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SOME PROPERTIES OF CUBIC FORMS OF AFFINE IMMERSIONS

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Abstract. It is proved that for affine immersions of an n-dimensional manifold into an affine space $\mathbb{R}^{n+p}$ with the maximal codimension $p = \frac{n(n+1)}{2}$ the condition that the cubic form is divisible by the second fundamental form implies that the cubic form vanishes. It is also proved that for immersions with pointwise planar normal sections the cubic form of the immersion is divisible by the second fundamental form.

1. Introduction. In this paper we compare three notions concerning affine immersions: a property of having planar normal sections, divisibility of the cubic form by the second fundamental form and being parallel (that is having vanishing cubic form). Our main interest is of affine immersions of the maximal codimension which were studied by Sasaki in [**5**] and by Vrancken in [**6**]. Sasaki introduced the unique equiaffine transversal bundle. The immersion endowed with this bundle has the property that it is a critical point of an appropriate area functional if and only if its mean curvature vanishes (compare [**4, 7**]). Immersions with general codimensions endowed with a transversal bundle constructed by Wiehe in [**7**] have the same property. The construction, however, requires an algebraic condition on the second fundamental form. This condition, known as regularity, does not depend on the choice of a transversal bundle.

We study the notion of divisibility of the cubic form C by the second fundamental form h. Although this notion is defined in a new way in [**8**], we follow the approach of Nomizu and Pinkall [**2**]. We prove that if we consider the unique equiaffine bundle constructed by Sasaki or by Wiehe, this condition implies vanishing of the cubic form. On the other hand, we verify that

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divisibility of C by h is the consequence of the geometric condition that the immersion has pointwise planar normal sections. The last fact is verified here for Sasaki immersions only.

2. Affine immersions with planar normal sections. Let M^n be an n-dimensional manifold with a torsion-free affine connection ∇. Let D be the canonical affine connection in the affine space $\mathbb{R}^{n+p}$ and $p = \frac{n(n+1)}{2}$. Consider a mapping $f : (M^n, \nabla) \to (\mathbb{R}^{n+p}, D)$. Let σ be a vector bundle on M^n. The mapping f is called an affine immersion (see [**3, 2**]) with a bundle σ as the transversal bundle if for every $x \in M^n$ we can write $\mathbb{R}^{n+p} = f_\star(T_x M^n) \oplus \sigma_x$. The equality

$$D_X Y = f_\star \nabla_X Y + h(X, Y)$$

defines a bilinear form h as the second fundamental form of the immersion by the condition that $h(X, Y) \in \sigma$ or tangent vector fields X and Y. We define transversal connection $\nabla^\perp$ in the following way. For a transversal vector field ξ, that is a local section ofσ, and a tangent vector X, $\nabla^\perp_X \xi$ is the transversal part of $D_X \xi$.

We recall the notion of a transversal (or normal) section [**1, 9**]. For a fixed $x \in M^n$ and a fixed vector $v \in T_x M^n$ let $E(x, v) = \mathrm{span}(\sigma_x \cup \{v\})$. The intersection of $E(x, v)$ with the manifold gives locally a curve γ, which we call a transversal section. We can assume that $\gamma(0) = x$ and $\gamma'(0) = v$. We say that the immersion f has (pointwise) planar normal sections if every transversal section at every point satisfies the condition

$$\gamma'(0) \wedge \gamma''(0) \wedge \gamma'''(0) = 0. \tag{2.1}$$

It is equivalent to the following equality:

$$h(v, v) \wedge C(v, v, v) = 0, \tag{2.2}$$

where C denotes the cubic form defined by $C(X, Y, Z) = \nabla^\perp_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$. It is well known that it is a σ-valued tensor and is totally symmetric.

