UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLVIII
2010

SEMICONTINUITY OF THE ŁOJASIEWICZ EXPONENT

by Arkadiusz Płoski

Abstract. We prove that the Łojasiewicz exponent $l_0(f)$ of a finite holomorphic germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is lower semicontinuous in any multiplicity-constant deformation of f.

1. Introduction. Let $\mathbb{C}\{z\}$ denote the ring of convergent power series in n variables $z = (z_1, \dots, z_n)$. Any sequence of convergent power series $h = (h_1, \dots, h_p) \in \mathbb{C}\{z\}^p$ without constant term defines the germ of a holomorphic mapping $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. We put $\operatorname{ord} h = \inf_k \{\operatorname{ord} h_k\}$, where $\operatorname{ord} h_k$ denotes the order of vanishing of h_k at 0 (by convention $\operatorname{ord} 0 = +\infty$). If $|\underline{z}| = \max_{j=1}^{n} |\underline{z}_j|$ for $\underline{z} = (\underline{z}_1, \dots, \underline{z}_n) \in \mathbb{C}^n$ then $\operatorname{ord} h$ for $h \neq 0$ is the largest $\alpha > 0$ such that $|h(\underline{z})| \leqslant c|\underline{z}|^\alpha$ with a constant $c > 0$ for $\underline{z} \in \mathbb{C}^n$ close to $0 \in \mathbb{C}^n$.

Let $f = (f_1, \dots, f_n) \in \mathbb{C}\{z\}^n$, $f(0) = 0$, define a finite holomorphic germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$; i.e., such that f has an isolated zero at the origin $0 \in \mathbb{C}^n$ and let $\mathrm{I}(f)$ be the ideal of $\mathbb{C}\{z\}$ generated by $f_1, \dots, f_n$. Then $\mathrm{I}(f)$ is of finite codimension in $\mathbb{C}\{z\}$ and the multiplicity $\mathrm{m}_0(f)$ of f is by definition equal to $\dim_{\mathbb{C}} {}^{\mathbb{C}\{z\}}/_{\mathrm{I}(f)}$. There exist arbitrary small neighbourhoods U and V of $0 \in \mathbb{C}^n$ such that the mapping $U \ni \underline{z} \to f(\underline{z}) \in V$ is an $\mathrm{m}_0(f)$-sheeted branched covering (see [**4**], chapter 5, § 2).

Another important characteristic of a finite germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ introduced and studied by M. Lejeune-Jalabert and B. Teissier in the 1973–1974 seminar at the Ecole Polytechnique (in a very general setting), see [**3**], is the Łojasiewicz exponent $l_0(f)$ defined to be the smallest $\theta > 0$ such that

2010 *Mathematics Subject Classification.* Primary: 32S05; Secondary 14B05.
Key words and phrases. Łojasiewicz exponent, multiplicity-constant deformation, Newton polygon.

there exist a neighbourhood U of $0 \in \mathbb{C}^n$ and a constant $c > 0$ such that

$$|f(\underline{z})| \geqslant c|\underline{z}|^\theta \quad \text{for all } \underline{z} \in U.$$

The Łojasiewicz exponent can be calculated by means of analytic arcs (see [**3**], § 5 and [**8**], § 2) $\phi(s) = (\phi_1(s), \ldots, \phi_n(s)) \in \mathbb{C}\{s\}^n$, $\phi(0) = 0$, $\phi(s) \neq 0$ in $\mathbb{C}\{s\}^n$:

$$l_0(f) = \sup_\phi \left\{ \frac{\operatorname{ord} f \circ \phi}{\operatorname{ord} \phi} \right\}.$$

The following lemma ([**7**], Corollary 1.4) will be useful for us.

Lemma 1.1. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a finite holomorphic germ. Then $l_0(f) \leqslant \mathrm{m}_0(f)$ with equality if and only if* $\operatorname{rank}\left(\frac{\partial f_i}{\partial z_j}(0)\right) \geqslant n - 1$.

