A FACTORISABLE DERIVATION OF POLYNOMIAL RINGS
IN n VARIABLES

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Abstract. Let \( k[x_1, \ldots, x_n] \) be the polynomial ring in \( n \geq 3 \) variables over a field \( k \) of characteristic zero, and let \( \Delta \) be the factorisable derivation of \( k[x_1, \ldots, x_n] \) defined by \( \Delta(x_i) = x_i(S - x_i) \), for \( i = 1, \ldots, n \), where \( S = x_1 + \cdots + x_n \). We prove that this derivation has no nontrivial polynomial constants, and we describe the field of its rational constants.

Introduction. Throughout this paper \( k \) is a field of characteristic zero, \( k[X] = k[x_1, \ldots, x_n] \) is the polynomial ring in \( n \geq 3 \) variables over \( k \), and \( k(X) = k(x_1, \ldots, x_n) \) is the field of quotients of \( k[X] \), that is \( k(X) \) is the field of rational functions in \( n \) variables over \( k \).

If \( R \) is a commutative \( k \)-algebra, then a \( k \)-linear mapping \( d : R \to R \) is said to be a \( k \)-derivation (or simply a derivation) of \( R \) if \( d(ab) = ad(b) + bd(a) \) for all \( a, b \in R \). In this case we denote by \( R^d \) the \( k \)-algebra of constants of \( R \) with respect to \( d \), that is, \( R^d = \{ r \in R ; d(r) = 0 \} \). Note that if \( R \) is a field, then \( R^d \) is a subfield of \( R \) containing \( k \).

If \( f_1, \ldots, f_n \) are polynomials belonging to \( k[X] \), then there exists exactly one derivation \( d : k[X] \to k[X] \) such that \( d(x_1) = f_1, \ldots, d(x_n) = f_n \). This derivation is of the form

\[
d = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}.
\]

Every derivation \( d \) of \( k[X] \) has a unique extension to a derivation of the field \( k(X) \); also this extension we denote by \( \bar{d} \). Thus, for any derivation \( d \) of \( k[X] \), there is the ring \( k[X]^d = \{ f \in k[X] ; d(f) = 0 \} \) and the field \( k(X)^d = \{ \varphi \in \}

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$k(X); \ d(\varphi) = 0\}$. Of course, $k(X)^d$ contains the field of quotients of $k[X]^d$, but in many cases these fields are different (see [21, 20]). We are mainly interested in some descriptions of $k[X]^d$ and $k(X)^d$. However, we know that, in general, such descriptions are very difficult to obtain. Rings and fields of constants appear in various classical problems; for details we refer the reader to [3] and [20].

The mentioned problems are also very difficult for factorisable derivations. We say that a derivation $d : k[X] \rightarrow k[X]$ is factorisable (or factorizable) if

$$d(x_i) = x_i(a_{i1}x_1 + \cdots + a_{in}x_n),$$

for all $i = 1, \ldots, n$, where each $a_{ij}$ belongs to $k$. Factorisable derivations and factorisable systems of ordinary differential equations have intensively been studied for a long time; see for example [5, 4, 19] and [20], where numerous references to this subject can be found. With any given derivation $d : k[X]$, using a special procedure, we may associate a factorisable derivation $\delta$ (see [24] for details). There exist derivations for which the problem of descriptions of $k[X]^d$ or $k(X)^d$ reduces to the same problem for the factorisable derivation associated with a given derivation. We know from [22] and [18] that this is the case if the derivation $d$ is monomial, that is, if all the polynomials $d(x_1), \ldots, d(x_n)$ are monomials. Consider, for example, a cyclic monomial derivation $d : k[X] \rightarrow k[X]$ defined by

$$d(x_i) = x_i^s, \ d(x_2) = x_3^s, \ldots, \ d(x_{n-1}) = x_n^s, \ d(x_n) = x_1^s,$$

where $n \geq 3$ and $s \geq 2$. Such $d$ is called a Jouanolou derivation ([8, 19, 9, 26]). The factorisable derivation $\delta$, associated with $d$, is a derivation of $k[X]$ defined by

$$\delta(x_i) = x_i (sx_{i+1} - x_i),$$

for $i = 1, \ldots, n$, where $x_{n+1} = x_1$ ([9, 26]). In 2003, H. Žoladek [26] proved that the field of constants of the factorisable derivation $\delta$ is trivial, that is, $k(X)^\delta = k$. As a consequence of this fact (and some results from [9]), he proved that the above Jouanolou derivation $d$ has no Darboux polynomials; in particular, he proved that also the field of constants of $d$ is trivial. Let us recall ([19, 20]) that a polynomial $F \in k[X]$ is a Darboux polynomial of $d$ if $F \notin k$ and $d(F) = \Lambda F$ for some $\Lambda \in k[X]$. Derivations without Darboux polynomials are intensively studied in many papers ([16, 10, 17]).