3. Divisibility of the cubic form by the second fundamental form; Sasaki transversal bundle. This notion is thoroughly studied in [**2**]. In the definition we only require an affine immersion. We say that the cubic form C is divisible by the second fundamental form h if there exists a 1-form α on M^n such that

$$C(X, Y, Z) = \alpha(X)h(Y, Z) + \alpha(Y)h(Z, X) + \alpha(Z)h(X, Y). \tag{3.1}$$

It is equivalent to the existence of a 1-form α such that

$$C(X, X, X) = \alpha(X)h(X, X) \tag{3.2}$$

because of the linearity of α as well as multilinearity and symmetry of C and h. We first assume that σ is the unique equiaffine transversal bundle constructed by Sasaki in [**5**]. Since we study local properties only, we identify the submanifold M^n with its image. We assume that $\dim \operatorname{span}\{h(X,Y) : X, Y \in T_xM^n\} = p = \frac{n(n+1)}{2}$. This implies that $h(v,v) \neq 0$ for $v \neq 0$. We will use Greek letters to denote pairs (i,j), where $1 \leq i \leq j \leq n$. It is always possible to choose a local tangent frame $X_1, \ldots, X_n$ such that

$$\mathrm{Det}[X_1, \ldots, X_n, h(X_1,X_1), h(X_1,X_2), \ldots, (X_n,X_n)] = \pm 1,$$

where Det is the determinant in $\mathbb{R}^{n+p}$ and $h(X_i,X_j)$ are arranged in the lexicographic order in the formula. We can also see that $\xi_{ij} = h(X_i,X_j)$ form a local basis of the transversal bundle for $1 \leq i \leq j \leq n$. We will use this basis in the sequel. We define the symmetric bilinear forms h^ρ by the formula $h(X,Y) = h^\rho(X,Y)\xi_\rho$ and their components by $h^\rho_{ij} = h^\rho(X_i,X_j)$. We will identify ξ_{ij} with ξ_{ji} in our notation. Similarly, we define C^ρ as the components of the cubic form C and $C^\rho_{ijk} = C^\rho(X_i,X_j,X_k)$. In the following, we will often use double indices ij instead of the Greek indices. Notice that $h^{kl}_{ij} = h^{kl}(X_i,X_j) = \delta^{kl}_{ij} + \delta^{kl}_{ji}$, where δ^{kl}_{ij} is the Kronecker symbol. By definition, δ^{kl}_{ij} is equal to 1 if $(k,l) = (i,j)$, and to zero otherwise. We use so called pseudo-inverses H^{ij}_ρ of the set of matrices $[h^\rho_{ij}]$ for $\rho = (1,1),(1,2),\ldots,(n,n)$, which are given as follows:

$$H^{ij}_{ab} = \frac{1}{2}(\delta^{ij}_{ab} + \delta^{ji}_{ab}) \quad \text{if } i \neq j,$$

$$H^{ii}_{ab} = \frac{1}{2}\delta^{ii}_{ab}.$$

They have the following algebraic properties (we use the Einstein summation convention):

$$H^{ir}_\rho h^\rho_{rj} = p\delta^i_j, \tag{3.3}$$

$$H^{ij}_\rho h^\gamma_{ij} = n\delta^\gamma_\rho. \tag{3.4}$$

From [**5**] it follows that a transversal bundle is the unique equiaffine bundle defined by Sasaki if and only if

$$\sum_{\rho,k} H^{lk}_\rho C^\rho_{ijk} = 0, \tag{3.5}$$

for all i,j,l. We state

Proposition 1. *Let $f : M^n \to \mathbb{R}^{n+p}$ be an affine immersion for $p = \frac{n(n+1)}{2}$ with the Sasaki transversal bundle. Then the cubic form C is divisible by the second fundamental form h if and only if $C = 0$ everywhere.*

Proof. It is sufficient to prove the "if" part of the proposition only. We fix a transversal and a tangent frames as before the proposition. For every ρ we take $\rho's$ components in equality (3.1) and substitute $X = X_i$, $Y = X_j$, $Z = X_r$ in all the equalities for given $i, j, r = 1, \dots, n$. Taking all triples i, j, r, we obtain equalities of the form

$$C^\rho_{rij} = \alpha(X_r)h^\rho_{ij} + \alpha(X_i)h^\rho_{jr} + \alpha(X_j)h^\rho_{ri}.$$