Now, let $h \in \mathbb{C}\{z\}$, $h(0) = 0$, be a convergent power series defining an isolated singularity at $0 \in \mathbb{C}^n$, i.e., such that the gradient of h, $\nabla h = \left(\frac{\partial h}{\partial z_1}, \ldots, \frac{\partial h}{\partial z_n}\right) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is finite at $0 \in \mathbb{C}^n$. Then $\mu_0 := \mathrm{m}_0(\nabla h)$ is the Milnor number of the singularity $h = 0$. In [**9**], Teissier calculated $\mathcal{L}_0(h) := l_0(\nabla h)$ in terms of polar invariants of the singularity and proved that the Łojasiewicz exponent $\mathcal{L}_0(h)$ is lower semicontinuous in any μ-constant deformation of the singularity $h = 0$. He also showed that if we do not assume $\mu =$ constant, then $\mathcal{L}_0(h)$ is neither upper or lower semicontinuous (see [**10**]). The "jump phenomena" of the Łojasiewicz exponent were rediscovered by some authors (see [**5**]). The aim of this note is to prove that the Łojasiewicz exponent is lower semicontinuous in any multiplicity-constant deformation of the finite holomorphic germ. The proof is based on the formula for the Łojasiewicz exponent given by the author in [**8**] (see also Lemma 3.3 in Section 3).

2. Result. Let $f = (f_1, \ldots, f_n) \in \mathbb{C}\{z\}^n$, $f(0) = 0$, define a finite holomorphic germ. A sequence $F = (F_1, \ldots, F_n) \in \mathbb{C}\{t, z\}^n$ of convergent power series in $k + n$ variables $(t, z) = (t_1, \ldots, t_k, z_1, \ldots, z_n)$ is a deformation of f if $F(0, z) = f(z)$ in $\mathbb{C}\{z\}$ and $F(t, 0) = 0$ in $\mathbb{C}\{t\}$. Then the sequence $(t, F(t, z)) \in \mathbb{C}\{t, z\}^{k+n}$ defines a holomorphic germ $(\mathbb{C}^{k+n}, 0) \to (\mathbb{C}^{k+n}, 0)$ of multiplicity $\mathrm{m}_0(f)$. Indeed, it is easy to check that the algebras ${}^{\mathbb{C}\{z\}}/_{\mathrm{I}(f)}$ and ${}^{\mathbb{C}\{t,z\}}/_{\mathrm{I}(t,F)}$ are $\mathbb{C}$-isomorphic.

We put $F_{\underline{t}} = F(\underline{t}, z) \in \mathbb{C}\{z\}^n$ for $\underline{t} \in \mathbb{C}^k$ close to 0. Then $F_{\underline{t}}(0) = 0$ and $\mathrm{m}_0(F_{\underline{t}}) \leqslant \mathrm{m}_0(F_0) = \mathrm{m}_0(f)$ for $\underline{t} \in \mathbb{C}^k$ close to 0 (see [**13**], chapter 2, § 5). We say that F is a multiplicity-constant deformation of the germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ if $\mathrm{m}_0(F_{\underline{t}}) = \mathrm{m}_0(F_0)$ for $\underline{t}$ close to 0.

The main result of this note is

THEOREM 2.1. *Let $F \in \mathbb{C}\{t,z\}^n$ be a multiplicity-constant deformation of the germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. Then*

$$l_0(F_0) \leqslant l_0(F_{\underline{t}}) \quad \text{for } \underline{t} \in \mathbb{C}^k \text{ close to } 0 \in \mathbb{C}^k.$$

Moreover, if F is a one-parameter deformation ($k = 1$), then $l_0(F_{\underline{t}})$ is constant for $\underline{t} \neq 0$ close to $0 \in \mathbb{C}$.

The proof of the theorem is given in Section 4 of this note. The inequality stated above may be strict:

EXAMPLE 2.2 (see [**5**], § 5). Let $F(t, z_1, z_2) = (tz_1 + z_1^a + z_2^b, z_1^p - z_2^q) \in \mathbb{C}\{t, z_1, z_2\}^2$ be a one-parameter deformation of $f(z_1, z_2) = (z_1^a + z_2^b, z_1^p - z_2^q)$. Assume that $a, b, p, q > 1$ are integers such that $\mathrm{GCD}(p, q) = 1$ and $bp < q$. Then $\mathrm{m}_0(F_{\underline{t}}) = bp$ for all $\underline{t} \in \mathbb{C}$, i.e., F is a multiplicity-constant deformation. If $\underline{t} \neq 0$ then $\operatorname{ord} F_{\underline{t}} = 1$ and we get $l_0(F_{\underline{t}}) = \mathrm{m}_0(F_{\underline{t}}) = bp$ by Lemma 1.1. Since $\operatorname{ord} F_0 > 1$, by the second part of Lemma 1.1, we get that $l_0(F_0) < \mathrm{m}_0(F_0) = bp$.

