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A FACTORISABLE DERIVATION OF POLYNOMIAL RINGS IN n VARIABLES

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Abstract. Let $k[x_1,\dots,x_n]$ be the polynomial ring in $n \geqslant 3$ variables over a field k of characteristic zero, and let Δ be the factorisable derivation of $k[x_1,\dots,x_n]$ defined by $\Delta(x_i) = x_i(S - x_i)$, for $i = 1,\dots,n$, where $S = x_1 + \cdots + x_n$. We prove that this derivation has no nontrivial polynomial constants, and we describe the field of its rational constants.

Introduction. Throughout this paper k is a field of characteristic zero, $k[X] = k[x_1,\dots,x_n]$ is the polynomial ring in $n \geqslant 3$ variables over k, and $k(X) = k(x_1,\dots,x_n)$ is the field of quotients of $k[X]$, that is $k(X)$ is the field of rational functions in n variables over k.

If R is a commutative k-algebra, then a k-linear mapping $d : R \to R$ is said to be a k-*derivation* (or simply a *derivation*) of R if $d(ab) = ad(b) + bd(a)$ for all $a, b \in R$. In this case we denote by R^d the k-algebra of constants of R with respect to d, that is, $R^d = \{r \in R;\ d(r) = 0\}$. Note that if R is a field, then R^d is a subfield of R containing k.

If $f_1,\dots,f_n$ are polynomials belonging to $k[X]$, then there exists exactly one derivation $d : k[X] \to k[X]$ such that $d(x_1) = f_1,\dots,d(x_n) = f_n$. This derivation is of the form

$$d = f_1\frac{\partial}{\partial x_1} + \cdots + f_n\frac{\partial}{\partial x_n}.$$

Every derivation d of $k[X]$ has a unique extension to a derivation of the field $k(X)$; also this extension we denote by d. Thus, for any derivation d of $k[X]$, there is the ring $k[X]^d = \{f \in k[X]; d(f) = 0\}$ and the field $k(X)^d = \{\varphi \in$

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$k(X);\ d(\varphi) = 0\}$. Of course, $k(X)^d$ contains the field of quotients of $k[X]^d$, but in many cases these fields are different (see [**21, 20**]). We are mainly interested in some descriptions of $k[X]^d$ and $k(X)^d$. However, we know that, in general, such descriptions are very difficult to obtain. Rings and fields of constants appear in various classical problems; for details we refer the reader to [**3**] and [**20**].

The mentioned problems are also very difficult for factorisable derivations. We say that a derivation $d : k[X] \to k[X]$ is *factorisable* (or *factorizable*) if

$$d(x_i) = x_i(a_{i1}x_1 + \cdots + a_{in}x_n),$$

for all $i = 1, \ldots, n$, where each a_{ij} belongs to k. Factorisable derivations and factorisable systems of ordinary differential equations have intensively been studied for a long time; see for example [**5, 4, 19**] and [**20**], where numerous references to this subject can be found. With any given derivation d of $k[X]$, using a special procedure, we may associate a factorisable derivation δ (see [**24**] for details). There exist derivations for which the problem of descriptions of $k[X]^d$ or $k(X)^d$ reduces to the same problem for the factorisable derivation associated with a given derivation. We know from [**22**] and [**18**] that this is the case if the derivation d is monomial, that is, if all the polynomials $d(x_1), \ldots, d(x_n)$ are monomials. Consider, for example, a cyclic monomial derivation $d : k[X] \to k[X]$ defined by

$$d(x_1) = x_2^s,\ d(x_2) = x_3^s,\ \ldots,\ d(x_{n-1}) = x_n^s,\ d(x_n) = x_1^s,$$

where $n \geqslant 3$ and $s \geqslant 2$. Such d is called a Jouanolou derivation ([**8, 19, 9, 26**]). The factorisable derivation δ, associated with d, is a derivation of $k[X]$ defined by

$$\delta(x_i) = x_i(sx_{i+1} - x_i),$$

for $i = 1, \ldots, n$, where $x_{n+1} = x_1$ ([**9, 26**]). In 2003, H. Żołądek [**26**] proved that the field of constants of the factorisable derivation δ is trivial, that is, $k(X)^\delta = k$. As a consequence of this fact (and some results from [**9**]), he proved that the above Jouanolou derivation d has no Darboux polynomials; in particular, he proved that also the field of constants of d is trivial. Let us recall ([**19, 20**]) that a polynomial $F \in k[X]$ is a Darboux polynomial of d if $F \notin k$ and $d(F) = \Lambda F$ for some $\Lambda \in k[X]$. Derivations without Darboux polynomials are intensively studied in many papers ([**16, 10, 17**].

