UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLVII
2009

FOLIATIONS BY MINIMAL SUBMANIFOLDS AND RICCI CURVATURE

by Richard H. Escobales, Jr.

For Patrick and Teresa Collins

Abstract. Let (M^n, g) be a closed, connected, oriented, C^∞, Riemannian, n-manifold with a transversely oriented, codimension-2 foliation **F**. Suppose the transverse volume form μ is basic and $\{X, Y\}$ are basic vector fields, so $\mu(X, Y) = 1$. Then the leaf component of $[X, Y]$, $\mathcal{V}[X, Y]$, is globally defined on M and is independent of the basic pair of vector fields $\{X, Y\}$ satisfying the above equation as observed by Cairns in [**5**]. Using the Bochner technique, we show under appropriate assumptions on cohomology and on the Ricci curvature of the leaves of the foliation **F**, that the distribution orthogonal to that of the leaves, **H**, is integrable and the leaves of this new foliation are minimal surfaces of M. Using results from Milnor [**26**] and Gray [**19**], we apply this theorem to give a necessary and sufficient condition for certain principal bundles to admit a flat connection (Corollary 1.6). In the second section we provide some analogous results for the special case when **F** is a Riemannian foliation.

Section 1. Throughout this paper all maps, functions and morphisms are assumed to be at least of class C^∞. On a closed connected oriented C^∞ Riemannian manifold (M^n, g), let **F** be a transversely oriented foliation of leaf dimension p and codimension $q = n-p$. Let **V** denote the distribution tangent to the foliation **F**, and **H** the distribution orthogonal to **V** in TM determined by the metric g. If E is a vector field on M, $\mathcal{V}E$ and $\mathcal{H}E$ will denote the projections of E onto the distributions **V** and **H**, respectively. Call the vector field E *vertical* if $\mathcal{V}E = E$. Call E *horizontal* if $\mathcal{H}E = E$.

2000 *Mathematics Subject Classification.* primary 57R30; secondary 53C25.
Key words and phrases. Minimal or harmonic foliation, mean curvature, closed manifold, characteristic form.

In general a C^∞ *foliation* of *codimension-q* on an n-dimensional manifold M can be defined is a maximal family of C^∞ submersions $f_\alpha : U_\alpha \to f_\alpha(U_\alpha) \subset \mathbb{R}^q$ where $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover of M and where for each $\alpha, \beta \in \Lambda$ and each $x \in U_\alpha \cap U_\beta$, there exists a local diffeomorphism $\phi^x_{\beta\alpha}$ of R^q so $f_\beta = \phi^x_{\beta,\alpha} \circ f_\alpha$ in some neighborhood U_x of x (see [**24**], pp. 2–3).

A horizontal vector field Z defined on some open set U where $U \subset U_\alpha$ is called f_α- *basic* provided $f_{\alpha *}Z$ is a well defined vector field on $f_\alpha(U)$. As pointed out in [**12**] (for *any* metric g), if $U \subset U_\beta$, then Z is also f_β- *basic*, so one can speak of Z as a *local basic vector field*. We sometimes drop the word "local." Let $i(W)$ and $\theta(W)$ denote the interior product and the Lie derivative with respect to a vector field W. A differential form ϕ is called *basic* provided $i(W)\phi = 0$ and $\theta(W)\phi = 0$ for all vertical vector fields W ([**37**], p. 118). We follow the conventions of [**1**] for the formalism of differential forms and their exterior derivatives. Observe, if ϕ is basic of degree q, where q is the codimension of $\mathbf{F}$, then ϕ is closed.

D will denote the Levi–Civita connection on M and, following [**15**], we introduce the tensors T and A as follows. For vector fields E and F on M,

$$T_E F = \mathcal{V} D_{\mathcal{V}E} \mathcal{H}F + \mathcal{H} D_{\mathcal{V}E} \mathcal{V}F, \quad \text{and} \tag{1.1}$$

$$A_E F = \mathcal{V} D_{\mathcal{H}E} \mathcal{H}F + \mathcal{H} D_{\mathcal{H}E} \mathcal{V}F. \tag{1.2}$$

Then T and A are tensors of type $(1,2)$. These tensors satisfy the usual properties outlined in [**15**]. We note that if X and Y are horizontal,

$$A_X Y \neq -A_Y X, \quad \text{in general,} \tag{1.3}$$

unless the foliation $\mathbf{F}$ is *bundle-like* with respect to the metric g (see [**21**], Lemma (1.2)) that is, if X is a basic vector field, $Wg(X,X) = 0$ for every vertical vector field W. Note this means that the defining submersions f_α of the foliation $\mathbf{F}$ above are *Riemannian submersions* in the sense of [**29**].

If $\{V_1, V_2, V_3, \ldots, V_p\}$ is a local orthonormal frame tangent to the foliation, we define the *mean curvature one-form* κ as follows:

$$\kappa(E) = \sum_{i=1}^{p} g(E, T_{V_i} V_i) = g(E, \sum_{i=1}^{p} T_{V_i} V_i). \tag{1.4}$$

Here $\tau = \sum_{i=1}^{p} T_{V_i} V_i$ is the mean-curvature vector field of the leaves of $\mathbf{F}$. (Following the now standard practice in foliations ([**37, 38**]), we suppress the factor $(1/p)$.) Call κ *horizontally closed* if $d\kappa(Z_1, Z_2) = 0$ for any horizontal fields Z_1, Z_2.

