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NOTE ON THE ŁOJASIEWICZ EXPONENT OF WEIGHTED HOMOGENEOUS ISOLATED SINGULARITIES

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Abstract. We compute the Łojasiewicz exponent for some classes of weighted homogeneous isolated singularities.

1. Result. A polynomial $f = f(z_1, \dots, z_n)$ in n complex variables defines an *isolated singularity* at the origin if $f(0) = 0$ and there is an open neighborhood U of $0 \in \mathbb{C}^n$ such that $\{z \in U : \frac{\partial f}{\partial z_1}(z) = \dots = \frac{\partial f}{\partial z_n}(z) = 0\} = \{0\}$.

In singularity theory the invariants of singularities play an important part. The most known invariant is the Milnor number

$$\mu_0(f) = \dim_{\mathbb{C}} {}^{\mathbb{C}[[z_1,\dots,z_n]]}\big/_{\left(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n}\right)}$$

introduced by Milnor in [**3**].

Another important invariant is the *Łojasiewicz exponent* $\mathcal{L}_0(f)$ of f. It is by definition the smallest $\theta > 0$ such that there exists a neighborhood U of $0 \in \mathbb{C}^n$ and a constant $c > 0$ such that

$$|\nabla f(z)| \geqslant c|z|^\theta \quad \text{for all } z \in U.$$

In the above inequality, $|\cdot|$ stands for any norm on $\mathbb{C}^n$. The Łojasiewicz exponent was defined in [**7**] by B. Teissier, who proved the basic properties of $\mathcal{L}_0(f)$. Teissier's conjecture that $\mathcal{L}_0(f)$ is, as $\mu_0(f)$ is, a topological invariant of a singularity is still open.

An interesting class of singularities is defined by weighted homogeneous polynomials. Let $(w_1, \dots, w_n)$ be a sequence of positive rationals. A polynomial $f = f(z_1, \dots, z_n)$ is called *weighted homogeneous* of type $(w_1, \dots, w_n)$ if f may be written as a sum of monomials $cz_1^{a_1} \cdot \dots \cdot z_n^{a_n}$ $(c \neq 0)$ with $\frac{a_1}{w_1} + \dots + \frac{a_n}{w_n} = 1$.

If a weighted homogeneous polynomial f of type $(w_1, \ldots, w_n)$ defines an isolated singularity, then $w_i > 1$ for all $i = 1, \ldots, n$. In [**4**], the authors proved the following formula for the Milnor number of a weighted homogeneous polynomial f of type $(w_1, \ldots, w_n)$ with an isolated singularity:

$$\mu_0(f) = \prod_{i=1}^{n}(w_i - 1).$$

On the other hand, T. Krasiński, G. Oleksik and A. Płoski proved in [**2**] that

$$\mathcal{L}_0(f) \leqslant \min\left(\max_{i=1}^{n}(w_i - 1), \prod_{i=1}^{n}(w_i - 1)\right)$$

with equality for $n = 2$ or $n = 3$. If $n > 3$, the inequality may be strict.

Example 1.1. Let $f = z_1 z_4 + z_2^3 + z_3^3 + z_4^5$. It is easy to see that f is a weighted homogeneous polynomial of type $\left(\frac{5}{4}, 3, 3, 5\right)$ with an isolated singularity at $0 \in \mathbb{C}^4$.

We shall check that

$$\begin{aligned}\mathcal{L}_0(f) = 2 < \min&\left(\max_{i=1}^{4}(w_i - 1), \prod_{i=1}^{4}(w_i - 1)\right)\\ &= \min\left(\max\left\{\frac{1}{4}, 2, 2, 4\right\}, \frac{1}{4}\cdot 2 \cdot 2 \cdot 4\right) = 4.\end{aligned}$$

To compute $\mathcal{L}_0(f)$, let us put $|(z_1, z_2, z_3, z_4)| = |z_1| + |z_2| + |z_3| + |z_4|$ for $z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$. There is $\nabla f(z) = (z_4, 3z_2^2, 3z_3^2, z_1 + 5z_4^4)$ and

$$\begin{aligned}\nabla f(z) &= |z_4| + 3|z_2|^2 + 3|z_3|^2 + |z_1 + 5z_4^4|\\ &\geqslant |z_4| + 3|z_2|^2 + 3|z_3|^2 + |z_1| - 5|z_4|^4\\ &= |z_4|(1 - 5|z_4|^3) + 3|z_2|^2 + 3|z_3|^2 + |z_1|.\end{aligned}$$

If $|z_1| \geqslant 3|z_1|^2$ and $1 - 5|z_4|^3 \geqslant 3|z_4|$, these inequalities hold near $0 \in \mathbb{C}^4$ and

$$|\nabla f(z)| \geqslant 3(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2).$$

Since all norms are equivalent in $\mathbb{C}^4$,

$$|\nabla f(z)| \geqslant c|z|^2 \quad \text{for a } c > 0 \text{ near } 0 \in \mathbb{C}^4;$$

therefore, $\mathcal{L}_0(f) \leqslant 2$.

On the other hand, $\nabla f(0, z_2, 0, 0) = (0, 3z_2^2, 0, 0)$, which implies $\mathcal{L}_0(f) \geqslant 2$. Thus $\mathcal{L}_0(f) = 2$.

Now, let $H(w_1, \ldots, w_n)$ be the set of all weighted homogeneous polynomials of type $(w_1, \ldots, w_n)$ with isolated singularity at $0 \in \mathbb{C}^n$. If $H(w_1, \ldots, w_n) \neq \emptyset$ then we can ask how to compute $\mathcal{L}_0(f)$ for $f \in H(w_1, \ldots, w_n)$ (if Teissier's conjecture holds, then from Milnor–Orlik's formula there follows that $\mathcal{L}_0(f)$ is constant on $H(w_1, \ldots, w_n)$).

