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# ON THE HESSE MATRIX OF A QUASI-HOMOGENEOUS ISOLATED SINGULARITY

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**Abstract.** We calculate the rank of the Hesse matrix of a quasi-homogeneous isolated singularity of type $(w_1, \dots, w_n)$ in terms of the weights $w_i$.

Let $f = f(x_1, \dots, x_n) \in \mathbb{C}\{x_1, \dots, x_n\}$ be a power series convergent near $0 \in \mathbb{C}^n$. We call $f$ *an isolated singularity* at $0 \in \mathbb{C}^n$ if $f(0) = 0$ and $0 \in \mathbb{C}^n$ is an isolated solution of a system of equations $\frac{\partial f}{\partial x_1} = \dots = \frac{\partial f}{\partial x_n} = 0$. Let us recall, that $f$ is *a quasi-homogeneous singularity* of type $(w_1, \dots, w_n)$ if $f$ is a polynomial of form

$$f = \sum_{\frac{i_1}{w_1} + \dots + \frac{i_n}{w_n} = 1} c_{i_1 \dots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

for some positive rationals $w_1, \dots, w_n$. The number $w_i$ is called the weight of the variable $x_i$.

Put $H_f(0) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(0)\right]_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$. The main result of the paper is

Theorem 1. *If $f$ is an isolated, quasi-homogeneous singularity of type $(w_1, w_2, \dots, w_n)$ then* $\operatorname{rank} H_f(0) = 2 \cdot \#\{k : w_k < 2\} + \#\{k : w_k = 2\}$.

In order to prove the above theorem we need

Lemma 2. *If $f$ is an isolated singularity then for any $k \in \{1, \dots, n\}$ there exist an $l \in \{1, \dots, n\}$ such that the monomial $x_k^a x_l$, $a \geqslant 1$ appears in $f$ with a nonzero coefficient.*

and

LEMMA 3. *Let $f$ be an isolated, quasi-homogeneous singularity of type $(w_1, w_2, \dots, w_n)$. Let us put $A_i = \{k : w_k = w_i\}$. $A_i' = \{k : \frac{1}{w_k} + \frac{1}{w_i} = 1\}$ for $i = 1, \dots, n$. Then for each $i$:*

(i) *if $w_i < 2$ then $\#A_i' \geqslant \#A_i = \operatorname{rank} \left[\frac{\partial^2 f}{\partial x_k \partial x_l}(0)\right]_{k\in A_i, l\in A_i'}$,*

(ii) *if $w_i = 2$ then $A_i' = A_i$ and $\#A_i = \operatorname{rank} \left[\frac{\partial^2 f}{\partial x_k \partial x_l}(0)\right]_{k,l\in A_i}$.*

Lemma 2 is proved in the article [**1**] by Saito (see Korollar 1.6), (cf. [**2**]).

PROOF OF LEMMA 3. First let us fix an $i$ such that $w_i < 2$. It is easy to check that if $w_k = w_i$ for an index $k$ then $x_k$ does not appear in $f$ in the power strictly greater than 1 and from Lemma 2 we see that for any $k \in A_i$ there exist $l \in A_i'$ such, that the monomial $x_k x_l$ appears in $f$ with a nonzero coefficient. Thus $f$ is of the form

$$f(x_1, \dots, x_n) = \sum_{r\in A_i, s\in A_i'} c_{rs} x_r x_s \; + \sum_{r\notin A_i, s\notin A_i'} c_{rs} x_r x_s + b(x_1, \dots, x_n),$$

where $\operatorname{ord} b \geqslant 3$. Since the singularity is isolated then $0 \in \mathbb{C}^n$ is an isolated solution of the system of equations

$$\begin{aligned}
&\frac{\partial f}{\partial x_k} = \sum_{l\in A_i'} c_{kl} x_l + \frac{\partial b}{\partial x_k} = 0 && \text{for } k \in A_i,\\
(1)\qquad &\frac{\partial f}{\partial x_l} = \sum_{k\in A_i} c_{kl} x_k + \frac{\partial b}{\partial x_l} = 0 && \text{for } l \in A_i',\\
&\frac{\partial f}{\partial x_j} = \frac{\partial \sum_{r\notin A_i, s\notin A_i'} c_{rs} x_r x_s}{\partial x_j} + \frac{\partial b}{\partial x_j} = 0 && \text{for } j \notin A_i \cup A_i'.
\end{aligned}$$

The solution remains isolated if we take $x_j = 0$ for all $j \notin A_i$. Under this condition system (1) is of the form

$$\begin{aligned}
&\frac{\partial f}{\partial x_k} \equiv 0 && \text{for } k \in A_i,\\
&\frac{\partial f}{\partial x_l} = \sum_{k\in A_i} c_{kl} x_k = 0 && \text{for } l \in A_i',\\
&\frac{\partial f}{\partial x_j} \equiv 0 && \text{for } j \notin A_i \cup A_i'.
\end{aligned}$$

Hence $\#A_i' \geqslant \#A_i = \operatorname{rank} \left[\frac{\partial^2 f}{\partial x_k \partial x_l}(0)\right]_{k\in A_i, l\in A_i'}$.