Note that C. Bivià-Ausina (see [**2**], Corollary 2.5) proved a result on the semicontinuity of the Łojasiewicz exponent which, however, does not imply our Theorem 2.1.

One can also indicate the deformations for which the Łojasiewicz exponent is upper semicontinuous like multiplicity.

PROPOSITION 2.3. *Let $F \in \mathbb{C}\{t,z\}^n$ be a deformation of $f \in \mathbb{C}\{z\}^n$ such that* $\operatorname{rank}\left(\frac{\partial F_i}{\partial z_j}(\underline{t}, 0)\right) \geqslant n - 1$ *for $\underline{t} \in \mathbb{C}^k$ close to $0 \in \mathbb{C}^k$. Then*

$$l_0(F_{\underline{t}}) \leqslant l_0(F_0) \quad \text{for } \underline{t} \in \mathbb{C}^k \text{ close to } 0.$$

PROOF. By Lemma 1.1, we get $l_0(F_{\underline{t}}) = \mathrm{m}_0(F_{\underline{t}})$ for $\underline{t} \in \mathbb{C}^k$ close to 0 and the proposition follows from the upper semicontinuity of the multiplicity. □

EXAMPLE 2.4. Let $f(z) = (z_1^m, z_2, \ldots, z_n)$ with $m > 1$ and let $F(t, z) = f(z_1 + t, z_2, \ldots, z_n) - f(t, 0, \ldots, 0) = ((z_1 + t)^m - t^m, z_2, \ldots, z_n)$ be a one-parameter deformation of f. Then $F(t, z)$ satisfies the assumption of Proposition 2.3. Using Lemma 1.1, we check that $l_0(F_{\underline{t}}) = \mathrm{m}_0(F_{\underline{t}}) = 1$ for $\underline{t} \neq 0$ and $l_0(F_0) = \mathrm{m}_0(F_0) = m$.

In the example above, the deformation of f is given by the translation of coordinates. Even for such a deformation, the Łojasiewicz exponent may be not upper semicontinuous:

EXAMPLE 2.5. Let $f(z_1, z_2, z_3) = (z_1^2, z_2^3, z_3^3 - z_1 z_2) \in \mathbb{C}\{z_1, z_2, z_3\}^3$ and let $F(t, z_1, z_2, z_3) = f(t + z_1, z_2, z_3) - f(t, 0, 0) = (2tz_1 + z_1^2, z_2^3, -tz_2 + z_3^3 - z_1 z_2)$. Then by Lemma 1.1 we get $l_0(F_{\underline{t}}) = \mathrm{m}_0(F_{\underline{t}}) = 9$ for $\underline{t} \neq 0$. On the other hand

$\mathrm{m}_0(F_0) = 18$ and $l_0(F_0) = \frac{18}{5}$ (see Example 3.5 of this note). The exponent $l_0(F_0)$ is attained on the arc $\phi(s) = (s^9, s^6, s^5)$.

Remark 2.6. The case of μ-constant deformations of isolated hypersurface singularities is much more subtle. Teissier's conjecture that "μ-constant implies the constancy of the Łojasiewicz exponent" ([**9**], Question on p. 278) is still open.