Following [**37**], page 65–66, let $\chi_{\mathbf{F}}$ denote the characteristic form for the foliation $\mathbf{F}$. Then with $\{V_1, \ldots, V_p\}$ as above and for vector fields $\{E_1, \ldots, E_p\}$ on

M^n, we have:

$$\chi_{\mathbf{F}}(E_1, E_{2,\ldots}, E_p) = det(g(E_i, V_j)). \tag{1.5}$$

This characteristic differential form (see [**38**], p. 37) is independent of the local orthonormal frame, $\{V_1, \ldots, V_p\}$. When restricted to a leaf of **F**, $\chi_{\mathbf{F}}$ represents the canonical volume form for that leaf, as observed on page 37 of [**38**]. If any one of the arguments E_i is horizontal, then the left hand side of (1.5) vanishes. This fact will be used repeatedly in the computations below.

A foliation **F** is *taut* provided there exists a Riemannian metric g on M for so that all of the leaves of **F** are minimal submanifolds of M. The foliation, **F**, on (M, g) is then called a *minimal foliation* as in [**4**] and [**39**]. A minimal foliation is also called a *harmonic foliation* by some authors (see [**38**], p. 27 or [**6**], p. 261).

Lemma 1.1.

(*a*) *Let (M^n, g) be a connected, oriented, C^∞ Riemannian n-manifold with a transversely oriented codimension-q foliation* **F***, with $q \geq 2$. Suppose X and Y are basic vector fields. Then $\mathcal{V}[X, Y]$ has vanishing leaf divergence if and only if κ is horizontally closed.*

(*b*) *Let* **F** *be a transversely oriented codimension-2 foliation on a closed, oriented, Riemannian manifold (M^n, g) which admits a transverse volume form μ. Let $\{X, Y\}$ be a pair of basic vector fields so $\mu(X, Y) = 1$ and consider the globally defined vector field $\mathcal{V}[X, Y]$ on M^n. Then $div_{\mathbf{F}}\mathcal{V}[X, Y] = div_M \mathcal{V}[X, Y]$. Hence in this case, if κ is horizontally closed, $div_M \mathcal{V}[X, Y] = 0$.*

Proof. (a) As noted in [**3**], (a) follows immediately from formula (3) of [**15**] which can be expressed this way:

$$d\kappa(X, Y) = -div_{\mathbf{F}}\mathcal{V}[X, Y] = -\Sigma_{i=1}^{p} g(D_{V_i}\mathcal{V}[X, Y], V_i), \tag{1.6}$$

where the right hand side denotes the divergence of $\mathcal{V}[X, Y]$ along a leaf of **F**.

The proof of (b) goes this way. Let $\{X_1, X_2\}$ be two orthonormal horizontal vector fields (not necessarily basic, since the foliation **F** is not necessarily bundle-like with respect to g). As Grant Cairns pointed out in his thesis [**5**], if $\{X, Y\}$ are two basic vector fields so $\mu(X, Y) = 1$, then $\mathcal{V}[X, Y]$ is a globally defined vector field on M, independent of the basic pair $\{X', Y'\}$ so $\mu(X', Y') = 1$. Thus, $\mathcal{V}[X, Y] = \mathcal{V}[X', Y']$ for any two such pairs of basic vector fields because any element of $SL(2, \mathbf{R})$ has determinant 1. If μ is basic, then μ is closed, because μ is of degree $q = 2$ as observed above. Then $\kappa^\perp = 0$ by Theorem 6.32 on page 71 of [**37**], so the vector field dual to $\kappa^\perp$,

$\tau^{\perp} = \Sigma_{i=1}^{2} A_{X_i} X_i = 0$. Using the properties of the tensor A (but *not* the alternating property, that is, $A_X Y = -A_Y X$ for horizontal X,Y), it follows that $\Sigma_{i=1}^{2} g(D_{X_i}\mathcal{V}[X,Y], X_i) = 0$, so $div_{\mathbf{F}}\mathcal{V}[X,Y] = div_M \mathcal{V}[X,Y]$, as claimed. The rest of the lemma follows from part (a). $\square$

The form $\kappa \wedge \chi_{\mathbf{F}}$ plays an important role in the work of Kamber and Tondeur on Riemannian foliations ([**37**], pp. 121 and 152, [**38**], pp. 39 and 82). It turns out that when this form is closed, the following pleasant property obtains for *arbitrary* foliations on Riemannian manifolds of codimension $q \geq 2$. The result proven in [**3**] illustrates once more the tie between cohomology and geometry. Indeed, the form $\kappa \wedge \chi_{\mathbf{F}}$ will play a role in Theorems 1.5, 1.7, 2.1 and 2.2, so one can think of the form $\kappa \wedge \chi_{\mathbf{F}}$ as the differential form that keeps on giving.