EXAMPLE 1.2. Let $d > 1$ be an integer. Then $H(d, \ldots, d)$ is the space of homogeneous polynomials of degree d and $\mathcal{L}_0(f) = d - 1$ for $f \in H(d, \ldots, d)$.

If $n \leqslant 3$, then $\mathcal{L}_0(f) = \min\{\max_{i=1}^n (w_i - 1), \prod_{i=1}^n (w_i - 1)\}$ for $f \in H(w_1, w_2, w_3)$, according to [**2**], Theorem 3. In this note we shall prove

THEOREM 1.3. *Suppose that $H(w) \neq \emptyset$, $w = (w_1, \ldots, w_n)$ and let $r(w) = 2 \cdot \sharp\{i : w_i < 2\} + \sharp\{i : w_i = 2\}$. Then $\prod_{i=1}^{n}(w_i - 1) \geqslant 2^{n-r(w)}$ and $r(w) = n$ if and only if $\mathcal{L}_0(f) = 1$ for $f \in H(w)$.*
Suppose that $r(w) < n$. Then

(i) if $\prod_{i=1}^{n}(w_i - 1) = 2^{n-r(w)}$ then $\mathcal{L}_0(f) = 2$ for $f \in H(w)$,
(ii) if $\prod_{i=1}^{n}(w_i - 1) = 2^{n-r(w)} + 1$ then $\mathcal{L}_0(f) = 3$ for $f \in H(w)$.

We give the proof of Theorem 1.3 in Section 2 of this note. Example 1.1 shows that $H\left(\frac{5}{4}, 3, 3, 5\right) \neq \emptyset$. From Theorem 1.3 *(ii)* we get $\mathcal{L}_0(f) = 2$ for all $f \in H(\frac{5}{4}, 3, 3, 5)$.

2. Proof. Let $\mathrm{Hess}_0(f)$ be the Hesse matrix of f at 0.

LEMMA 2.1. *Suppose that $f = f(z_1, \ldots, z_n)$ has an isolated singularity at $0 \in \mathbb{C}^n$. Let $r = \operatorname{rank} \mathrm{Hess}_0(f) < n$. Then $\mu_0(f) \geqslant 2^{n-r}$,*

(i) if $\mu_0(f) = 2^{n-r}$ then $\mathcal{L}_0(f) = 2$,
(ii) if $\mu_0(f) = 2^{n-r} + 1$ then $\mathcal{L}_0(f) = 3$.

PROOF. Apply Lemma 3.13 from [**5**] to the gradient ∇f. □

LEMMA 2.2. *Let $f = f(z_1, \ldots, z_n)$ be a weighted homogeneous polynomial of type $(w_1, \ldots, w_n)$ with isolated singularity at $0 \in \mathbb{C}^n$. Then*

$$\operatorname{rank} \mathrm{Hess}_0(f) = 2 \cdot \sharp\{i : w_i < 2\} + \sharp\{i : w_i = 2\}.$$

PROOF. The proof is given in [**6**], Theorem 1. □

PROOF OF THEOREM 1.3. Using the Milnor–Orlik formula, Lemma 2.2 and the inequality $\mu_0(f) \geqslant 2^{n-r}$, we get $\prod_{i=1}^{n}(w_i - 1) \geqslant 2^{n-r(w)}$. By Lemma 2.2, $r(w) = n$ if and only if $\operatorname{rank} \mathrm{Hess}_0(f) = n$. If $\operatorname{rank} \mathrm{Hess}_0(f) = n$, then it is easy to check that $\mathcal{L}_0(f) = 1$ (see also [**5**], Corollary 3.8). On the other hand, if $\mathcal{L}_0(f) = 1$, then, by [**5**], Proposition 3.2, we get $\mu_0(f) \leqslant [\mathcal{L}_0(f)]^n = 1$ (here $[\cdot]$ stands for the integral part), thus $\mu_0(f) = 1$ and $\operatorname{rank} \operatorname{Hess}_0(f) = n$.

To show *(i)* and *(ii)*, let us observe that, by Lemma 2.2, $\operatorname{rank} \operatorname{Hess}_0(f) = r(w)$. Use Lemma 2.1. $\square$

Example 2.3 (see [**1**], Example 4.17). Let $f = z_1 z_3 + z_2^2 + z_1^2 z_2$. Then f is weighted homogeneous of type $w = (4, 2, \frac{4}{3})$. Here

$$r(w) = 2 \cdot \sharp\{i : w_i < 2\} + \sharp\{i : w_i = 2\} = 2 \cdot 1 + 1 = 3, \text{ and } n = 3.$$

Therefore, $\mathcal{L}_0(f) = 1$.

Example 2.4. Consider the polynomial $f = z_1 z_3 + z_2^4 + z_3^5$, weighted homogeneous of type $w = (\frac{5}{4}, 4, 5)$. Here we have

$$r = 2 \cdot \sharp\{i : w_i < 2\} + \sharp\{i : w_i = 2\} = 2 \cdot 1 + 0 = 2 \text{ and } n = 3,$$

thus $\prod_{i=1}^{3}(w_i - 1) = \frac{1}{4} \cdot 3 \cdot 4 = 3 = 2^{n-r} + 1$. By Theorem 1.3 *(ii)*, $\mathcal{L}_0(f) = 3$.

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