Examples of factorisable derivations are the famous Lotka–Volterra derivations for $n = 3$ (see for example: [11, 12, 15, 13, 14]). A Lotka–Volterra derivation is a derivation $d : k[x, y, z] \rightarrow k[x, y, z]$ such that

$$d(x) = x(Cy + z), \ d(y) = y(Az + x), \ d(z) = z(Bx + y),$$
where $A, B, C \in k$. There also exist some specific generalizations of Lotka–Volterra derivations, for polynomial rings in $n \geq 4$ variables. One of such generalizations is the derivation $D : k[X] \to k[X]$ defined by

$$D(x_i) = x_i (x_{i-1} - x_{i+1}),$$

for $i = 1, \ldots, n$, where $x_0 = x_n$ and $x_{n+1} = x_1$. Such $D$ is called either a Lotka–Volterra derivation ([6, 1, 23]) or a Volterra derivation ([2, 25]). It is not easy to describe the ring of constants of $D$ for arbitrary $n \geq 3$. If $n = 3$, then some description is given in [15]. P. Ossowski and J. Zieliński ([23]) determined all polynomial constants for $n = 4$. Recently, Zieliński ([25]) presented such description for $n = 5$. Hence, we know a structure of $k[X]^D$ for $n \leq 5$ only. For $n \geq 6$, the problem is open. There are similar open problems concerning the field $k(X)^D$, even for $n < 6$. It is a natural question what happens if in the above derivation $D$ we change the sign before $x_{i+1}$, that is, if

$$D(x_i) = x_i (x_{i-1} + x_{i+1}),$$

for $i = 1, \ldots, n$. In particular, if $n = 3$, then $D$ is a cyclic derivation of $k[x, y, z]$ such that

$$D(x) = x(y + z), \quad D(y) = y(z + x), \quad D(z) = z(x + y).$$

There are no results concerning $k[X]^D$ and $k(X)^D$ for an arbitrary $n$.

In this paper, we consider a similar factorisable derivation $\Delta : k[X] \to k[X]$, defined by

$$\Delta(x_i) = x_i (S - x_i),$$

for $i = 1, \ldots, n$, where $S$ is the sum $x_1 + \cdots + x_n$. We prove that, for an arbitrary $n \geq 3$, the ring of constant of $\Delta$ is trivial, that is, $k[X]^\Delta = k$. Moreover, we prove that the field $k(X)^\Delta$ is generated by $n - 1$ algebraically independent rational functions; we also present some explicit formulas for generators. Note that if $n = 3$, then $\Delta$ coincides with the above mentioned derivation $D$ of $k[x, y, z]$.

1. Polynomial constants. Let us recall that $\Delta : k[X] \to k[X]$ is the factorisable derivation of the polynomial ring $k[X] = k[x_1, \ldots, x_n]$ defined by

$$\Delta(x_i) = x_i (S - x_i),$$

for $i = 1, \ldots, n$, where $n \geq 3$, $k$ is a field of characteristic zero and $S$ is the sum $x_1 + \cdots + x_n$. In this section, using a method described in [19] and [20], we prove that the ring of constants of $\Delta$ is equal to $k$. Note that the derivation $\Delta$ is homogeneous; all the elements $\Delta(x_1), \ldots, \Delta(x_n)$ are nonzero homogeneous polynomials of degree 2. Hence, if there exists a nontrivial polynomial constant of $\Delta$, then there exists such a constant which is homogeneous.
Let us assume that $F \in k[X]$ is a nonzero homogeneous polynomial of degree $m \geq 1$ such that $\Delta(F) = 0$. Then $x_1(S-x_1) \frac{\partial F}{\partial x_1} + \cdots + x_n(S-x_n) \frac{\partial F}{\partial x_n} = 0$ and, since $F$ is homogeneous, $x_1 \frac{\partial F}{\partial x_1} + \cdots + x_n \frac{\partial F}{\partial x_n} = mF$. As a combination of these two equalities we obtain the equality

\[
(1) \quad x_1(x_1-x_n) \frac{\partial F}{\partial x_1} + \cdots + x_n(x_n-x_n) \frac{\partial F}{\partial x_n} = m(S-x_n)F,
\]

which does not include the last partial derivative.