3. Characteristic polynomial and the Łojasiewicz exponent. Let $f = (f_1, \ldots, f_n) \in \mathbb{C}\{z\}^n$ be a sequence of convergent power series defining a finite holomorphic germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. Then the extension $\mathbb{C}\{z\} \supset \mathbb{C}\{f\}$ is a finite $\mathbb{C}\{f\}$-module. For any $h \in \mathbb{C}\{z\}$ there is a unique irreducible polynomial $Q_{f,h} = s^{m_h} + c_1(w)s^{m_h-1} + \cdots + c_{m_h}(w) \in \mathbb{C}\{w\}[s]$ in $n+1$ variables $(w, s) = (w_1, \ldots, w_n, s)$ such that $Q_{f,h}(f, h) = 0$. It is called the minimal polynomial of h relative to f. Its degree $m_{f,h} := \deg_s Q_{f,h}$ divides the multiplicity $\mathrm{m}(f)$; we put $P_{f,h} = Q^r_{f,h}$, where $r = \frac{\mathrm{m}(f)}{m_{f,h}}$ and call $P_{f,h}$ the characteristic polynomial of h relative to f. If $h(0) = 0$ then $Q_{f,h}$ and consequently $P_{f,h}$ is a distinguished polynomial.

Remark 3.1. Let $L = \mathbb{C}\{z\}_{(0)}$ and $K = \mathbb{C}\{f\}_{(0)}$ be fields of fractions of the ring $\mathbb{C}\{z\}$ and $\mathbb{C}\{f\}$, respectively. Then $Q_{f,h}(f, s) \in K[s]$ is the monic minimal polynomial of h relative to the field extension L/K and $P_{f,h}(f, s)$ is the characteristic polynomial of h relative to L/K. For the various equivalent definitions of the characteristic polynomial (see Zariski–Samuel [**14**], chapter II, § 10).

The lemma below follows immediately from the Rückert–Weierstrass parametrization theorem (see [**1**], § 31, (31.23)).

Lemma 3.2. *Let $P(w, s) = s^m + a_1(w)s^{m-1} + \cdots + a_m(w) \in \mathbb{C}\{w\}[s]$ be a distinguished polynomial of degree $m = \mathrm{m}_0(f)$ and let $h \in \mathbb{C}\{z\}$, $h(0) = 0$. Then the following two conditions are equivalent:*

(i) *$P(w, s)$ is the characteristic polynomial of h relative to f,*
(ii) *Let U and V be neighbourhoods of $0 \in \mathbb{C}^n$ such that the mapping $U \ni \underline{z} \to f(\underline{z}) \in V$ is a $\mathrm{m}_0(f)$-sheeted branched covering and $h = h(\underline{z})$ is convergent in V. Then the set $\{(\underline{w}, \underline{s}) \in V \times \mathbb{C} : P(\underline{w}, \underline{s}) = 0\}$ is the image of U by the mapping $U \ni \underline{z} \to (f(\underline{z}), h(\underline{z})) \in V \times \mathbb{C}$, provided that U, V are small enough.*

To study the Łojasiewicz exponent $l_0(f)$, it is useful to consider the inequalities of the type

$$(\text{Ł}) \qquad |h(\underline{z})| \leqslant c|f(\underline{z})|^\theta \quad \text{near the origin } 0 \in \mathbb{C}^n.$$

The least upper bound of the set of all $\theta > 0$ for which (Ł) holds for some constant $c > 0$ in a neighbourhood $U \subset \mathbb{C}^n$ of 0 will be denoted $o_f(h)$ and called the Łojasiewicz exponent of h relative to f.

LEMMA 3.3. *Let $P_{f,h}(w,s) = s^m + a_1(w)s^{m-1} + \cdots + a_m(w) \in \mathbb{C}\{w,s\}$ be the characteristic polynomial of $h \in \mathbb{C}\{z\}$, $h \neq 0$, relative to f. Let $I = \{i \in \{1,\ldots,m\} : a_i \neq 0\}$. Then*

$$o_f(h) = \min_{i\in I}\left\{\frac{1}{i}\operatorname{ord} a_i\right\}.$$

PROOF (after [**8**], proof of Theorem 2.3). Let U and V be neighbourhoods of $0 \in \mathbb{C}^n$ such that the mapping $U \ni \underline{z} \to f(\underline{z}) \in V$ is an $\mathrm{m}_0(f)$-sheeted branched covering and $h = h(z)$ converges in V. Let $P(w,s)$ be the characteristic polynomial of h relative to f. Then by Lemma 3.2, the inequality $|h(\underline{z})| \leqslant c|f(\underline{z})|^\theta$, $\underline{z} \in U$, is equivalent to the estimate

$$(*)\qquad \{(\underline{w},\underline{s}) \in V \times \mathbb{C} : P(\underline{w},\underline{s}) = 0\} \subset \{(\underline{w},\underline{s}) \in V \times \mathbb{C} : |\underline{s}| \leqslant |\underline{w}|^\theta\}$$

for U, V small enough.