THEOREM 1.2. *Let (M^n, g) be a closed, connected, oriented, C^∞ Riemannian n-manifold with a transversely oriented codimension-q foliation $\mathbf{F}$. Suppose X and Y are basic vector fields. Then $\mathcal{V}[X,Y]$ has vanishing leaf divergence (equivalently κ is horizontally closed) whenever $\kappa \wedge \chi_{\mathbf{F}}$ is a closed (possibly zero) de Rham cohomology $(p+1)$-form. In fact, if the codimension of $\mathbf{F}$, $q = 2$, then κ is horizontally closed if and only if $\kappa \wedge \chi_{\mathbf{F}}$ is closed.*

The key formula developed in [**3**] is

$$d(\kappa \wedge \chi_{\mathbf{F}})(V_1, V_2, \ldots, V_p, X, Y) = d\kappa(X,Y). \tag{1.7}$$

This means in the case of Lemma 1.1 (b),

$$d(\kappa \wedge \chi_{\mathbf{F}})(V_1, V_2, \ldots, V_p, X, Y) = d\kappa(X,Y) = -div_{\mathbf{F}}\mathcal{V}[X,Y] = -div_M \mathcal{V}[X,Y]. \tag{1.8}$$

The following simple result may be useful. It holds for *any* foliation on a Riemannian manifold with codimension $q \geq 1$.

PROPOSITION 1.3. *Let (M^n, g) be any oriented Riemannian manifold that admits a foliation $\mathbf{F}$ with mean curvature one-form, κ. Then κ is closed provided κ is basic and $\kappa \wedge \chi_{\mathbf{F}}$ is closed. If the codimension of the foliation is 2, κ is closed if and only if κ is basic and $\kappa \wedge \chi_{\mathbf{F}}$ is closed.*

PROOF. If X is basic and V is vertical, then κ is basic if and only if $d\kappa(X,V) = 0$ as pointed out in [**15**]. If $\kappa \wedge \chi_{\mathbf{F}}$ is closed, then $d\kappa(X,Y) = 0$ for basic $\{X,Y\}$, by 1.7. The proof in the special case when $q = 1$ is easy and hence omitted. $\square$

For most of the rest of this paper we assume the foliation, $\mathbf{F}$, has codimension $q = 2$ and that $\mathbf{F}$ admits a basic transverse volume form μ. Note, the condition that μ is basic is weaker than the condition that the foliation $\mathbf{F}$ be bundle-like with respect to the Riemannian metric, g, since when the metric is bundle-like, the transverse volume form can be computed explicitly in terms of the metric. Later, we will sketch how this is done.

Let β be the one-form on M dual to $\mathcal{V}[X,Y]$ with respect to the metric g. Then for any vector field E on M,

$$\beta(E) = g(\mathcal{V}[X,Y], E). \tag{1.9}$$

Note, β is independent of the choice of basic vector fields $\{X,Y\}$ so that $\mu(X,Y) = 1$.

The following property of β is as elementary as it is striking.

Proposition 1.4. *Let* $\mathbf{F}$ *be a transversely oriented codimension-2 foliation on a closed, oriented, Riemannian manifold* (M^n, g) *which admits a basic transverse volume form* μ. *Then* β *is co-closed on each of the leaves of* $\mathbf{F}$ *or on* M *if and only if* $\kappa \wedge \chi_{\mathbf{F}}$ *is closed on* M.

Remark. This means that the existence of a cohomology (p+1) form, namely $\kappa \wedge \chi_{\mathbf{F}}$, encodes the co-closedness of β.

Proof. β is co-closed on the leaves of $\mathbf{F}$ provided its dual vector field has zero divergence along a leaf, that is provided $-div_{\mathbf{F}}\mathcal{V}[X,Y] = 0$ by [**33**], page 168. Likewise, β is co-closed on M, provided $-div_M \mathcal{V}[X,Y] = 0$. Each of these occurs if and only if $\kappa \wedge \chi_{\mathbf{F}}$ is closed by equation (1.8). Specifically, if $\delta\beta$ denotes the codifferential of β, then

$$\begin{aligned} d(\kappa \wedge \chi_{\mathbf{F}})(V_1, V_2, \ldots, V_p, X, Y) &= d\kappa(X,Y) = -div_{\mathbf{F}}\mathcal{V}[X,Y] \\ &= -div_M \mathcal{V}[X,Y] = \delta\beta. \end{aligned} \tag{1.10}$$

□

We now come to one of the main result of the paper, Theorem 1.5. Note, in this section, we do *not* assume that $\mathbf{F}$ is bundle like with respect to the metric g. The result appears to be pleasant because three key conditions force the conclusion: two involve cohomology and one involves Ricci curvature. Note, if β itself is closed then, for $\{X,Y\}$, above, $d\beta(X,Y) = -g(\mathcal{V}[X,Y], \mathcal{V}[X,Y]) = 0$, and $\mathbf{H}$ is integrable. A weaker condition which only requires that the pullback of β to the leaves of $\mathbf{F}$ be closed, suffices to establish the integrability of $\mathbf{H}$ in the presence of appropriate conditions (and one hopes pleasing conditions) on the Ricci curvature of the leaves of $\mathbf{F}$.