Examples of factorisable derivations are the famous Lotka–Volterra derivations for $n = 3$ (see for example: [**11, 12, 15, 13, 14**]). A *Lotka–Volterra derivation* is a derivation $d : k[x, y, z] \to k[x, y, z]$ such that

$$d(x) = x(Cy + z), \quad d(y) = y(Az + x), \quad d(z) = z(Bx + y),$$

where $A, B, C \in k$. There also exist some specific generalizations of Lotka–Volterra derivations, for polynomial rings in $n \geqslant 4$ variables. One of such generalizations is the derivation $D : k[X] \to k[X]$ defined by

$$D(x_i) = x_i\,(x_{i-1} - x_{i+1})\,,$$

for $i = 1, \dots, n$, where $x_0 = x_n$ and $x_{n+1} = x_1$. Such D is called either a *Lotka–Volterra derivation* ([**6, 1, 23**]) or a *Volterra derivation* ([**2, 25**]). It is not easy to describe the ring of constants of D for arbitrary $n \geqslant 3$. If $n = 3$, then some description is given in [**15**]. P. Ossowski and J. Zieliński ([**23**]) determined all polynomial constants for $n = 4$. Recently, Zieliński ([**25**]) presented such description for $n = 5$. Hence, we know a structure of $k[X]^D$ for $n \leqslant 5$ only. For $n \geqslant 6$, the problem is open. There are similar open problems concerning the field $k(X)^D$, even for $n < 6$. It is a natural question what happens if in the above derivation D we change the sign before x_{i+1}, that is, if

$$D(x_i) = x_i\,(x_{i-1} + x_{i+1})$$

for $i = 1, \dots, n$. In particular, if $n = 3$, then D is a cyclic derivation of $k[x, y, z]$ such that

$$D(x) = x(y + z), \quad D(y) = y(z + x), \quad D(z) = z(x + y).$$

There are no results concerning $k[X]^D$ and $k(X)^D$ for an arbitrary n.

In this paper, we consider a similar factorisable derivation $\Delta : k[X] \to k[X]$, defined by

$$\Delta(x_i) = x_i\,(S - x_i)\,,$$

for $i = 1, \dots, n$, where S is the sum $x_1 + \cdots + x_n$. We prove that, for an arbitrary $n \geqslant 3$, the ring of constant of Δ is trivial, that is, $k[X]^\Delta = k$. Moreover, we prove that the field $k(X)^\Delta$ is generated by $n - 1$ algebraically independent rational functions; we also present some explicit formulas for generators. Note that if $n = 3$, then Δ coincides with the above mentioned derivation D of $k[x, y, z]$.

1. Polynomial constants. Let us recall that $\Delta : k[X] \to k[X]$ is the factorisable derivation of the polynomial ring $k[X] = k[x_1, \dots, x_n]$ defined by

$$\Delta(x_i) = x_i\,(S - x_i)\,,$$

for $i = 1, \dots, n$, where $n \geqslant 3$, k is a field of characteristic zero and S is the sum $x_1 + \cdots + x_n$. In this section, using a method described in [**19**] and [**20**], we prove that the ring of constants of Δ is equal to k. Note that the derivation Δ is homogeneous; all the elements $\Delta(x_1), \dots, \Delta(x_n)$ are nonzero homogeneous polynomials od degree 2. Hence, if there exists a nontrivial polynomial constant of Δ, then there exists such a constant which is homogeneous.

Let us assume that $F \in k[X]$ is a nonzero homogeneous polynomial of degree $m \geqslant 1$ such that $\Delta(F) = 0$. Then $x_1(S-x_1)\frac{\partial F}{\partial x_1}+\cdots+x_n(S-x_n)\frac{\partial F}{\partial x_n} = 0$ and, since F is homogeneous, $x_1\frac{\partial F}{\partial x_1} + \cdots + x_n\frac{\partial F}{\partial x_n} = mF$. As a combination of these two equalities we obtain the equality

$$(1) \qquad x_1(x_1 - x_n)\frac{\partial F}{\partial x_1} + \cdots + x_{n-1}(x_{n-1} - x_n)\frac{\partial F}{\partial x_{n-1}} = m(S - x_n)F,$$

which does not include the last partial derivative.