Let $\varphi : k[X] = k[x_1, \ldots, x_n] \rightarrow k[x_1, \ldots, x_n-1]$ be the $k$-algebra homomorphism such that $\varphi(x_i) = x_i$ for $i = 1, \ldots, n-1$, and $\varphi(x_n) = 1$. This homomorphism commutes with the partial derivatives $\partial_{x_1}, \ldots, \partial_{x_{n-1}}$, that is, $\varphi \circ \partial_{x_i} = \partial_{x_i} \circ \varphi$, for $i = 1, \ldots, n-1$. Denote by $\overline{F}$ the image of $F$ with respect to $\varphi$, that is,

\[
\overline{F} = \varphi(F) = F(x_1, \ldots, x_{n-1}, 1).
\]

Note that $\overline{F}$ is a polynomial belonging to $k[x_1, \ldots, x_{n-1}]$. Put $z = x_n$ and let

\[
F = F_0 z^m + F_1 z^{m-1} + \cdots + F_{m-1} z + F_m,
\]

where each $F_i$ (for $i = 0, 1, \ldots, m$) is either zero or a nonzero homogeneous polynomial, belonging to $k[x_1, \ldots, x_{n-1}]$, of degree $i$. Then we obtain the equality

\[
\overline{F} = F_0 + F_1 + \cdots + F_m,
\]

which is the decomposition of $\overline{F}$ into homogeneous components. Since $F \neq 0$, there exists $i \in \{0, 1, \ldots, m\}$ such that $F_i \neq 0$, and this implies that $\overline{F}$ is a nonzero polynomial. Suppose that $F_0 \neq 0$ and $F_1 = F_2 = \cdots = F_m = 0$. Then $F = az^m$, where $0 \neq a \in k$, $z = x_n$. But $\Delta(F) = 0$, so $0 = \Delta(ax_n^m(S-x_n)) \neq 0$; a contradiction. Therefore, $\overline{F}$ is a nonzero polynomial of degree $p$, where $1 \leq p \leq m$. Moreover, by (1), there follows:

\[
m(x_1 + \cdots + x_{n-1}) \overline{F} = m \varphi(S-x_n) \varphi(F) = \varphi\left(m(S-x_n)F\right) = \varphi\left(x_1(x_1-x_n) \frac{\partial F}{\partial x_1} + \cdots + x_n(x_n-x_n) \frac{\partial F}{\partial x_n-n} \right) = x_1(x_1-1) \frac{\partial \overline{F}}{\partial x_1} + \cdots + x_n(x-n-1) \frac{\partial \overline{F}}{\partial x_n-n}.
\]

Hence, the polynomial $\overline{F}$ satisfies the equality

\[
(2) \quad x_1(x_1-1) \frac{\partial \overline{F}}{\partial x_1} + \cdots + x_n(x_n-1) \frac{\partial \overline{F}}{\partial x_n-n} = m(x_1 + x_2 + \cdots + x_{n-1}) \overline{F}.
\]
Let \( \sigma : k[x_1, \ldots, x_{n-1}] \to k[x_1, \ldots, x_{n-1}] \) be the affine \( k \)-algebra automorphism defined by

\[
\sigma(x_i) = x_i + 1, \quad \text{for} \quad i = 1, \ldots, n-1.
\]

This homomorphism also commutes with all the partial derivatives \( \frac{\partial}{\partial x_i}, \ldots, \frac{\partial}{\partial x_{n-1}} \), that is, \( \sigma \circ \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i} \circ \sigma \), for \( i = 1, \ldots, n-1 \). Consider the polynomial

\[
G = \sigma(F) = F(x_1 + 1, \ldots, x_{n-1} + 1).
\]

It is clear that \( G \neq 0 \), \( \deg G = \deg F = p \) with \( 1 \leq p \leq m \). Moreover, by (2), we obtain

\[
(x_1 + 1)x_1 \frac{\partial G}{\partial x_1} + \cdots + (x_{n-1} + 1)x_{n-1} \frac{\partial G}{\partial x_{n-1}} = m \left( x_1 + x_2 + \cdots + x_{n-1} + (n-1) \right) G.
\]

(3)

It follows from the above equality that \( G(0, \ldots, 0) = 0 \).