Recall the Ricci curvature of a manifold is *quasi-positive* provided it is positive semi-definite everywhere and positive definite at a point. The Ricci curvature is *quasi-negative* on a manifold if it is negative semi-definite everywhere negative definite at a point. The results below can be viewed as a companion to results of earlier papers, like [**12, 13**] and [**7**]. The author learned about quasi-positive Ricci curvature and quasi-negative Ricci curvature from Wu's article [**40**], and the special beauty of codimension-2 foliations admitting a basic transverse volume form from [**5**]. He became interested in $\kappa \wedge \chi_{\mathbf{F}}$ as a result of [**38**]. Other authors have looked at minimal foliations and Ricci curvature and have obtained beautiful results like those of [**4**] and [**39**]. The interested reader might also profit from [**26, 28**] and [**20**]. Related but different results that recently came to the author's attention appear in [**9**].

If L is a leaf of $\mathbf{F}$, then $i_L : L \to M$ denotes the inclusion map. $i_L(L)$ is always an *immersed submanifold* of M, following Definition 3.7.7 of [**10**], page 93 (see also page 18). If i_L is a homeomorphism from L onto $i_L(L)$, then $i_L(L)$ is called an *imbedded submanifold* of M as in [**10**].

Remark. The next two results highlight the utility of having all the leaves of the codimension-2 foliation $\mathbf{F}$ possess either quasi-positive or quasi-negative Ricci curvature.

Theorem 1.5. *Let* $\mathbf{F}$ *be a transversely oriented codimension-2 foliation on a closed, oriented, Riemannian manifold* (M^n, g) *which admits a basic transverse volume form* μ. *Suppose the following two conditions obtain.*

(*i*) *Restricted to each leaf* L *of* $\mathbf{F}$, β *is a closed one-form, that is,* $i_L^*\beta$ *is closed on* L.

(*ii*) $\kappa \wedge \chi_{\mathbf{F}}$ *is closed on* M.

If the Ricci curvature of each leaf L *of* $\mathbf{F}$, Ric_L, *is quasi-positive on* L, *then* $\mathbf{H}$ *is integrable and the leaves of* $\mathbf{H}$ *are minimal surfaces of* M.

Proof. Set $f = (1/2)g(\mathcal{V}[X,Y], \mathcal{V}[X,Y])$. Then f attains a maximum at some $p \in M$, since M is compact. Let L be the leaf of $\mathbf{F}$ containing $p \in M$. Now β is closed on L by (i) and is co-closed on L by (ii) and Proposition 1.4. Since the gradient of $div_L \mathcal{V}[X,Y]$ $(div_{\mathbf{F}} \mathcal{V}[X,Y])$ is zero by formula (1.9), one of the terms in part 3 of Proposition 3.3 on page 175 of [**31**] vanishes. Observe that ∇ in [**31**] is replaced below by $\tilde{D}$, the covariant derivative on L induced from D on M. Then that formula of [**31**] yields the following equation for the Laplacian of f on L, $\triangle_L f$.

$$\triangle_L f = |\tilde{D}\mathcal{V}[X,Y]|_L^2 + Ric_L(\mathcal{V}[X,Y], \mathcal{V}[X,Y]) \geq 0. \tag{1.11}$$

Thus, f is subharmonic on L. This means f is constant on L by the maximum principle of E. Hopf. (A nice proof of this principle appears in Matsushima [**25**], pages 296–299). Hence, $\triangle_L f \equiv 0$. If $f \not\equiv 0$ on L, then by the above $f \equiv c > 0$ on L. Then at the $x \in L$ where Ric_L is positive definite , $\triangle_L f > 0$, which is a contradiction. Hence, $f \equiv 0$ on L, and, in particular, $f(p) = 0$. Since f attained its maximum on M at $p \in L$, $f \equiv 0$ on M. Hence by definition of f, $\mathcal{V}[X,Y] \equiv 0$ on M and $\mathbf{H}$ is integrable by the Frobenius Theorem. As observed earlier, since μ is basic and of degree $q = 2$, μ is closed. Hence, by

Theorem 6.32 on page 71 of [**37**], the leaves of **H** are minimal surfaces of M, since then $\kappa^{\perp} = 0$, and in our context $\kappa^{\perp}$ is the mean curvature one-form of our now integrable **H**. This proves the Theorem 1.5. $\square$

EXAMPLE.

(a) For $n \geq 4$, $S^{n-2} \times T^2$, the direct product of the $(n-2)$ sphere of radius 1 and a flat 2-torus, illustrates Theorem 1.5.

(b) Our second application of Theorem 1.5 is somewhat surprising. Unhappily, it seems only to work for certain principal bundles over connected, closed surfaces.