Let $\varphi : k[X] = k[x_1, \dots, x_n] \to k[x_1, \dots, x_{n-1}]$ be the k-algebra homomorphism such that $\varphi(x_i) = x_i$ for $i = 1, \dots, n-1$, and $\varphi(x_n) = 1$. This homomorphism commutes with the partial derivatives $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}$, that is, $\varphi \circ \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i} \circ \varphi$, for $i = 1, \dots, n-1$. Denote by $\overline{F}$ the image of F with respect to φ, that is,

$$\overline{F} = \varphi(F) = F(x_1, \dots, x_{n-1}, 1).$$

Note that $\overline{F}$ is a polynomial belonging to $k[x_1, \dots, x_{n-1}]$. Put $z = x_n$ and let

$$F = F_0 z^m + F_1 z^{m-1} + \cdots + F_{m-1} z + F_m,$$

where each F_i (for $i = 0, 1, \dots, m$) is either zero or a nonzero homogeneous polynomial, belonging to $k[x_1, \dots, x_{n-1}]$, of degree i. Then we obtain the equality

$$\overline{F} = F_0 + F_1 + \cdots + F_m,$$

which is the decomposition of $\overline{F}$ into homogeneous components. Since $F \neq 0$, there exists $i \in \{0, 1, \dots, m\}$ such that $F_i \neq 0$, and this implies that $\overline{F}$ is a nonzero polynomial. Suppose that $F_0 \neq 0$ and $F_1 = F_2 = \cdots = F_m = 0$. Then $F = az^m$, where $0 \neq a \in k$, $z = x_n$. But $\Delta(F) = 0$, so $0 = \Delta(ax_n^m) = amx_n^m(S - x_n) \neq 0$; a contradiction. Therefore, $\overline{F}$ is a nonzero polynomial of degree p, where $1 \leqslant p \leqslant m$. Moreover, by (1), there follows:

$$\begin{aligned} m(x_1 + \cdots + x_{n-1})\overline{F} &= m\varphi(S - x_n)\varphi(F) = \varphi\Big(m(S - x_n)F\Big) \\ &= \varphi\Big(x_1(x_1 - x_n)\frac{\partial F}{\partial x_1} + \cdots + x_{n-1}(x_{n-1} - x_n)\frac{\partial F}{\partial x_{n-1}}\Big) \\ &= x_1(x_1 - 1)\frac{\partial \overline{F}}{\partial x_1} + \cdots + x_{n-1}(x_{n-1} - 1)\frac{\partial \overline{F}}{\partial x_{n-1}}. \end{aligned}$$

Hence, the polynomial $\overline{F}$ satisfies the equality

$$(2) \quad x_1(x_1 - 1)\frac{\partial \overline{F}}{\partial x_1} + \cdots + x_{n-1}(x_{n-1} - 1)\frac{\partial \overline{F}}{\partial x_{n-1}} = m(x_1 + x_2 + \cdots + x_{n-1})\overline{F}.$$

Let $\sigma : k[x_1, \ldots, x_{n-1}] \to k[x_1, \ldots, x_{n-1}]$ be the affine k-algebra automorphism defined by

$$\sigma(x_i) = x_i + 1, \quad \text{for} \quad i = 1, \ldots, n-1.$$

This homomorphism also commutes with all the partial derivatives $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}$, that is, $\sigma \circ \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i} \circ \sigma$, for $i = 1, \ldots, n-1$. Consider the polynomial

$$G = \sigma\left(\overline{F}\right) = \overline{F}(x_1 + 1, \ldots, x_{n-1} + 1).$$

It is clear that $G \neq 0$, $\deg G = \deg \overline{F} = p$ with $1 \leqslant p \leqslant m$. Moreover, by (2), we obtain

$$\begin{aligned}(x_1+1)x_1\frac{\partial G}{\partial x_1} + \cdots + (x_{n-1}+1)x_{n-1}\frac{\partial G}{\partial x_{n-1}} \\ = m\Big(x_1 + x_2 + \cdots + x_{n-1} + (n-1)\Big)G.\end{aligned} \tag{3}$$

It follows from the above equality that $G(0, \ldots, 0) = 0$.