Let $\Theta_0 = \min_{i\in I}\left\{\frac{1}{i}\operatorname{ord} a_i\right\}$. It is easy to check (see [**6**], Proposition 2.2) that Θ_0 is the largest number $\theta > 0$ for which $(*)$ holds. This proves the lemma. □

LEMMA 3.4. $$l_0(f) = \left(\min_{i=1}^{n}\{o_f(z_i)\}\right)^{-1}.$$

PROOF. Obvious. □

EXAMPLE 3.5. Let us get back to Example 2.5. Let $f = (f_1, f_2, f_3) = (z_1^2, z_2^3, z_3^3 - z_1z_2)$. There is $\mathrm{m}_0(f) = 18$. The characteristic polynomials of z_1 and z_2 are $(s_1^2 - w_1)^9$ and $(s_2^3 - w_2)^6$, respectively; hence $o_f(z_1) = \frac{1}{2}$, $o_f(z_2) = \frac{1}{3}$. To calculate $o_f(z_3)$, let us observe that

$$P(w,s) = (s^3 - w_3)^6 - w_1^3w_2^2$$

is the characteristic polynomial of $h = z_3$ relative to f. Indeed, $P(f, z_3) = 0$ in $\mathbb{C}\{z\}$ and $P(w,s)$ is irreducible: if u is a variable, then $P(u,u,0,s) = s^{18} - u^5$ is irreducible, whence $P(w,s)$ is irreducible.

Write $P(w,s) = s^{18} - 6w_3s^{15} + \cdots + (w_3^6 - w_1^3w_2^2)$. Using Lemma 3.3, we check that $o_f(z_3) = \frac{\operatorname{ord}(w_3^6 - w_1^3w_2^2)}{18} = \frac{5}{18}$. Then we get $l_0(f) = \left(\min\{\frac{1}{2}, \frac{1}{3}, \frac{5}{18}\}\right)^{-1} = \frac{18}{5}$.

LEMMA 3.6. *Let $F = F(t,z) \in \mathbb{C}\{t,z\}^n$ be a multiplicity-constant deformation of a finite germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and let $h \in \mathbb{C}\{z\}$, $h(0) = 0$. Let $P_h(t,w,s) = s^m + a_1(t,w)s^{m-1} + \cdots + a_m(t,w) \in \mathbb{C}\{t,w\}[s]$ be the characteristic polynomial of h relative to $(t, F(t,z))$. Then for $\underline{t} \in \mathbb{C}^k$ close enough to $0 \in \mathbb{C}^k$*

the polynomial $P_h(\underline{t}, w, s) = s^m + a_1(\underline{t}, w)s^{m-1} + \cdots + a_m(\underline{t}, w) \in \mathbb{C}\{w\}[s]$ *is the characteristic polynomial of* h *relative to* $F(\underline{t}, z) \in \mathbb{C}\{z\}^n$.

PROOF. There exist arbitrary small neighbourhoods U and V of $0 \in \mathbb{C}^n$ and W of $0 \in \mathbb{C}^k$ such that the mapping $W \times U \ni (\underline{t}, \underline{z}) \to (\underline{t}, F(\underline{t}, \underline{z})) \in W \times V$ is $\mathrm{m}_0(f)$-sheeted branched covering. Since $F = F(t, z)$ is a multiplicity-constant deformation, also the mappings $U \ni \underline{z} \to F(\underline{t}, \underline{z}) \in V$ for $\underline{t} \in W$ are $\mathrm{m}_0(f)$-sheeted branched coverings. Fix $h = h(z) \in \mathbb{C}\{z\}$, $h(0) = 0$. Shrinking the neighbourhoods $W \times U$ and $W \times V$, by Lemma 3.2, we get that the image of $W \times U$ under the mapping $W \times U \ni (\underline{t}, \underline{z}) \to (\underline{t}, F(\underline{t}, \underline{z}), h(\underline{z})) \in W \times V \times \mathbb{C}$ has the equation $P_h(t, w, s) = 0$ in $W \times V \times \mathbb{C}$. Therefore, the image of U under the mapping $U \ni \underline{z} \to (F(\underline{t}, \underline{z}), h(\underline{z})) \in V \times \mathbb{C}$ has the equation $P_h(\underline{t}, w, s) = 0$ in $V \times \mathbb{C}$. Using again Lemma 3.2, we conclude that $P_h(\underline{t}, w, s)$ is the characteristic polynomial of h relative to $F(\underline{t}, z)$. □