Let G be a Lie group admitting a biinvariant metric $\langle\,,\rangle$ which unlike [**19**] we require to be positive definite. Then by section 5 of [**19**], the principal bundle, $G \to P \to B$ with bundle map $\pi : P \to B$ has the structure of a Riemannian submersion. The leaves of the foliation are the inverse images $\pi^{-1}(b)$ for $b \in B$. As above, **V** is the distribution tangent to the leaves of the foliation, and **H**, the distribution orthogonal to **V** induced by the submersion metric. This metric on P Gray calls the *natural metric*, denoted by $< E, F >$ for vector fields E and F on P. We will call the connection **H** on P, the *Gray connection.* Recall from [**22**], page 51, a vertical vector field A^* on a principal bundle P is called a *fundamental vector field* provided $A^* = \sigma(A)$ where $A \in \mathbf{g}$ where $\mathbf{g}$ is the Lie algebra of G and σ is the homomorphism from $\mathbf{g}$ to $\chi(P)$, the Lie algebra of vector fields on P. According to Corollary 7.7 of [**26**], a Lie group with compact universal covering admits a biinvariant metric of constant Ricci curvature 1. When G is a Lie Group with this structure we will say the Lie group G is of *special Milnor type.*

COROLLARY 1.6. *Let B, be a connected, closed oriented surface, and let $G \to P \to B$ be a principal bundle over B where G is a Lie group of special Milnor type above. Then the Gray connection* **H** *determined by its natural metric is flat if and only if for any two local orthonormal basic vector fields $\{X, Y\}$, the globally defined vector field $\mathcal{V}[X, Y]$ is a fundamental vector field on P.*

PROOF. It is easy to see that P is connected. The connection **H** is flat if and only if $\Omega(X, Y) = 0$ for any basic orthonormal pair $\{X, Y\}$, where Ω is the curvature of the connection. By (5.6) of [**19**] this occurs if and only if in our notation above, $A_X Y = (1/2)\mathcal{V}[X, Y] = 0$. Now the zero vector field is a fundamental vector field so the condition on $\mathcal{V}[X, Y]$ is necessary. We need to show that the condition is sufficient. This will be the case if we can apply Theorem 1.5. We need to show that restricted to a leaf of the foliation by the fibers of π ,β, the one form dual to $\mathcal{V}[X, Y]$ is closed, that is if $i_L^*\beta$ is closed. As noted on page 167 of [**33**], this will occur when given fundamental vector fields A^* and B^* on P, one has:

$$\langle D_{A^*}\mathcal{V}[X,Y],B^*\rangle - \langle A^*, D_{B^*}\mathcal{V}[X,Y]\rangle = 0, \tag{1.12}$$

where D is the Levi–Civita connection on the leaf of π. Note, since $T = 0$ by equation (5.2) of [**19**], the induced connection on the leaves of π and the connection D on P coincide. But then, by equation (5.3) of [**19**], $D_{A^*}B^* = (1/2)[A^*, B^*]$. If X is basic and A^* is a fundamental vector field, then $[X, A^*] = 0$, as pointed out in (5.1) of [**19**]. Moreover, the Jacobi identity for vector fields, yields that if X and Y are basic, then $[[X,Y], A^*] = 0$. In fact, for Riemannian submersions, $\mathcal{H}[X,Y]$ is basic when X and Y are basic [**29**], so $[\mathcal{H}[X,Y], A^*] = 0$. It follows easily that $[\mathcal{V}[X,Y], A^*] = 0$. Thus, if $\mathcal{V}[X,Y]$ is a fundamental vector field, one can exploit equation (1.12) above and $i_L^*\beta$ is indeed closed.

Since, as already noted, $T = 0$, $\kappa \wedge \chi_{\mathbf{F}}$ vanishes on M by equation (1.4) above. Theorem 1.5 above applies. Thus, **H** is integrable, that is, $\mathcal{V}[X,Y] = 0$ and the Gray connection on P is flat. Pages 304–305 of [**30**] is a nice companion to [**19**]. □

Recall a foliation **F** on a Riemannian manifold (M^n, g) is *totally umbilic* provided there exists a horizontal vector field N so that for all vectors $\{U, V\}$ tangent to the leaves of **F** one has $T_U V = g(U,V)N$, where T is the tensor defined in (1.1) above. If $\{X, Y\}$ are basic vector fields then for a totally umbilic foliation of leaf dimension p, one has by a formula on page 59 of [**7**] that $(\theta(\mathcal{V}[X,Y])g)(U,V) = -(2/p)d\kappa(X,Y)g(U,V)$. If κ is horizontally closed, then $\mathcal{V}[X,Y]$ is an infinitesimal isometry along each leaf. But by (1.7), this occurs whenever $\kappa \wedge \chi_{\mathbf{F}}$ is closed. Note, for totally umbilic foliations, $\tau = pN$, where τ is the mean curvature vector field of **F**. When the leaf dimension $p = n-2$ and when the transverse volume form μ is basic, one has the following improvement of part of Theorem 3.3 of [**7**].

Theorem 1.7. *Let* **F** *be a transversely oriented codimension-2 foliation with totally umbilic leaves on a closed, oriented, Riemannian manifold* (M^n, g) *which admits a basic transverse volume form* μ. *Suppose* $\kappa \wedge \chi_{\mathbf{F}}$ *is closed on* M.

If the Ricci curvature of each leaf L *of* **F**, Ric_L, *is quasi-negative on* L, *then* **H** *is integrable and the leaves of* **H** *are minimal surfaces of* M.

Proof. From the remarks before the statement of the theorem, under the stated conditions $\mathcal{V}[X,Y]$ is a local infinitesimal isometry of the leaves of **F**. When μ is basic, the codimension $q = 2$, and $\{X, Y\}$ are basic vector fields so $\mu(X,Y) = 1$, then $\mathcal{V}[X,Y]$ is a *global* infinitesimal isometry for each of the leaves of **F**. Now the proof of Theorem 3.3 of [**7**] yields that **H** is integrable.