Let H be the nonzero homogeneous component of G of the minimal degree. Put $q = \deg H$. Since $G \neq 0$ and $G(0, \ldots, 0) = 0$, then $q \geqslant 1$. Thus, H is a nonzero homogeneous polynomial, belonging to $k[x_1, \ldots, x_{n-1}]$, and $\deg H = q$ with $1 \leqslant q \leqslant p \leqslant m$.

Comparing in the homogeneous components of the smallest degree in (3), we obtain the equality

$$x_1\frac{\partial H}{\partial x_1} + \cdots + x_{n-1}\frac{\partial H}{\partial x_{n-1}} = m(n-1)H.$$

But H is homogeneous, so by Euler's identity we have

$$x_1\frac{\partial H}{\partial x_1} + \cdots + x_{n-1}\frac{\partial H}{\partial x_{n-1}} = qH.$$

Hence, $q = (n-1)m$ and we have a contradiction: $2m \leqslant (n-1)m = q \leqslant m$.

Thus we have proved the following theorem.

Theorem 1.1. *For any $n \geqslant 3$, the derivation Δ has no nontrivial polynomial constants. In other words:*

$$k[X]^{\Delta} = \Big\{F \in k[X];\ d(F) = 0\Big\} = k.$$

2. An extension of Δ. In this section we will show that the derivation Δ is associated with some simple monomial derivation δ of a polynomial ring in n variables over k.

We denote by $k[Y]$ the polynomial ring $k[y_1,\dots,y_n]$, by $k(Y)=k(y_1,\dots,y_n)$ the field of quotients of $k[Y]$, and by π the product $y_1y_2\cdots y_n$. Moreover, we use notations:

$$u_1 = y_1 - y_n,\ u_2 = y_2 - y_n,\ \dots,\ u_{n-1} = y_{n-1} - y_n.$$

Let us consider the unique derivation $\delta : k[Y] \to k[Y]$ such that $\delta(y_i) = \pi$ for $i = 1,\dots,n$. This derivation is of the form $\delta = \pi\delta_0$, where

$$\delta_0 = \frac{\partial}{\partial y_1} + \cdots + \frac{\partial}{\partial y_n}.$$

The polynomials $u_1,\dots,u_{n-1}$ are constants with respect to δ, and we have the following proposition holds.

Proposition 2.1. $k[Y]^\delta = k[u_1,\dots,u_{n-1}]$, $k(Y)^\delta = k(u_1,\dots,u_{n-1})$.

Proof. Observe that $\sigma\delta_0\sigma^{-1} = \frac{\partial}{\partial y_n}$, where $\sigma : k[Y] \to k[Y]$ is the k-algebra automorphism defined by $\sigma(y_i) = y_i + y_n$ for $i = 1,\dots,n-1$, and $\sigma(y_n) = y_n$. Hence,

$$\begin{aligned} k[Y]^\delta = k[Y]^{\delta_0} &= \sigma^{-1}\Big(k[Y]^{\partial/\partial y_n}\Big) = \sigma^{-1}\Big(k[y_1,\dots,y_{n-1}]\Big) \\ &= k[\sigma^{-1}(y_1),\dots,\sigma^{-1}(y_{n-1})] = k[u_1,\dots,u_{n-1}] \end{aligned}$$

and, by the same argument, $k(Y)^\delta = k(u_1,\dots,u_{n-1})$. □

Now we introduce the elements $x_1,\dots,x_n$, which are polynomials, belonging to $k[Y]$, defined by $x_i = \frac{\pi}{y_i}$ for $i = 1,\dots,n$, that is,

$$x_1 = y_2y_3\cdots y_n,\quad x_2 = y_1y_3y_4\cdots y_n,\quad \dots,\quad x_n = y_1y_2\cdots y_{n-1}.$$

Proposition 2.2. *The above polynomials $x_1,\dots,x_n$ are algebraically independent over k.*