Let \( H \) be the nonzero homogeneous component of \( G \) of the minimal degree. Put \( q = \deg H \). Since \( G \neq 0 \) and \( G(0, \ldots, 0) = 0 \), then \( q \geq 1 \). Thus, \( H \) is a nonzero homogeneous polynomial, belonging to \( k[x_1, \ldots, x_{n-1}] \), and \( \deg H = q \) with \( 1 \leq q \leq p \leq m \).

Comparing in the homogeneous components of the smallest degree in (3), we obtain the equality

\[
x_1 \frac{\partial H}{\partial x_1} + \cdots + x_{n-1} \frac{\partial H}{\partial x_{n-1}} = m(n-1)H.
\]

But \( H \) is homogeneous, so by Euler’s identity we have

\[
x_1 \frac{\partial H}{\partial x_1} + \cdots + x_{n-1} \frac{\partial H}{\partial x_{n-1}} = qH.
\]

Hence, \( q = (n-1)m \) and we have a contradiction: \( 2m \leq (n-1)m = q \leq m \).

Thus we have proved the following theorem.

**Theorem 1.1.** For any \( n \geq 3 \), the derivation \( \Delta \) has no nontrivial polynomial constants. In other words:

\[
k[X]^\Delta = \left\{ F \in k[X] : d(F) = 0 \right\} = k.
\]
2. An extension of $\Delta$. In this section we will show that the derivation $\Delta$ is associated with some simple monomial derivation $\delta$ of a polynomial ring in $n$ variables over $k$.

We denote by $k[Y]$ the polynomial ring $k[y_1, \ldots, y_n]$, by $k(Y) = k(y_1, \ldots, y_n)$ the field of quotients of $k[Y]$, and by $\pi$ the product $y_1y_2 \cdots y_n$. Moreover, we use notations:

$$u_1 = y_1 - y_n, \ u_2 = y_2 - y_n, \ldots, \ u_{n-1} = y_{n-1} - y_n.$$

Let us consider the unique derivation $\delta : k[Y] \to k[Y]$ such that $\delta(y_i) = \pi$ for $i = 1, \ldots, n$. This derivation is of the form $\delta = \pi \delta_0$, where

$$\delta_0 = \frac{\partial}{\partial y_1} + \cdots + \frac{\partial}{\partial y_n}.$$ 

The polynomials $u_1, \ldots, u_{n-1}$ are constants with respect to $\delta$, and we have the following proposition holds.

**Proposition 2.1.** $k[Y]^\delta = k[u_1, \ldots, u_{n-1}], \ k(Y)^\delta = k(u_1, \ldots, u_{n-1}).$

**Proof.** Observe that $\sigma \delta \sigma^{-1} = \frac{\partial}{\partial y_n}$, where $\sigma : k[Y] \to k[Y]$ is the $k$-algebra automorphism defined by $\sigma(y_i) = y_i + y_n$ for $i = 1, \ldots, n - 1$, and $\sigma(y_n) = y_n$. Hence,

$$k[Y]^\delta = k[Y]^{\delta_0} = \sigma^{-1}
\left( k[Y] \left( \frac{\partial}{\partial y_n} \right) \right)
= \sigma^{-1}
\left( k[y_1, \ldots, y_{n-1}] \right)
= k[\sigma^{-1}(y_1), \ldots, \sigma^{-1}(y_{n-1})]
= k[u_1, \ldots, u_{n-1}]$$

and, by the same argument, $k(Y)^\delta = k(u_1, \ldots, u_{n-1})$. \hfill $\square$

Now we introduce the elements $x_1, \ldots, x_n$, which are polynomials, belonging to $k[Y]$, defined by $x_i = \frac{y_i}{y_n}$ for $i = 1, \ldots, n$, that is,

$$x_1 = y_2y_3 \cdots y_n, \ x_2 = y_1y_3y_4 \cdots y_n, \ldots, \ x_n = y_1y_2 \cdots y_{n-1}.$$