4. Proof of the main result. Let us begin with

THEOREM 4.1. *Let* $F = F(t, z) \in \mathbb{C}\{t, z\}^n$ *be a multiplicity-constant deformation of a finite germ* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. *Let* $h \in \mathbb{C}\{z\}$, $h \neq 0$. *Then*

$$o_{F_{\underline{t}}}(h) \leqslant o_{F_0}(h) \quad \textit{for } \underline{t} \in \mathbb{C}^k \textit{ close to } 0 \in \mathbb{C}^k.$$

Moreover, if F *is a one-parameter deformation* ($k = 1$), *then* $o_{F_{\underline{t}}}(h)$ *is constant for* $\underline{t} \neq 0$ *close to* $0 \in \mathbb{C}$.

PROOF. Let $P_h(t, w, s) = s^m + a_1(t, w)s^{m-1} + \cdots + a_m(t, w) \in \mathbb{C}\{t, w\}[s]$ be the characteristic polynomial of h relative to $(t, F(t, z)) \in \mathbb{C}\{t, z\}^{k+n}$. Then by Lemma 3.6, for $\underline{t} \in \mathbb{C}^k$ close to $0 \in \mathbb{C}^k$, $P_h(\underline{t}, w, s) = s^m + a_1(\underline{t}, w)s^{m-1} + \cdots + a_m(\underline{t}, w) \in \mathbb{C}\{w\}[s]$ is the characteristic polynomial of h relative to $F_{\underline{t}}$. By Lemma 3.3, $o_{F_{\underline{t}}}(h) = \inf_i \left\{ \frac{\operatorname{ord} a_i(\underline{t}, w)}{i} \right\} \leqslant \inf_i \left\{ \frac{\operatorname{ord} a_i(0, w)}{i} \right\} = o_{F_0}(h)$ for $\underline{t} \in \mathbb{C}^k$ close to $0 \in \mathbb{C}^k$, since $\operatorname{ord} a_i(\underline{t}, w) \leqslant \operatorname{ord} a_i(0, w)$ if $|\underline{t}|$ is small. If $k = 1$, then $\operatorname{ord} a_i(\underline{t}, w) \equiv \mathit{const}$ for $\underline{t} \neq 0$ close to $0 \in \mathbb{C}$ and $o_{F_{\underline{t}}}(h) = \mathit{const}$. □

PROOF OF THEOREM 2.1. Use Theorem 4.1 and Lemma 3.4. □

5. Łojasiewicz exponent and the Newton polygon. Let $P(w, s) = s^m + a_1(w)s^{m-1} + \cdots + a_m(w) \in \mathbb{C}\{w, s\}$ be a distinguished polynomial in variables $(w, s) = (w_1, \ldots, w_n, s)$. Put $a_0(w) = 1$ and $I = \{i : a_i \neq 0\}$. The Newton polygon $\mathcal{N}(P)$ of P is defined to be

$$\mathcal{N}(P) = \operatorname{convex} \bigcup_{i \in I} \left((\operatorname{ord} a_i, m - i) + \mathbb{R}_+^2 \right), \quad \text{where } \mathbb{R}_+ = \{a \in \mathbb{R} : a \geqslant 0\}.$$

Then $\mathcal{N}(P)$ intersects the vertical axis at point $(0, m)$ and the horizontal axis at point $(\operatorname{ord} a_m, 0)$ provided that $a_m \neq 0$. Note that $\theta(P) := \inf_i \left\{ \frac{\operatorname{ord} a_i}{i} \right\}$ is equal to the slope of the first side of the Newton polygon $\mathcal{N}(P)$, see [**12**].