Proof. It is enough to prove (see for example [**7**]) that the Jacobian $\det[\partial x_i/\partial y_j]$ is nonzero. Observe that

$$\det[\partial x_i/\partial y_j] = \begin{vmatrix} 0 & \frac{x_1}{y_2} & \frac{x_1}{y_3} & \cdots & \frac{x_1}{y_n} \\ \frac{x_2}{y_1} & 0 & \frac{x_2}{y_3} & \cdots & \frac{x_2}{y_n} \\ \frac{x_3}{y_1} & \frac{x_3}{y_2} & 0 & \cdots & \frac{x_3}{y_n} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{x_n}{y_1} & \frac{x_n}{y_2} & \frac{x_n}{y_3} & \cdots & 0 \end{vmatrix} = \frac{x_1\cdots x_n}{y_1\cdots y_n}\det M,$$

where M is the following $n \times n$ matrix:

$$\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}.$$

It is easy to check that $\det M = (-1)^{n-1}(n-1)$. Hence, the Jacobian $\det[\partial x_i/\partial y_j]$ is nonzero. □

PROPOSITION 2.3. *If $x_1, \dots, x_n$ are the polynomials described above, then*

$$\delta(x_i) = x_i(S - x_i),$$

for $i = 1, \dots, n$, where $S = x_1 + \cdots + x_n$.

PROOF. Let us recall that $\delta(y_i) = \pi$ and $x_i = \pi/y_i$ for $i = 1, \dots, n$, where $\pi = y_1 \cdots y_n$. For $i = 1$:

$$\begin{aligned} \delta(x_1) &= \delta(y_2 y_3 \cdots y_n) \\ &= \delta(y_2) y_3 y_4 \cdots y_n + y_2 \delta(y_3) y_4 \cdots y_n + \cdots + y_2 y_3 \cdots y_{n-1} \delta(y_n) \\ &= \pi y_3 y_4 \cdots y_n + y_2 \pi y_4 \cdots y_n + \cdots + y_2 y_3 \cdots y_{n-1} \pi \\ &= \frac{\pi}{y_1}\frac{\pi}{y_2} + \frac{\pi}{y_1}\frac{\pi}{y_3} + \cdots + \frac{\pi}{y_1}\frac{\pi}{y_n} \\ &= x_1 x_2 + x_1 x_3 + \cdots + x_1 x_n = x_1(S - x_1). \end{aligned}$$

We may repeat the same for any $i = 2, \dots, n$ and hence $\delta(x_i) = x_i(S - x_i)$ for $i = 1, \dots, n$. □

Since $x_1, \dots, x_n$ are algebraically independent over the field k (see Proposition 2.2), we have the polynomial ring $k[X] = k[x_1, \dots, x_n]$. Thus, we have two polynomial rings:

$$k[X] = k[x_1, \dots, x_n] \quad \text{and} \quad k[Y] = k[y_1, \dots, y_n],$$

and $k[X]$ is a subring of $k[Y]$. We also have the field extension $k(X) \subset k(Y)$, where $k(X) = k(x_1, \dots, x_n)$ and $k(Y) = k(y_1, \dots, y_n)$. It follows from [**18**] that the extension $k(X) \subset k(Y)$ is Galois, and $\dim_{k(X)} k(Y) = n - 1$, but we do not need such information.

Observe that, by Proposition 2.3, $\delta(k[X]) \subseteq k[X]$ and the restriction of the derivation δ to $k[X]$ is exactly equal to the derivation Δ. We already know that $k[X]^\Delta = k$ (see Theorem 1.1). Our aim is to describe the field $k(X)^\Delta$. Now we know that $k(X)^\Delta$ is a subfield of the field $k(Y)^\delta$, which, by Proposition 2.1, is equal to the field $k(u_1, \dots, u_{n-1})$. Moreover, $k(X)^\Delta = k(Y)^\delta \cap k(X)$.

3. Rational constants. We use the same notations as in the previous section. Put also

$$W := x_1 \cdots x_n \quad \text{and} \quad N := n-1.$$

Observe that $W = \frac{\pi}{y_1} \cdots \frac{\pi}{y_n} = \frac{\pi^n}{\pi} = \pi^N$. Moreover,

$$x_i^N = \left(\frac{\pi}{y_i}\right)^N = \frac{W}{y_i^N},$$

so $y_i^N = \frac{W}{x_i^N}$ for $i = 1, \ldots, n$. Thus, the following lemma is true.

LEMMA 3.1. *The powers* $y_1^N, \ldots, y_n^N$ *and* π^N *belong to* $k(X)$.

Note also that

$$\frac{x_j}{x_i} = \frac{\pi/y_j}{\pi/y_i} = \frac{y_i}{y_j},$$

and hence each quotient $\frac{y_i}{y_j}$, for $i, j \in \{1, \ldots, n\}$, belongs to the field $k(X)$.