Then we fix r, multiply each of these equations by H^{ij}_ρ and sum them over i, j and ρ. We obtain

$$0 = H^{ij}_\rho C^\rho_{rij} = \alpha(X_i)H^{ij}_\rho h^\rho_{jr} + \alpha(X_j)H^{ij}_\rho h^\rho_{ri} + \alpha(X_r)H^{ij}_\rho h^\rho_{ij}.$$

Using (3.3) and (3.4), we get $0 = \sum \alpha(X_i)p\delta^i_r + \sum \alpha(X_j)p\delta^j_r + np\alpha(X_r)$, whence $0 = p(n+2)\alpha(X_r)$. Because it is valid for every r, $\alpha = 0$ identically and, as a consequence, $C = 0$ identically. □

Remark 1. Proposition 1 is also true for immersions $f : M^n \to \mathbb{R}^{n+p}$ with $2 \leq p \leq \frac{n(n+1)}{2}$, which are regular in the sense of Wiehe and are equipped with a Wiehe transversal bundle. Equation (5.2) and Theorem 5.3 in [**7**] indicate that h and C come from this bundle if and only if equality (3.5) is satisfied, where H^{lk}_ρ are pseudo-inverse elements with respect to the components h^ρ of the second fundamental form (see [**7**] for details). Moreover, pseudo-inverses satisfy equations (3.3) and (3.4). Thus, we can apply the proof of the Proposition 1 to this situation.

4. Main results. We now consider immersions with the maximal codimension, with pointwise planar normal sections. In the following proposition, we do not give any assumption about a transversal bundle.

Proposition 2. *Let $f : M^n \to \mathbb{R}^{n+p}$, where $p = \frac{n(n+1)}{2}$ be an immersion with an arbitrary transversal bundle σ. Assume that for every $x \in M^n$* $\dim \operatorname{span}\{h(X,Y) : X, Y \in T_xM^n\} = p$. *Then the immersion has pointwise planar normal sections if and only if its cubic form C is divisible by the second fundamental form h.*

Proof. First, we notice that the "only if" part is true by equality (2.2). Assume that the immersion has pointwise planar normal sections. Fix $x \in M^n$ and restrict ourselves to vector spaces T_xM^n and σ_x. Since $h(v,v) \neq 0$ for $v \neq 0$, because of the maximality of the codimension p, the equality (2.2) allows us to define a map $\alpha : T_xM^n \to \mathbb{R}$ such that

$$C(v,v,v) = \alpha(v)h(v,v) \tag{4.1}$$

for $v \neq 0$. We put $\alpha(0) = 0$. It is enough to prove that α is linear. We observe that if we put tv instead of v in (4.1) for a real number t, we get $\alpha(tv) = t\alpha(v)$.

Then we fix two linearly independent vectors $u, w \in T_xM^n$ and put a linear combination $v = v(t,s) = tu + sw$ into (4.1). We can see that the mapping $(t,s) \mapsto \alpha(tu+sw)$ is smooth for $(t,s) \neq (0,0)$ by definition of α. We obtain

$$\begin{aligned} t^3C(u,u,u) + 3t^2sC(u,u,w) + ts^2C(u,w,w) + s^3C(w,w,w) \\ = \alpha(tu+sw)(t^2h(u,u) + 2tsh(u,w) + s^2h(w,w)). \end{aligned} \tag{4.2}$$

Since p is maximal, vectors $h(u,u)$, $h(u,w)$ and $h(w,w)$ are linearly independent in the transversal space σ_x. We can assume that $h(u,u) = \xi_1$. Then (4.2) becomes

$$t^3C^1(u,u,u)+3t^2sC^1(u,u,w)+ts^2C^1(u,w,w)+s^3C^1(w,w,w) = t^2\alpha(tu+sw).$$