Now, let $w_i = 2$. Then obviously $A_i = A'_i$. We can write

$$f(x_1, \dots, x_n) = \sum_{r,s \in A_i} c_{rs} x_r x_s + \sum_{r,s \notin A_i} c_{rs} x_r x_s + b(x_1, \dots, x_n),$$

with $\operatorname{ord} b \geqslant 3$. Taking in (1) $x_j = 0$ for all $j \notin A_i$ we get

$$\frac{\partial f}{\partial x_k} = \sum_{l \in A_i} c_{kl} x_l = 0 \qquad \text{for } k \in A_i = A'_i,$$

$$\frac{\partial f}{\partial x_j} \equiv 0 \qquad \text{for } j \notin A_i = A'_i,$$

hence $\# A_i = \# A'_i = \operatorname{rank} \left[ \frac{\partial^2 f}{\partial x_k \partial x_l}(0) \right]_{k,l \in A_i = A'_i}$. This completes the proof of Lemma 3. □

Proof of Theorem 1. Of course, if all weights $w_i > 2$ then $\operatorname{rank} H_f(0) = 0$. Indeed, this is because if a monomial $x_k x_l$, $k, l \in \{1, \dots, n\}$ appears in $f$ with a nonzero coefficient then $\frac{1}{w_k} + \frac{1}{w_l} = 1$, what is impossible when $w_k > 2$ and $w_l > 2$.

Now let us suppose that $w_1 \leqslant \dots \leqslant w_n$. Take a sequence $w_{i_1}, w_{i_2}, \dots, w_{i_s}$ such that

$$\{w_{i_1}, w_{i_2}, \dots, w_{i_s}\} = \{w_i : w_i \leqslant 2\},$$
$$w_{i_1} < w_{i_2} < \dots < w_{i_s} \leqslant 2.$$

We easily see that the sets $A_{i_j}$, $j = 1, \dots, s$ are pairwise disjoint and also the sets $A'_{i_j}$, $j = 1, \dots, s$. Let $H_j = \left[ \frac{\partial^2 f}{\partial x_k \partial x_l}(0) \right]_{k \in A_{i_j}, l \in A'_{i_j}}$ for $j = 1, \dots, s$. Then $H_f(0)$ is of the form

$$H_f(0) = \begin{bmatrix}
0 & & & \cdots & & & 0 & 0 & 0 & H_1 & 0 \\
0 & & & \cdots & & & 0 & H_2 & 0 & 0 & 0 \\
0 & & & \cdots & & & 0 & 0 & 0 & 0 & 0 \\
0 & & & \cdots & & & 0 & 0 & 0 & 0 & 0 \\
\vdots & & & & & \iddots & & & & & \vdots \\
0 & & \cdots & 0 & H_s & 0 & & \cdots & & & 0 \\
\vdots & & & \iddots & & & & & & & \vdots \\
0 & 0 & 0 & & & \cdots & & & & & 0 \\
0 & H_2^* & 0 & & & \cdots & & & & & 0 \\
0 & 0 & 0 & & & \cdots & & & & & 0 \\
H_1^* & 0 & 0 & & & \cdots & & & & & 0 \\
0 & 0 & 0 & & & \cdots & & & & & 0
\end{bmatrix},$$

where $H_j^*$ means $H_j$ transposed.

Hence $\operatorname{rank} H_f(0) = 2 \cdot \sum_{j=1}^{s-1} \operatorname{rank} H_j + \operatorname{rank} H_s$. By Lemma 3

$$\sum_{j=1}^{s-1} \operatorname{rank} H_i = \sum_{j=1}^{s-1} \# A_{i_j} = \#\{k : w_k < 2\},$$

$$\operatorname{rank} H_s = \# A_{i_s} = \#\{k : w_k = 2\}$$

and Theorem 1 follows. □

## References


1. Saito K., *Quasihomogene isolierte Singularitäten von Hyperflächen*, Inventiones Math., **14** (1971), Springer-Verlag, 123–142.
2. Sękalski M., *On the multiplicity of a quasi-homogeneous isolated singularity*, Univ. Iagel. Acta Math., **XLVI** (2008), 95–97.


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