LEMMA 3.2. *If* $i_1, \ldots, i_n$ *are integers such that the sum* $i_1 + \cdots + i_n$ *is divisible by* N, *then* $y_1^{i_1} \cdots y_n^{i_n}$ *belongs to* $k(X)$.

PROOF. Let $i_1 + \cdots + i_n = aN$ with $a \in \mathbb{Z}$. Then

$$y_1^{i_1} \cdots y_n^{i_n} = y_1^{aN} y_1^{-(i_1+\cdots+i_n)} y_1^{i_1} \cdots y_n^{i_n} = \left(y_1^N\right)^a \left(\frac{y_2}{y_1}\right)^{i_2} \cdots \left(\frac{y_n}{y_1}\right)^{i_n},$$

and this lemma follows from the previous observations. □

Let us recall that $u_i = y_i - y_n$ for $i = 1, \ldots, N$.

LEMMA 3.3. *The powers* $u_1^N, \ldots, u_N^N$ *belong to* $k(X)$.

PROOF. Since $u_i^N = (y_i - y_n)^N = \sum_{p+q=N} a_{pq} y_i^p y_n^q$, where each a_{pq} is an integer, the result follows from Lemma 3.2. □

LEMMA 3.4. *Each quotient* $\frac{u_i}{u_j}$, *for* $i, j \in \{1, \ldots, N\}$, *belongs to* $k(X)$.

PROOF. The result follows from the equalities

$$\frac{u_i}{u_j} = \frac{y_i - y_n}{y_j - y_n} = \frac{(y_i - y_n)\left(y_j^{N-1} + y_j^{N-2} y_n + \cdots + y_n^{N-1}\right)}{y_j^N - y_n^N}$$

and the previous lemmas. □

As a consequence of the above lemmas, we obtain the following proposition.

Proposition 3.5. *If $i_1, \dots, i_N$ are integers such that the sum $i_1 + \cdots + i_N$ is divisible by N, then $u_1^{i_1} \cdots u_N^{i_N}$ belongs to the field of constants $k(X)^\Delta$. In particular, all rational functions of the forms u_i^N and $\frac{u_i}{u_j}$, for $i, j \in \{1, \dots, N\}$, belong to $k(X)^\Delta$.*

Proof. Let $i_1 + \cdots + i_N = aN$ with $a \in \mathbb{Z}$, and put $\gamma = u_1^{i_1} \cdots u_N^{i_N}$. Then

$$\gamma = u_1^{aN} u_1^{-(i_1+\cdots+i_N)} u_1^{i_1} \cdots u_N^{i_N} = \left(u_1^N\right)^a \left(\frac{u_2}{u_1}\right)^{i_2} \cdots \left(\frac{u_N}{u_1}\right)^{i_N},$$

and hence, by Lemmas 3.3 and 3.4, the element γ belongs to $k(X)$. But γ belongs also to the field $k(u_1, \dots, u_N)$, which is equal to $k(Y)^\delta$ (see Proposition 2.1). Recall that $k(X)^\Delta = k(Y)^\delta \cap k(X)$. Therefore, $\gamma \in k(X)^\Delta$. □

Lemma 3.6. *The element u_1 is algebraic over $k(X)$ and the degree of its minimal polynomial over $k(X)$ is equal to N.*

Proof. Since $u_1^N \in k(X)^\Delta \subset k(X)$, the element u_1 is algebraic over $k(X)$, and the degree of its minimal polynomial over $k(X)$ is not greater than N. Suppose that this degree is equal to m and $m < N$. Then there exist elements $a_0, a_1, \dots, a_m$, belonging to $k[X]$, such that $a_m \neq 0$, and $a_m u_1^m + \cdots + a_1 u_1^1 + a_0 = 0$. Let us recall that $x_i = \frac{\pi}{y_i}$, for $i = 1, \dots, n$, where $\pi = y_1 y_2 \dots y_n$. Hence, in the polynomial ring $k[Y] = k[y_1, \dots, y_n]$, the following equality holds:

$$a_m\left(\frac{\pi}{y_1}, \dots, \frac{\pi}{y_n}\right)(y_1 - y_n)^m + \cdots + a_1\left(\frac{\pi}{y_1}, \dots, \frac{\pi}{y_n}\right)(y_1 - y_n)^1 + a_0\left(\frac{\pi}{y_1}, \dots, \frac{\pi}{y_n}\right) = 0.$$