**Proposition 2.2.** The above polynomials $x_1, \ldots, x_n$ are algebraically independent over $k$.

**Proof.** It is enough to prove (see for example [7]) that the Jacobian $\det(\partial x_i/\partial y_j)$ is nonzero. Observe that

$$\det(\partial x_i/\partial y_j) = \begin{vmatrix}
0 & \frac{y_1}{y_2} & \frac{y_1}{y_3} & \cdots & \frac{y_1}{y_n} \\
\frac{y_2}{y_1} & 0 & \frac{y_2}{y_3} & \cdots & \frac{y_2}{y_n} \\
\frac{y_3}{y_1} & \frac{y_3}{y_2} & 0 & \cdots & \frac{y_3}{y_n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{y_n}{y_1} & \frac{y_n}{y_2} & \frac{y_n}{y_3} & \cdots & 0
\end{vmatrix} = \frac{x_1 \cdots x_n}{y_1 \cdots y_n} \det M,$$
where $M$ is the following $n \times n$ matrix:

$$
\begin{bmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & 1 \\
1 & 1 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 0
\end{bmatrix}.
$$

It is easy to check that $\det M = (-1)^{n-1}(n-1)$. Hence, the Jacobian $\det[\partial x_j/\partial y_j]$ is nonzero. \hfill \square

**Proposition 2.3.** If $x_1, \ldots, x_n$ are the polynomials described above, then

$$
\delta(x_i) = x_i(S - x_i),
$$

for $i = 1, \ldots, n$, where $S = x_1 + \cdots + x_n$.

**Proof.** Let us recall that $\delta(y_i) = \pi$ and $x_i = \pi/y_i$ for $i = 1, \ldots, n$, where $\pi = y_1 \cdots y_n$. For $i = 1$:

$$
\delta(x_1) = \delta(y_2y_3 \cdots y_n)
= \delta(y_2)y_3y_4 \cdots y_n + y_2\delta(y_3)y_4 \cdots y_n + \cdots + y_2y_3 \cdots y_{n-1}\delta(y_n)
= \pi y_3y_4 \cdots y_n + \pi y_2y_4 \cdots y_n + \cdots + \pi y_2y_3 \cdots y_{n-1}\pi
= \frac{\pi}{y_1y_2} + \frac{\pi}{y_1y_3} + \cdots + \frac{\pi}{y_1y_n}
= x_1x_2 + x_1x_3 + \cdots + x_1x_n = x_1(S - x_1).
$$

We may repeat the same for any $i = 2, \ldots, n$ and hence $\delta(x_i) = x_i(S - x_i)$ for $i = 1, \ldots, n$. \hfill \square

Since $x_1, \ldots, x_n$ are algebraically independent over the field $k$ (see Proposition 2.2), we have the polynomial ring $k[X] = k[x_1, \ldots, x_n]$. Thus, we have two polynomial rings:

$$
k[X] = k[x_1, \ldots, x_n] \quad \text{and} \quad k[Y] = k[y_1, \ldots, y_n],
$$

and $k[X]$ is a subring of $k[Y]$. We also have the field extension $k(X) \subset k(Y)$, where $k(X) = k(x_1, \ldots, x_n)$ and $k(Y) = k(y_1, \ldots, y_n)$. It follows from $[18]$ that the extension $k(X) \subset k(Y)$ is Galois, and $\dim_{k(X)} k(Y) = n - 1$, but we do not need such information.

Observe that, by Proposition 2.3 $\delta(k[X]) \subset k[X]$ and the restriction of the derivation $\delta$ to $k[X]$ is exactly equal to the derivation $\Delta$. We already know that $k[X]^\Delta = k$ (see Theorem 1.1). Our aim is to describe the field $k(X)^\Delta$. Now we know that $k(X)^\Delta$ is a subfield of the field $k(Y)^\delta$, which, by Proposition 2.1 is equal to the field $k(u_1, \ldots, u_{n-1})$. Moreover, $k(X)^\Delta = k(Y)^\delta \cap k(X)$. 

3. Rational constants. We use the same notations as in the previous section. Put also

\[ W := x_1 \cdots x_n \quad \text{and} \quad N := n - 1. \]

Observe that \( W = \frac{\pi \cdots \pi}{y_1 \cdots y_n} = \pi^n = \pi^N. \) Moreover,

\[ x_i^N = \left( \frac{\pi}{y_i} \right)^N = \frac{W}{y_i^N}, \]

so \( y_i^N = \frac{W}{x_i} \) for \( i = 1, \ldots, n. \) Thus, the following lemma is true.