Putting $t = 0, s \neq 0$, we get $C^1(w,w,w) = 0$ and $t^2C^1(u,u,u)+3tsC^1(u,u,w)+s^2C^1(u,w,w) = t\alpha(tu+sw)$ for $t \neq 0$. As (t,s) tends to $(0,1)$, we get $C^1(u,w,w) = 0$ and $\alpha(tu+sw) = tC^1(u,u,u)+3sC^1(u,u,w)$. If $(t,s) \to (0,1)$, then $\alpha(w) = 3C^1(u,u,w)$ and for $(t,s) \to (1,0)$ we get $\alpha(u) = C^1(u,u,u)$. Thus, $\alpha(tu+sw) = t\alpha(u) + s\alpha(w)$ for $(t,s) \neq (0,0)$. Defining α in the same way for every $x \in M^n$, we conclude that it is a one form on M^n. The proof is completed. □

Combining Propositions 1 and 2, we can formulate the main theorem of the paper.

Theorem 1. *Let $f : M^n \to \mathbb{R}^{n+p}$, where $p = \frac{n(n+1)}{2}$, be an immersion with the Sasaki transversal bundle. Then the immersion has planar normal sections if and only if $C = 0$ identically.*

The next result involves properties of the function α in (3.2). Here, a transversal bundle is arbitrary.

Proposition 3. *Let $f : M^n \to \mathbb{R}^{n+p}$ be an affine immersion with an arbitrary transversal bundle. Assume that there is a function α such that $C(v,v,v) = \alpha(v)h(v,v)$ for non-zero tangent vectors v and $\alpha(0) = 0$. Moreover, let h be non-degenerate in the sense that for every tangent vector v there is a tangent vector w such that $h(v,w) \neq 0$. If α is a differentiable function on TM^n, then it is a one form.*

Proof. Let α be a differentiable function. Let u, w be linearly independent tangent vectors and let $v = v(t,s) = tu + sw$, where $t, s \in \mathbb{R}$. Consider the two cases: either $h(u,u) = h(u,w) = h(w,w) = 0$ or not. In the second case, we get equation (4.2) with $v = tu + sw$. Looking at its non-vanishing component, we can see that a function $(t,s) \mapsto \alpha(v(t,s))$ is a quotient of two homogeneous polynomials for $(t,s) \neq (0,0)$. Since this function is also differentiable at $(0,0)$, this quotient has to be a first degree polynomial. Since $\alpha(tv) = t\alpha(v)$ for any tangent vector v and $t \in \mathbb{R}$, $\alpha(tu+sw) = t\alpha(u) + s\alpha(w)$. In the first case, we fix (t,s) and consider a tangent vector z such that $h(v(t,s),z) \neq 0$ and, consequently, it is non-zero for (t,s) in an open set. Then we put a vector of the form $kv(t,s)+lz$ in (3.2) for $k,l \in \mathbb{R}$, substitute $k=1$ and $l=0$, obtaining $C(v(t,s),v(t,s),v(t,s))=0$. After dividing the equation by l and letting (k,l) converge to $(1,0)$, we get

$$3C(v(t,s),v(t,s),z) = 2\alpha(v(t,s))h(v(t,s),z).$$

If we treat the non-vanishing component of $h(v(t,s),z)$ and the same component of $C(v(t,s),v(t,s),z)$ as homogeneous polynomials, the differentiability of α implies its linearity. Thus, α is a one form. □

We recall (see [**7**]) that regularity of an immersion in the sense of Wiehe implies that the second fundamental form h is non-degenerate as in the assumption of the previous proposition. Combining the results of Proposition 3 and Remark 1, we get

Corollary 1. Let $f : M^n \to \mathbb{R}^{n+p}$ be a regular affine immersion with a Wiehe transversal bundle. Assume that there is a function α such that $C(v,v,v) = \alpha(v)h(v,v)$ for non-zero tangent vectors v and $\alpha(0)=0$. If α is a differentiable function on TM^n, then $C=0$ identically.

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