Consider the total degrees with respect to the variables $y_1, \dots, y_n$. Such degree of each polynomial $a_i\left(\frac{\pi}{y_1}, \dots, \frac{\pi}{y_n}\right)$, for $i = 0, 1, \dots, m$, is divisible by N. This means that in the above equality all the summands have degrees which are pairwise incongruent modulo N. Hence,

$$a_i\left(\frac{\pi}{y_1}, \dots, \frac{\pi}{y_n}\right) = 0,$$

for all $i = 0, 1, \dots, m$. In particular, $a_m\left(\frac{\pi}{y_1}, \dots, \frac{\pi}{y_n}\right) = 0$, that is, $a_m(x_1, \dots, x_n) = 0$. But, by the assumption, $a_m \neq 0$ and, by Proposition 2.2, the elements $x_1, \dots, x_n$ are algebraically independent over k. Thus we have a contradiction. □

Consider the field

$$L := k\left(u_1^N, \frac{u_2}{u_1}, \frac{u_3}{u_1}, \dots, \frac{u_N}{u_1}\right).$$

It is obvious that the generators $u_1^N, \frac{u_2}{u_1}, \dots, \frac{u_N}{u_1}$ are algebraically independent over k. Note that, by Proposition 3.5, all the generators belong to $k(X)^\Delta$. Thus, L is a subfield of the field $k(X)^\Delta$. We will show that $k(X)^\Delta = L$.

Since $u_1^N \in L$ and $L \subset k(X)$, immediately from Lemma 3.6 we obtain the following new lemma.

LEMMA 3.7. *The element u_1 is algebraic over L and the degree of its minimal polynomial over L is equal to N.*

Observe that

$$k(u_1, \dots, u_N) = L(u_1).$$

In fact, the inclusion $\supseteq$ is obvious. The inclusion $\subseteq$ is obvious too, because $u_1 \in L(u_1)$, and $u_i = \frac{u_i}{u_1} u_1 \in L(u_1)$ for $i = 2, \dots, N$. Now, by Lemma 3.7, the following proposition is true.

PROPOSITION 3.8. *Every element φ of the field $k(u_1, \dots, u_N)$ has a unique presentation of the form*

$$\varphi = a_{N-1} u_1^{N-1} + \cdots + a_1 u_1^1 + a_0,$$

where $a_0, \dots, a_m \in L$.

Now we are ready to prove that $k(X)^\Delta = L$.

THEOREM 3.9. *$k(X)^\Delta = L$. In other words, for any $n \geqslant 3$ there exist $n-1$ rational functions $\varphi_1, \dots, \varphi_{n-1} \in k(X)$, algebraically independent over k, such that the field of constants of the derivation Δ is equal to $k(\varphi_1, \dots, \varphi_{n-1})$.*

PROOF. We already know that $k(X)^\Delta$ contains L. To prove the inclusion in the opposite direction, let us assume that $\varphi \in k(X)^\Delta$. Then $\varphi \in k(Y)$ (because $k(X)^\Delta \subset k(X) \subset k(Y)$), and $\delta(\varphi) = 0$, where δ is the derivation defined in Section 2. Hence, $\varphi \in k(Y)^\delta$. But $k(Y)^\delta = k(u_1, \dots, u_N)$ (see Proposition 2.1), so $\varphi \in k(u_1, \dots, u_N)$, and, by Proposition 3.8, we obtain an equality of the form

$$\varphi = a_{N-1} u_1^{N-1} + \cdots + a_1 u_1^1 + a_0,$$

for some $a_0, \dots, a_{N-1} \in L$. But $L \subset k(X)$, whence the elements $a_0, \dots, a_{N-1}$ belong to $k(X)$, and moreover, $\varphi \in k(X)$. Hence, by Lemma 3.6, the equalities $a_1 = a_2 = \cdots = a_{N-1} = 0$, and $\varphi = a_0 \in L$. Therefore, $k(X)^\Delta \subseteq L$, and consequently, $k(X)^\Delta = L$. □

We have proved that $k(X)^\Delta = k(\varphi_1, \dots, \varphi_N)$, where $N = n - 1$, and

$$\varphi_1 = u_1^N, \quad \varphi_2 = \frac{u_2}{u_1}, \quad \varphi_3 = \frac{u_3}{u_1}, \quad \dots, \quad \varphi_N = \frac{u_N}{u_1}.$$

