} : \max_{S \in \Gamma^{n-1}(f) \setminus E_f} m(S) < x_j(S_j)\}.$$

If $J = \emptyset$ then

$$(11) \qquad \max_{S \in \Gamma^{n-1}(f) \setminus E_f} m(S) = \max_{S \in \Gamma^{n-1}(f)} m(S)$$

and our theorem is an immediate consequence of Theorem 1.8. So it can be assumed that $J \neq \emptyset$. Without lost of generality we can suppose $J = \{1, \dots, l\}$ for some $l \leq k$. Put

$$g(z_1, \dots, z_n) := f(z_1, \dots, z_n) - \sum_{i=1}^{l} f(0, \dots, 0, z_i, 0, \dots, 0). \tag{12}$$

Of course $g(0) = 0$, $\nabla g(0) = 0$, so g is a singularity. Denote by S_j^w the family of all $S \in \Gamma(f)$, such that $S \subset S_j$ and S includes at least one exceptional segment with respect to axis Ox_j. It is easy to check $\Gamma(g) = \Gamma(f) \setminus \sum_{i=1}^{l} S_i^w$, so g is a nondegenerate in Kouchnirenko's sense. We show that g is an isolated singularity. Suppose to the contrary that g is not isolated singularity. It follows that there exists a nonzero parametrization $\Phi = (\phi_1, \dots, \phi_n) \in \mathbb{C}\{t\}^n$, $\Phi(0) = 0$ such that $\nabla g \circ \Phi = 0$. We have the following cases:

(a) $\phi_i \neq 0$, $i = 1, \dots, n$. Then from Lemma 2.2 we get that there is $S \in \Gamma(g)$, on which g is degenerate, a contradiction.

(b) There exists $i \in \{1, \dots, n\}$ such that $\phi_i = 0$. Since f is an isolated singularity, then by the form of g we easily check that $\phi_{i_1} \neq 0$ for some $i_1 \in \{1, \dots l\}$. Since $S_{i_1} \in \Gamma(f)$, so $g'_{z_k}(0, \dots, 0, \phi_{i_1}, 0, \dots, 0) \neq 0$ for every $k \neq i_1$, so $\phi_{i_2} \neq 0$ for some $i_2 \neq i_1$. Let $\{i_1, \dots, i_n\}$ be permutation of set $\{1, \dots, n\}$ such that $\phi_{i_1}, \dots, \phi_{i_m} \neq 0$, and $\phi_{i_{m+1}}, \dots, \phi_{i_n} = 0$. Write now the function g in the form

$$g(z_1, \dots, z_n) = h(z_{i_1}, \dots, z_{i_m}) + z_{i_{m+1}} h_{m+1}(z_1, \dots, z_n) + \dots + z_{i_n} h_n(z_1, \dots, z_n),$$

$m \geq 2$. By the form of g we conclude that h is a singularity and $\nabla h(\phi_{i_1}, \dots, \phi_{i_m}) = 0$. Hence from Lemma 2.2 there exists $S \in \Gamma(h)$ such that $\nabla h_S(\operatorname{in} \phi_{i_1}, \dots, \operatorname{in} \phi_{i_m}) = 0$, so h is degenerate on S. From Lemma 2.1 we have that $S \in \Gamma(g)$. Of course $g_S = h_S$. Define the parametrization $\Psi = (\psi_1, \dots, \psi_n)$, $\psi_i \colon = \operatorname{in} \phi_i$ for $i = i_1, \dots, i_m$, $\psi_i \colon = t$ for remaining i. Note that $\psi_i \neq 0$ for $i = 1, \dots, n$. Since $g_S = h_S$ and $\nabla h_S(\operatorname{in} \phi_{i_1}, \dots, \operatorname{in} \phi_{i_m}) = 0$, then

$$(g_S)'_{z_i}(\Psi) = 0, \; i = i_1, \dots, i_m. \tag{13}$$

On the other hand $S \subset \{(x_1, \dots, x_n) \in \mathbb{R}^n \colon x_{i_{m+1}} = \dots = x_{i_n} = 0\}$, so $(g_S)'_{z_i} \equiv 0$ for $i = i_{m+1}, \dots i_n$. Reasumming $\nabla g_S \circ \Psi = 0$. It follows that g is degenerate on face S, a contradiction. Hence g is an isolated and nondegenerate in Kouchnirenko's sense singularity. By Theorem 1.8 we have that