As before, the leaves of $\mathbf{H}$ are minimal surfaces of M thanks to Theorem 6.32 on page 71 of [**37**]. □

REMARKS. In our formalism (see [**31**], page 166), formula (**) on page 60 of [**7**] reads:

$$\triangle_L f = |\tilde{D}\mathcal{V}[X,Y]|_L^2 - Ric_L(\mathcal{V}[X,Y],\mathcal{V}[X,Y]) \geq 0, \tag{1.13}$$

where the function f is given by $f = (1/2)g(\mathcal{V}[X,Y],\mathcal{V}[X,Y])$. The equation (**) on page 60 of [**7**] incorrectly omits the factor (1/2), but the argument there goes through without any trouble. Formulas (1.11) and (1.13) go right back to Bochner himself. Remarkably, for these formulas to come into play here, $\kappa \wedge \chi_{\mathbf{F}}$ must be closed.

Section 2. A foliation $\mathbf{F}$ is a *Riemannian foliation* of leaf dimension p and codimension-q, provided that there is some Riemannian metric g on M^n with respect to which $\mathbf{F}$ is bundle-like in the sense above. If $\mathbf{F}$ is a Riemannian foliation on a compact manifold M^n, then a fundamental result of Dominguez, [**11**], shows that there always exists a metric g for which $\mathbf{F}$ is bundle-like and for which the associated mean curvature one-form, κ, is basic. We call this metric, a *Dominguez metric*. In this section we will assume that the original foliation $\mathbf{F}$ is Riemannian and that the metric g chosen for M is a Dominguez metric. First note, that when $\mathbf{F}$ is bundle-like with respect to g, it is well known that we can choose a basic *orthonormal* frame $\{X,Y\}$ for $\mathbf{H}$ [**29**], since in the bundle-like case the local submersions, f_α, defining $\mathbf{F}$ are Riemannian submersions. Now the transverse volume form μ for $\mathbf{F}$ can be expressed explicitly in terms of the basic components of the Dominguez metric g by [**27**] pages 38–39 combined with [**22**], page 283.

Now the results in Section 1 simplify considerably. We have the following theorems.

THEOREM 2.1. *Let $\mathbf{F}$ be a transversely oriented codimension-2 Riemannian foliation leaves on a closed, oriented, Riemannian manifold (M^n,g) where g is a Dominguez metric for $\mathbf{F}$. Suppose further, that restricted to each leaf L of $\mathbf{F}$, β is a closed one-form: that is, $i_L^*\beta$ is closed on L.*

(*a*) *If the Ricci curvature of each leaf L of $\mathbf{F}$, Ric_L, is quasi-positive on L, then $\mathbf{H}$ is integrable and the leaves of $\mathbf{H}$ are totally geodesic surfaces in M. Moreover, if $\kappa\wedge\chi_{\mathbf{F}}$ is harmonic on M with respect to the Dominguez metric, then in fact, the leaves of $\mathbf{F}$ are minimal codimension-2 submanifolds of M.*

(*b*) *If additionally, the sectional curvatures of M are non-negative, then the leaves of $\mathbf{F}$ are necessarily totally geodesic.*

PROOF. The proof of (a) proceeds in this way. Since κ is basic and $\mathbf{F}$ is bundle-like with respect to g, κ is closed by a result of Kamber–Tondeur (see [**37**], page 150). By equation (1.7) above, this means $\kappa \wedge \chi_{\mathbf{F}}$ is closed on M. From the above remarks, μ is basic. Then, Theorem 1.5 applies and $\mathbf{H}$ is integrable and its leaves are minimal. In fact, the leaves of $\mathbf{H}$ are in this case totally geodesic because when $\mathbf{F}$ is bundle-like with respect to g, equation (1.3) becomes $A_X Y = -A_Y X$, for *any* horizontal vector fields $\{X, Y\}$. By equation (1.2), $A_X Y$ is the second fundamental form for the leaves of $\mathbf{H}$ when $\mathbf{H}$ is integrable. Hence, $A \equiv 0$ and the leaves of $\mathbf{H}$ are not only integrable but totally geodesic. If $\kappa \wedge \chi_{\mathbf{F}}$ is harmonic on M with respect to the Dominguez metric, then the leaves of $\mathbf{F}$ are also minimal submanifolds of M by Theorem 1.11 of [**3**].