All the elements $\varphi_1, \ldots, \varphi_N$ are rational functions belonging to $k(X) = k(x_1, \ldots, x_n)$. We may present explicit formulas for these functions. Observe that

$$\begin{aligned}\varphi_1 = u_1^N = (y_1 - y_n)^N = y_1^N \left(1 - \frac{y_n}{y_1}\right)^N = \frac{W}{x_1^N}\left(1 - \frac{x_1}{x_n}\right)^N \frac{W}{x_1^N x_n^N}(x_n - x_1)^N \\ = \frac{x_1 x_2 \cdots x_n}{x_1^N x_n^N}(x_n - x_1)^N = \frac{x_2 x_3 \cdots x_{n-1}}{x_1^{N-1} x_n^{N-1}}(x_n - x_1) \\ = \frac{x_2 x_3 \cdots x_{n-1}}{x_1^{n-2} x_n^{n-2}}(x_n - x_1)^{n-1},\end{aligned}$$

and, if $i \in \{2, \ldots, N\}$, there is:

$$\varphi_i = \frac{u_i}{u_1} = \frac{y_i - y_n}{y_1 - y_n} = \frac{\frac{y_i}{y_n} - 1}{\frac{y_1}{y_n} - 1} = \frac{\frac{x_n}{x_i} - 1}{\frac{x_n}{x_1} - 1} = \frac{\frac{x_n - x_i}{x_i}}{\frac{x_n - x_1}{x_1}} = \frac{x_1(x_n - x_i)}{x_i(x_n - x_1)}.$$

Let us rewrite Theorems 1.1 and 3.9 as a single theorem in the following final version.

THEOREM 3.10. *Let $k[X] = k[x_1, \ldots, x_n]$ be the polynomial ring in $n \geqslant 3$ variables over a field k of characteristic zero, and let $\Delta : k[X] \to k[X]$ be the derivation defined by*

$$\Delta(x_i) = x_i(S - x_i),$$

for $i = 1, \ldots, n$, where $S = x_1 + \cdots + x_n$. The derivation Δ has no nontrivial polynomial constants. The field of constants of Δ is equal to $k(\varphi_1, \ldots, \varphi_{n-1})$, where

$$\varphi_1 = \frac{x_2 x_3 \cdots x_{n-1}}{x_1^{n-2} x_n^{n-2}}(x_n - x_1)^{n-1},$$

$$\varphi_2 = \frac{x_1(x_n - x_2)}{x_2(x_n - x_1)}, \quad \varphi_3 = \frac{x_1(x_n - x_3)}{x_3(x_n - x_1)}, \quad \ldots, \quad \varphi_{n-1} = \frac{x_1(x_n - x_{n-1})}{x_{n-1}(x_n - x_1)}.$$

The rational constants $\varphi_1, \ldots, \varphi_{n-1}$ are algebraically independent over k.

Note the specific cases of the above theorem, for $n = 3$ and $n = 4$.

COROLLARY 3.11. *Let $k[x, y, z]$ be the polynomial ring in three variables over a field k of characteristic zero. Let $\Delta : k[x, y, z] \to k[x, y, z]$ be the derivation defined by*

$$\begin{cases} \Delta(x) = x(y + z), \\ D(y) = y(x + z), \\ \Delta(z) = z(x + y). \end{cases}$$

Then $k[x, y, z]^\Delta = k$, and $k(x, y, z)^\Delta = k\left(\frac{y(z-x)^2}{xz}, \frac{x(y-z)}{y(x-z)}\right)$.

Corollary 3.12. *Let $k[x,y,z,t]$ be the polynomial ring in four variables over a field k of characteristic zero. Let $\Delta : k[x,y,z,t] \to k[x,y,z,t]$ be the derivation defined by*

$$\begin{cases} \Delta(x) = x(y+z+t), \\ D(y) = y(x+z+t), \\ \Delta(z) = z(x+y+t), \\ \Delta(t) = t(x+y+z). \end{cases}$$

Then $k[x,y,z,t]^{\Delta} = k$, and $k(x,y,z,t)^{\Delta} = k\left(\frac{yz(z-t)^3}{xt},\ \frac{x(y-t)}{y(x-t)},\ \frac{x(z-t)}{z(x-t)}\right)$.

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