**Lemma 3.1.** The powers \( y_1^N, \ldots, y_n^N \) and \( \pi^N \) belong to \( k(X). \)

Note also that

\[ \frac{x_j}{x_i} = \frac{\pi/y_j}{\pi/y_i} = \frac{y_i}{y_j}, \]

and hence each quotient \( \frac{y_i}{y_j}, \) for \( i, j \in \{1, \ldots, n\}, \) belongs to the field \( k(X). \)

**Lemma 3.2.** If \( i_1, \ldots, i_n \) are integers such that the sum \( i_1 + \cdots + i_n \) is divisible by \( N, \) then \( y_1^{i_1} \cdots y_n^{i_n} \) belongs to \( k(X). \)

**Proof.** Let \( i_1 + \cdots + i_n = aN \) with \( a \in \mathbb{Z}. \) Then

\[ y_1^{i_1} \cdots y_n^{i_n} = y_1^{aN} y_1^{-(i_1+i_2+\cdots+i_n)} y_1^{i_1} \cdots y_n^{i_n} = (y_1^N)^a \left( \frac{y_2}{y_1} \right)^{i_2} \cdots \left( \frac{y_n}{y_1} \right)^{i_n}, \]

and this lemma follows from the previous observations. \( \square \)

Let us recall that \( u_i = y_i - y_n \) for \( i = 1, \ldots, N. \)

**Lemma 3.3.** The powers \( u_1^N, \ldots, u_N^N \) belong to \( k(X). \)

**Proof.** Since \( u_i^N = (y_i - y_n)^N = \sum_{p+q=N} a_{pq} y_1^p y_n^q, \) where each \( a_{pq} \) is an integer, the result follows from Lemma 3.2. \( \square \)

**Lemma 3.4.** Each quotient \( \frac{u_i}{u_j}, \) for \( i, j \in \{1, \ldots, N\}, \) belongs to \( k(X). \)

**Proof.** The result follows from the equalities

\[ \frac{u_i}{u_j} = \frac{y_i - y_n}{y_j - y_n} = \frac{(y_i - y_n) \left( y_j^{N-1} + y_j^{N-2} y_n + \cdots + y_n^{N-1} \right)}{y_j^N - y_n^N} \]

and the previous lemmas. \( \square \)

As a consequence of the above lemmas, we obtain the following proposition.
PROPOSITION 3.5. If \( i_1, \ldots, i_N \) are integers such that the sum \( i_1 + \cdots + i_N \) is divisible by \( N \), then \( u_1^{i_1} \cdots u_N^{i_N} \) belongs to the field of constants \( k(X)^\Delta \). In particular, all rational functions of the forms \( u_i^N \) and \( u_i \), for \( i, j \in \{1, \ldots, N\} \), belong to \( k(X)^\Delta \).

**Proof.** Let \( i_1 + \cdots + i_N = aN \) with \( a \in \mathbb{Z} \), and put \( \gamma = u_1^{i_1} \cdots u_N^{i_N} \). Then

\[
\gamma = u_1^{a N} u_1^{-(i_1 + \cdots + i_N)} u_1^{i_1} \cdots u_N^{i_N} = \left( \frac{u_1^N}{u_1} \right)^a \left( \frac{u_2}{u_1} \right)^{i_2} \cdots \left( \frac{u_N}{u_1} \right)^{i_N},
\]

and hence, by Lemmas 3.3 and 3.4, the element \( \gamma \) belongs to \( k(X) \). But \( \gamma \) belongs also to the field \( k(u_1, \ldots, u_N) \), which is equal to \( k(Y)^\delta \) (see Proposition 2.1). Recall that \( k(X)^\Delta = k(Y)^\delta \cap k(X) \). Therefore, \( \gamma \in k(X)^\Delta \). □

**Lemma 3.6.** The element \( u_1 \) is algebraic over \( k(X) \) and the degree of its minimal polynomial over \( k(X) \) is equal to \( N \).

**Proof.** Since \( u_1^N \in k(X)^\Delta \subset k(X) \), the element \( u_1 \) is algebraic over \( k(X) \), and the degree of its minimal polynomial over \( k(X) \) is not greater than \( N \). Suppose that this degree is equal to \( m \) and \( m < N \). Then there exist elements \( a_0, a_1, \ldots, a_m \), belonging to \( k[X] \), such that \( a_m \neq 0 \), and \( a_m u_1^m + \cdots + a_1 u_1 + a_0 = 0 \). Let us recall that \( x_i = \frac{\pi}{y_i} \) for \( i = 1, \ldots, n \), where \( \pi = y_1 y_2 \cdots y_n \).