Part (b) follows from (a) and from the proposition 5.87 on page 66 of [**38**], which asserts that when $\mathbf{H}$ is integrable and M^n has non-negative sectional curvatures, then the harmonicity of the leaves of $\mathbf{F}$ force those leaves to be totally geodesic. □

THEOREM 2.2. *Let* $\mathbf{F}$ *be a transversely oriented codimension-2 Riemannian foliation with totally umbilic leaves on a closed, oriented, Riemannian manifold* (M^n, g) *where* g *is a Dominguez metric for* $\mathbf{F}$*.*

(*a*) *If the Ricci curvature of each leaf* L *of* $\mathbf{F}$*,* Ric_L*, is quasi-negative on* L*, then* $\mathbf{H}$ *is integrable and the leaves of* $\mathbf{H}$ *are totally geodesic of* M*.*

(*b*) *If additionally,* $\kappa \wedge \chi_{\mathbf{F}}$ *is harmonic on* M *with respect to the Dominguez metric, then in fact, the leaves of* $\mathbf{F}$ *are totally geodesic codimension-2 submanifolds of* M *and locally* M *is isometric to a product of the plaques of the leaves of* $\mathbf{H}$ *and* $\mathbf{F}$*.*

PROOF. That κ is closed follows from the argument in the proof of Theorem 2.1. Then the fact that for totally umbilic foliations one has

$$(\theta(\mathcal{V}[X, Y])g)(U, V) = -(2/p)d\kappa(X, Y)g(U, V),$$

as noted in the paragraph above the statement of Theorem 1.7, yields that the globally defined $\mathcal{V}[X, Y]$ is an infinitesimal isometry when restricted to the leaves of $\mathbf{F}$. As noted before the statement of Theorem 2.1, the existence of the Dominguez metric also guarantees that μ is basic. Theorem 1.7 applies and $\mathbf{H}$ is integrable and its leaves are minimal surfaces of M. In fact, by the argument just made in the proof of Theorem 2.1, the leaves of $\mathbf{H}$ are totally geodesic surfaces of M. If additionally, $\kappa \wedge \chi_{\mathbf{F}}$ is harmonic on M with respect to the Dominguez metric, then the leaves of $\mathbf{F}$ are minimal submanifolds of M just as in the proof of Theorem 2.1.

To show part (b), observe that if $\kappa \wedge \chi_{\mathbf{F}}$ is harmonic, then κ vanishes identically. Hence, the mean curvature vector field τ vanishes. But when the leaves are totally umbilic, $\tau = pN$, where p is the leaf dimension and N is the normal curvature vector field. This means for arbitrary vertical vector fields U and V,

$T_U V = g(U, V)N = 0$, so the leaves are totally geodesic. The last part of part (b) follows from part (a) together with the argument in last paragraph of the proof of Theorem 4.1 in [**12**], page 338. □

REMARK. If the partial Ricci curvature of M in the sense of page 67 of [**38**] is negative definite at a single point, then $\mathbf{F}$ cannot be totally geodesic, as follows from proposition 5.91 also on page 67 of [**38**]. This means of course that in this case $\kappa \wedge \chi_{\mathbf{F}}$ cannot be harmonic in part (b) of Theorem 2.2.

Acknowledgements. The author thanks Gabriel Baditoiu for a helpful comment in an earlier version of this manuscript, Chris Kinsey for her help with Tex, and referees for helpful comments. Part of this paper was presented at the 9th Conference on Geometry and Topology of Manifolds Workshop – Kraków 30th June 2008 – 5th July 2008. The author wishes to thank the organizers of that conference, and Professor Robert Wolak, in particular, for extending an invitation to talk at the conference.

References

1. Abraham R., Marsden J. E., Ratiu T., *Manifolds, Tensor Analysis and Applications*, 2^{nd} ed., Springer-Verlag, New York, Berlin, Heidelberg, 1988.
2. Borel A., *Compact Clifford–Klein forms of symmetric spaces*, Topology, **2** (1963), 111–122.
3. Baditoiu G., Escobales Jr. R., Ianus S., *A cohomology* $(p+1)$ *form canonically associated with certain codimension-q foliations on a Riemannian manifold*, Tokyo Journal of Mathematics, **29**, No. **1** (June 2006), 247–270.
4. Brito F., *A remark on minimal foliations of codimension two*, Tohoku Math. J., (2), **36**, No. **3** (1984), 341–350.
5. Cairns G., Thèse, *Feuilletages géodésibles*, l'Universite des Sciences du Languedoc, 1987.
6. Candel A., Conlon L., *Foliations I*, Graduate Studies in Mathematics, **23**. American Mathematical Society, Providence, RI, 2000, xiv+402 pp.
7. Cairns G., Escobales R., *Further geometry of the mean curvature one-form and the normal plane field one-form on a foliated Riemannian manifold*, J. Australian Math. Soc. (Series A), **62** (1997), 46–63.
8. Cairns G., Escobales R., *Note on a theorem of Gromoll-Grove*, Bull. Austral. Math Soc., **55** (1997), 1–5.
9. Chaouch M. A., *Théorème de Bochner et feuilletage minimal. (French) [Bochner's theorem and minimal foliations]*, Balkan J. Geom. Appl., **12**, No. **2** (2007), 32–50. (Reviewer: James J. Hebda) 53C12 (53C20 53C50).
10. Conlon L., *Differentiable manifolds: a first course*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks] Birkhäuser Boston, Inc., Boston, MA, 1993, xiv+395 pp. ISBN: 0-8176-3626-9.
11. Dominguez D., *Finiteness and tenseness theorems for Riemannian foliations*, Amer. J. Math., **120**, No. **6** (1998), 1237–1276.
12. Escobales, Jr. R., *The integrability tensor for bundle-like foliations*, Trans. Amer. Math. Soc., **270** (1982), 333–339.