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a finite holomorphic germ and let $h \in \mathbb{C}\{z\}$, $h(0) = 0$, $h \neq 0$ in $\mathbb{C}\{z\}$. We put

$$\mathcal{N}(f, h) = \sigma(\mathcal{N}(P_{f,h})),$$

where σ is the symmetry of $\mathbb{R}^2_+$ given by $\sigma(\alpha, \beta) = (\beta, \alpha)$, and call $\mathcal{N}(f, h)$ the Newton polygon of h relative to f.

From the proof of Theorem 4.1 there follows the semicontinuity of the Newton polygon in Teissier's sense (see [**11**] and [**9**]).

Theorem 5.1. *Let $F = F(t, z) \in \mathbb{C}\{t, z\}^n$ be a multiplicity-constant deformation of f. Then*

$$\mathcal{N}(F_{\underline{t}}, h) \subset \mathcal{N}(F_0, h) \quad \textit{for } \underline{t} \in \mathbb{C}^k \textit{ close to } 0.$$

If $k = 1$ then $\mathcal{N}(F_{\underline{t}}.h)$ does not depend on $\underline{t}$ provided that $\underline{t} \neq 0$ is close to $0 \in \mathbb{C}$.

Observe that $\mathcal{N}(f, h)$ intersects the horizontal axis at point $(\mathrm{m}_0(f), 0)$. The intersection of the last edge (with vertex at $(\mathrm{m}_0(f), 0)$) of $\mathcal{N}(f, h)$ is equal to $\frac{1}{o_f(h)}$. We will elsewhere prove that $\mathcal{N}(f, h)$ is identical to the Newton polygon of the pair of ideals $\mathrm{I}(f)$, $\mathrm{I}(h) = (h)\mathbb{C}\{z\}$ introduced by Teissier in [**10**]. In the notation of [**3**], Complément 2 we have $\mathcal{N}(f, h) = \mathcal{N}_{\mathrm{I}(f)}(h)$.

References

1. Abhyankar S. S., *Local Analytic Geometry*, Academic Press, 1964.
2. Bivià-Ausina C., *Local Lojasiewicz exponents, Milnor numbers and mixed multiplicities of ideals*, Math. Z., **262** (2009), 389–409.
3. Lejeune-Jalabert M., Teissier B., *Clôture intégrale des idéaux et équisingularité*, Centre Mathématiques, Université Scientifique et Medical de Grenoble (1974). See also Ann. Fac. Sci. Toulouse Math., (**6**) **17**, No. **4** (2008), 781–859.
4. Łojasiewicz S., *Introduction to Complex Analytic Geometry*, Birkhäuser Verlag, 1991.
5. Mc Neal J., Némethi A., *The order of contact of a holomorphic ideal in* $\mathbb{C}^2$, Math. Z., **250** (**4**) (2005), 873–883.
6. Płoski A., *Une évaluation pour les sous-ensembles analytiques complexes*, Bull. Pol. Acad. Sci. Math., **31** (1983), 259–262.
7. ______, *Sur l'exposant d'une application analytique I*, Bull. Ac. Pol.: Math., **32** (1984), 669–673.
8. ______, *Multiplicity and the Lojasiewicz exponent*, Singularities (Warsaw, 1985), 353–364, Banach Center Publications, 20, PWN, Warsaw, 1985.

9. Teissier B., *Variétés polaires I – Invariant polaires de singularités d'hypersurfaces*, Invent. Math., **40** (1977), 267–292.
10. ______, *Jacobian Newton polyhedra and equisingularity*, Preceedings R.I.M.S. Conference on singularities, Kyoto, April 1978, (Publ. R.I.M.S. 1978).
11. ______, *The hunting of invariants in the geometry of discriminants*, in: *Real and Complex Singularities*, Oslo 1976, Per Holm (1977), 565–677.
12. ______, *Complex Curve Singularities: a biased introduction*, in: *Singularities in geometry and topology*, World Scientific Publishing, Hackensack, NJ, 2007, 825–887.
13. Tougeron J. C., *Idéaux de fonctions différentiables*, Springer-Verlag, 1972.
14. Zariski O., Samuel P., *Commutative Algebra*, Vol. I, Van Nostrand Company, 1958.

Received November 30, 2010

Department of Mathematics
Kielce University of Technology
Al. 1000 L PP 7
25-314 Kielce, Poland
e-mail: matap@tu.kielce.pl