$Ł_0(g) \leq m_0(g) - 1$. We easily get $g'_{z_i} = f'_{z_i}$, $i = l+1, \ldots, n$ and $g'_{z_i} = f'_{z_i} - f'_{z_i}(0, \ldots, 0, z_i, 0, \ldots, 0)$, $i = 1, \ldots, l$. Note further that $\operatorname{ord} f(0, \ldots, 0, z_i, 0, \ldots, 0) = x_i(S_i)$, $i = 1, \ldots, l$. Then

$$\begin{aligned} &\operatorname{ord}(\nabla g - \nabla f) = \min_{i=1}^{n}[\operatorname{ord}(g'_{z_i} - f'_{z_i})] = \min_{i=1}^{l}[\operatorname{ord}(g'_{z_i} - f'_{z_i})] \\ &= \min_{i=1}^{l}[\operatorname{ord} f'_{z_i}(0, \ldots, 0, z_i, 0, \ldots 0)] = \min_{i=1}^{l}[x_i(S_i)] - 1 > m_0(g) - 1 \geq Ł_0(g), \end{aligned}$$

so $\operatorname{ord}(\nabla g - \nabla f) > Ł_0(g)$. Hence from Lemma 2.3 we get that $Ł_0(f) = Ł_0(g)$. Note that

$$x_i(S_i) > \max_{S \in \Gamma^{n-1}(f) \setminus E_f} \{m(S)\} \geq x_j(S_j),\ i = 1, \ldots, l,\ j = l+1, \ldots, n. \tag{14}$$

Reasumming

$$\begin{aligned} Ł_0(f) = Ł_0(g) &\leq m_0(g) - 1 = \max\{m(S) \colon S \in \Gamma^{n-1}(f) \setminus \{S_1, \ldots, S_l\}\} - 1 \\ &= \max\{m(S) \colon S \in \Gamma^{n-1}(f) \setminus E_f\} - 1, \end{aligned}$$

which finishes the proof. □

Example 2.4. Let $f(z_1, z_2, z_3) := z_3^{20} + z_1^2 + z_2^2 + z_3^4 z_1 + z_3^4 z_2$. It is easy to check that f is an isolated singularity and nondegenerate in Kouchnirenko's sense. It is easy to check that $\Gamma^2(f)$ consists of two faces. One of them $S = \operatorname{conv}\{(1,0,4), (0,1,4), (0,0,20)\}$ is exceptional with respect to Ox_3 axis and $m_0(f) = 20$. By Theorem 1.8 we get $Ł_0(f) \leq m_0(f) - 1 = 19$, and by the main theorem we have

$$Ł_0(f) \leq \max_{S \in \Gamma^2(f) \setminus \{S\}} \{m(S)\} - 1 = 8 - 1 = 7. \tag{15}$$

Hence this estimation is really better. It can be shown by result of paper [**4**] and Lemma 2.3 that we have $Ł_0(f) = 7$. Hence the obtained estimation is optimal.

Note that for $n = 2$ immediate corollary from the theorem above is inequality "in one side" in Theorem 1.6.

Corollary 2.5. *Let $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be an isolated singularity and nondegenerate in Kouchnirenko's sense and let $\Gamma^1(f) \setminus E_f \neq \emptyset$. Then*

$$Ł_0(f) \leq \max_{S \in \Gamma^1(f) \setminus E_f} \{m(S)\} - 1. \tag{16}$$

Problem. How to define an exceptional face to get an equality in Theorem 1.9?

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