Hence, in the polynomial ring \( k[X] = k[y_1, \ldots, y_n] \), the following equality holds:

\[
a_m \left( \frac{\pi}{y_1}, \ldots, \frac{\pi}{y_n} \right) (y_1 - y_m)^m + \cdots + a_1 \left( \frac{\pi}{y_1}, \ldots, \frac{\pi}{y_n} \right) (y_1 - y_n)^1 + a_0 \left( \frac{\pi}{y_1}, \ldots, \frac{\pi}{y_n} \right) = 0.
\]

Consider the total degrees with respect to the variables \( y_1, \ldots, y_n \). Such degree of each polynomial \( a_i \left( \frac{\pi}{y_1}, \ldots, \frac{\pi}{y_n} \right) \), for \( i = 0, 1, \ldots, m \), is divisible by \( N \). This means that in the above equality all the summands have degrees which are pairwise incongruent modulo \( N \). Hence,

\[
a_i \left( \frac{\pi}{y_1}, \ldots, \frac{\pi}{y_n} \right) = 0,
\]

for all \( i = 0, 1, \ldots, m \). In particular, \( a_m \left( \frac{\pi}{y_1}, \ldots, \frac{\pi}{y_n} \right) = 0 \), that is, \( a_m(x_1, \ldots, x_n) = 0 \). But, by the assumption, \( a_m \neq 0 \) and, by Proposition 2.2, the elements \( x_1, \ldots, x_n \) are algebraically independent over \( k \). Thus we have a contradiction. □

Consider the field

\[
L := k \left( u_1^{N}, \frac{u_2}{u_1}, \frac{u_3}{u_1}, \ldots, \frac{u_N}{u_1} \right).
\]
It is obvious that the generators $u_1^N, \frac{u_2}{u_1^2}, \ldots, \frac{u_N}{u_1}$ are algebraically independent over $k$. Note that, by Proposition 3.5, all the generators belong to $k(X)^\Delta$. Thus, $L$ is a subfield of the field $k(X)^\Delta$. We will show that $k(X)^\Delta = L$.

Since $u_1^N \in L$ and $L \subset k(X)$, immediately from Lemma 3.6 we obtain the following new lemma.

**Lemma 3.7.** The element $u_1$ is algebraic over $L$ and the degree of its minimal polynomial over $L$ is equal to $N$.

Observe that

$$k(u_1, \ldots, u_N) = L(u_1).$$

In fact, the inclusion $\supseteq$ is obvious. The inclusion $\subseteq$ is obvious too, because $u_1 \in L(u_1)$, and $u_i = \frac{u_i}{u_1} u_1 \in L(u_1)$ for $i = 2, \ldots, N$. Now, by Lemma 3.7, the following proposition is true.

**Proposition 3.8.** Every element $\varphi$ of the field $k(u_1, \ldots, u_N)$ has a unique presentation of the form

$$\varphi = a_{N-1} u_1^{N-1} + \cdots + a_1 u_1 + a_0,$$

where $a_0, \ldots, a_m \in L$.

Now we are ready to prove that $k(X)^\Delta = L$.

**Theorem 3.9.** $k(X)^\Delta = L$. In other words, for any $n \geq 3$ there exist $n-1$ rational functions $\varphi_1, \ldots, \varphi_{n-1} \in k(X)$, algebraically independent over $k$, such that the field of constants of the derivation $\Delta$ is equal to $k(\varphi_1, \ldots, \varphi_{n-1})$.

**Proof.** We already know that $k(X)^\Delta$ contains $L$. To prove the inclusion in the opposite direction, let us assume that $\varphi \in k(X)^\Delta$. Then $\varphi \in k(Y)$ (because $k(X)^\Delta \subset k(X) \subset k(Y)$), and $\delta(\varphi) = 0$, where $\delta$ is the derivation defined in Section 2. Hence, $\varphi \in k(Y)^\delta$. But $k(Y)^\delta = k(u_1, \ldots, u_N)$ (see Proposition 2.1), so $\varphi \in k(u_1, \ldots, u_N)$, and, by Proposition 3.8, we obtain an equality of the form

$$\varphi = a_{N-1} u_1^{N-1} + \cdots + a_1 u_1 + a_0,$$

for some $a_0, \ldots, a_{N-1} \in L$. But $L \subset k(X)$, whence the elements $a_0, \ldots, a_{N-1}$ belong to $k(X)$, and moreover, $\varphi \in k(X)$. Hence, by Lemma 3.6, the equalities $a_1 = a_2 = \cdots = a_{N-1} = 0$, and $\varphi = a_0 \in L$. Therefore, $k(X)^\Delta \subseteq L$, and consequently, $k(X)^\Delta = L$. 