13. Escobales Jr. R., *Bundle-like foliations with Kählerian leaves*, Trans. Amer. Math. Soc., **276**, No. **2** (1983), 853–859.
14. Escobales Jr. R., *Foliations by minimal surfaces and contact structures on certain closed 3-manifolds*, International Journal of Mathematics and Mathematical Sciences, Volume 2003, No. **21**, 11 April 2003, 1323–1330.
15. Escobales R., Parker P., *Geometric consequences of the normal curvature cohomology class in umbilic foliations*, Indiana University Mathematics Journal, **37** (1988), 389–408.
16. Freedman M., Luo F., *Selected applications of geometry to low-dimensional topology*, Marker Lectures in the Mathematical Sciences held at The Pennsylvania State University, University Park, Pennsylvania, February 2–5, 1987. University Lecture Series, 1. American Mathematical Society, Providence, RI, 1989, xii+79 pp.
17. Greub W., Halperin S., Vanstone R., *Connections, curvature, and cohomology*, Vol. II: *Lie groups, principal bundles, and characteristic classes*, Pure and Applied Mathematics, Vol. **47-II**, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York–London, 1973, xxi+541 pp.
18. Goldberg S. I., *Curvature and homology*, Pure and Applied Mathematics, Vol. XI, Academic Press, New York–London, 1962, xvii+315 pp. 53.72 (53.80).
19. Gray A., *Pseudo-Riemannian almost product manifolds and submersions*, Journal of Mathematics and Mechanics, **16**, No. **7** (1966), 715–737.
20. Haefliger A., *Some remarks on foliations with minimal leaves*, J. Differential Geometry, **15**, No. **2** (1980), 269–284.
21. Johnson D. L., Whitt L. B., *Totally geodesic foliations*, J. Differential Geometry, **15**, No. **2** (1980), 225–235.
22. Kobayashi S., Nomizu K., *Foundations of differential geometry*, Vol. I, Interscience Publishers, a division of John Wiley & Sons, New York–London, 1963, xi+329 pp.
23. Kobayashi S., Nomizu K., *Foundations of differential geometry*, Vol. II, Interscience Tracts in Pure and Applied Mathematics, No. **15**, Vol. **II**, Interscience Publishers John Wiley & Sons, Inc., New York–London–Sydney, 1969, xv+470 pp.
24. Lawson Jr. H. B., *The quantitative theory of foliations*, CBMS Regional Conferences in Mathematics, American Mathematical Society, **27** (1977).
25. Matsushima Y., *Differentiable manifolds*, Translated from the Japanese by E. T. Kobayashi, Pure and Applied Mathematics, **9**, Marcel Dekker, Inc., New York, 1972, vii+303 pp.
26. Milnor J., *Curvature on left invariant metrics on Lie groups*, Advances in Math., **21** (1976), 293–329.
27. Molino P., *Riemannian foliations*, Translated from the French by Grant Cairns. With appendices by G. Cairns, Y. Carrière, É. Ghys, E. Salem and V. Sergiescu. Progress in Mathematics, **73**, Birkhuser Boston, Inc., Boston, MA, 1988, xii+339 pp.
28. Macias-Virgos E., Sanmartin-Carbón E., *Minimal foliations on Lie groups*, Indag. Mathem., N.S., **3** (**1**), 41–46.
29. O'Neill B., *The fundamental equations of a submersion*, Michigan Math. J., **13** (1966), 459–469.
30. O'Neill B., *Semi-Riemannian geometry. With applications to relativity*, Pure and Applied Mathematics, **103**, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983, xiii+468 pp.
31. Petersen P., *Riemannian geometry*, Graduate Texts in Mathematics, **171**, Springer-Verlag, New York, 1998, xvi+432 pp. ISBN: 0-387-98212-4.

32. Pittie H. V., *Characteristic classes of foliations*, Research Notes in Mathematics, No. **10**, Pitman Publishing, London-San Francisco, Calif.–Melbourne, 1976, v+107 pp.
33. Poor W., *Differential Geometric Structures*, McGraw-Hill Book Company, New York, 1981.
34. Ranjan A., *Structural equations and an integral formula for foliated manifolds*, Geom. Dedicata, **20**, No. **1** (1986), 85–91.
35. Rummler H., *Quelques notions simples en géométrie riemannienne et leurs applications aux feuilletages compacts*, Comment. Math. Helv., **54**, No. **2** (1979), 224–239.
36. Sullivan D., *A homological characterization of foliations consisting of minimal surfaces*, Comment. Math. Helv., **54** (1979), 218–223.
37. Tondeur Ph., *Foliations on Riemannian manifolds*, Springer-Verlag, New York, 1988.
38. Tondeur Ph., *Geometry of Foliations*, Birkhäuser, Basel, Boston, Berlin, 1997.
39. Takagi R., Yorozu S., *Minimal foliations on Lie groups*, Tohoku Math. J. (2), **36**, No. **4** (1984), 541–554.
40. Wu H., *A remark on the Bochner technique in differential geometry*, Proc. Amer. Math. Soc., **78** (1980), 403–408.

Received May 10, 2009

Canisius College
Buffalo, NY 14208
USA
e-mail: escobalr@canisius.edu