We have proved that $k(X)^\Delta = k(\varphi_1, \ldots, \varphi_N)$, where $N = n - 1$, and

$$\varphi_1 = u_1^N, \quad \varphi_2 = \frac{u_2}{u_1}, \quad \varphi_3 = \frac{u_3}{u_1}, \quad \ldots, \quad \varphi_N = \frac{u_N}{u_1}.$$
All the elements $\varphi_1, \ldots, \varphi_N$ are rational functions belonging to $k(X) = k(x_1, \ldots, x_n)$. We may present explicit formulas for these functions. Observe that

$$
\varphi_1 = u_1^N = (y_1 - y_n)^N = y_1^N \left(1 - \frac{y_n}{y_1}\right)^N = \frac{W}{x_1^N} \left(1 - \frac{x_1}{x_n}\right)^N \frac{W}{x_1^N x_n^N} (x_n - x_1)^N
$$

$$
= \frac{x_1 x_2 \cdots x_n}{x_1^N x_n^N} (x_n - x_1)^N = \frac{x_2 x_3 \cdots x_{n-1}}{x_1^N x_n^N} (x_n - x_1)
$$

and, if $i \in \{2, \ldots, N\}$, there is:

$$
\varphi_i = \frac{u_i}{u_1} = \frac{y_i - y_n}{y_1 - y_n} = \frac{y_i - y_n}{y_1 - y_n} = \frac{x_i - 1}{x_i - 1} = \frac{x_n - x_i}{x_n - x_i} = \frac{x_i (x_n - x_i)}{x_n - x_i}. 
$$

Let us rewrite Theorems 1.1 and 3.9 as a single theorem in the following final version.

**Theorem 3.10.** Let $k[X] = k[x_1, \ldots, x_n]$ be the polynomial ring in $n \geq 3$ variables over a field $k$ of characteristic zero, and let $\Delta : k[X] \rightarrow k[X]$ be the derivation defined by

$$
\Delta(x_i) = x_i (S - x_i),
$$

for $i = 1, \ldots, n$, where $S = x_1 + \cdots + x_n$. The derivation $\Delta$ has no nontrivial polynomial constants. The field of constants of $\Delta$ is equal to $k(\varphi_1, \ldots, \varphi_{n-1})$, where

$$
\varphi_1 = \frac{x_2 x_3 \cdots x_{n-1}}{x_n^2 x_{n-1}^2} (x_n - x_1)^{n-1},
$$

$$
\varphi_2 = \frac{x_1 (x_n - x_2)}{x_2 (x_n - x_1)}, \quad \varphi_3 = \frac{x_1 (x_n - x_3)}{x_3 (x_n - x_1)}, \quad \ldots, \quad \varphi_{n-1} = \frac{x_1 (x_n - x_{n-1})}{x_{n-1} (x_n - x_1)}.
$$

The rational constants $\varphi_1, \ldots, \varphi_{n-1}$ are algebraically independent over $k$.

Note the specific cases of the above theorem, for $n = 3$ and $n = 4$.

**Corollary 3.11.** Let $k[x, y, z]$ be the polynomial ring in three variables over a field $k$ of characteristic zero. Let $\Delta : k[x, y, z] \rightarrow k[x, y, z]$ be the derivation defined by

$$
\begin{align*}
\Delta(x) &= x(y + z), \\
D(y) &= y(x + z), \\
\Delta(z) &= z(x + y).
\end{align*}
$$

Then $k[x, y, z]^\Delta = k$, and $k(x, y, z)^\Delta = k\left(\frac{y(z-x)^2}{xz}, \frac{x(y-z)}{y(z-x)}\right)$. 

Corollary 3.12. Let $k[x, y, z, t]$ be the polynomial ring in four variables over a field $k$ of characteristic zero. Let $\Delta : k[x, y, z, t] \to k[x, y, z, t]$ be the derivation defined by

\[
\begin{align*}
\Delta(x) &= x(y + z + t), \\
\Delta(y) &= y(x + z + t), \\
\Delta(z) &= z(x + y + t), \\
\Delta(t) &= t(x + y + z).
\end{align*}
\]

Then $k[x, y, z, t]^\Delta = k$, and $k(x, y, z, t)^\Delta = k\left(\frac{yz(z-t)^3}{xt}, \frac{z(y-t)}{y(x-t)}, \frac{z(z-t)}{z(t)}\right)$.

References


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