

DEGENERATE SINGULARITIES AND THEIR MILNOR NUMBERS

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Abstract. We give an example of a curious behaviour of the Milnor number with respect to evolving degeneracy of an isolated singularity in \mathbb{C}^2 .

1. Introduction. Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be an *isolated singularity*, i.e. let f be a holomorphic function germ with an isolated critical point at 0. The *Milnor number of f* is defined as

$$\mu(f) := \dim_{\mathbb{C}} \left(\frac{\mathcal{O}^n}{(\partial f / \partial x_1, \dots, \partial f / \partial x_n) \mathcal{O}^n} \right).$$

The number turns out to be a topological invariant of a singularity (see [7]). We also put $\mu(f) = 0$ for a smooth f .

To the singularity f a combinatorial object is also associated: – its Newton diagram $\mathcal{N}(f)$. Under some non-degeneracy conditions on f (see Section 2) the Milnor number $\mu(f)$ can be computed from its Newton diagram. It is the celebrated Kouchnirenko Theorem (see [5] or [8] for the case $n = 2$).

THEOREM 1. *There exists a number $\nu(f)$, called the Newton number of f , depending on the Newton diagram of f only and such that*

1. $\nu(f) \leq \mu(f)$,
2. if f is non-degenerate, $\nu(f) = \mu(f)$.

Although Theorem 1 is valid in any dimension, the inverse implication in (2), as observed by Płoski [8, 9] and [1], is true for $n = 2$ only.

THEOREM 2. *If $n = 2$ and $\nu(f) = \mu(f)$, then f is non-degenerate.*

The fact that Theorem 2 is not true in general, was already noticed by Kouchnirenko [5, Remarque 1.21], see also Example 1.

In light of Theorem 2, the two-variable case seems to be very special. Indeed, it turns out that in this case there exists a complete characterisation of non-degeneracy of a singularity f in a coordinate system, in terms of intrinsic topological invariants of f (see [1]).

Let us explicitly list some other properties of Milnor numbers.

- i. $\mu(\cdot)$ is upper semi-continuous w.r.t. holomorphic unfoldings (see [2, Theorem 2.6]).
- ii. $\mu(\cdot)$ is an increasing function on the set of *non-degenerate* singularities partially ordered by the relation

$$f \preceq g \Leftrightarrow \mathcal{N}(f) \supset \mathcal{N}(g),$$

where $f, g \in \mathcal{O}^n$; a simple proof of this fact can be found in [3].

- iii. Let $n = 2$ and f be non-degenerate. A simple consequence of Theorem 1 and Theorem 2. is that if f ‘gets degenerated’ on *any* face \mathcal{S} of $\mathcal{N}(f)$, then its Milnor number increases. Precisely, if $g = f + r$ is another isolated singularity such that $\mathcal{N}(f) = \mathcal{N}(g)$ and g is degenerate on $\mathcal{S} \in \mathcal{N}_0(f)$ (see Section 2 for definitions) then $\mu(g) > \mu(f)$.

In the paper we examine the possibility of extending property (iii) onto the case of degenerate singularities (Section 3). Our first result is that it cannot be done in a verbatim way. Namely, we give an example (Example 2) of a singularity f such that, f having been degenerated on one segment of its Newton boundary, its Milnor number decreases. The second result gives such a method of degenerating f under which the Milnor number increases (Proposition 2).

2. Definitions and auxiliary properties. In this section we briefly recall the necessary basics. Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ with an expansion of the form

$$f = \sum_{\alpha \in \mathbb{N}_0^n} f_\alpha x^\alpha,$$

where the usual multi-index notation is applied. We define the *support* of f as $\text{Supp } f := \{\alpha \in \mathbb{N}_0^n : f_\alpha \neq 0\}$ and the *Newton diagram* of f as $\mathcal{N}(f) := \text{conv}(\text{Supp } f + \mathbb{N}_0^n)$. The set of the compact faces of $\mathcal{N}(f)$ of positive dimension is called the *Newton boundary* of f and is denoted by $\mathcal{N}_0(f)$. f is said to be *convenient* if $\mathcal{N}_0(f)$ meets each of the coordinate axes. For a convenient f we denote by $\mathcal{N}_-(f)$ the compact polytope defined as $\overline{\mathbb{R}_+^n \setminus \mathcal{N}(f)}$. Then the *Newton number* $\nu(f)$ of f is defined by

$$\nu(f) := n!V_n - (n-1)!V_{n-1} + \dots + (-1)^{n-1}V_1 + (-1)^n,$$

where V_n is the n -dimensional volume of $\mathcal{N}_-(f)$ and for $1 \leq k \leq n-1$, V_k is the sum of the k -dimensional volumes of the intersections of $\mathcal{N}_-(f)$ with the coordinate planes of dimension k . If f is an isolated singularity and f is not

convenient, it can be made convenient by adding to it high enough powers of the missing variables and then the formula above makes sense for the changed f . It can be shown that such operations on f lead to the same Newton numbers and so – to the definition of $\nu(f)$ in the general case of isolated singularities (cf. [5, 6, 11]).

The non-degeneracy condition, which is the key to the Kouchnirenko Theorem, can be formulated as follows.

DEFINITION 1. For $\mathcal{S} \in \mathcal{N}_0(f)$ let

$$\text{in}_{\mathcal{S}}(f) := \sum_{\alpha \in \mathcal{S} \cap \text{Supp } f} f_{\alpha} x^{\alpha}.$$

We say that f is non-degenerate on \mathcal{S} if the system

$$\nabla \text{in}_{\mathcal{S}}(f) = 0$$

has no solutions in $(\mathbb{C}^*)^n$, where ∇ denotes the gradient of a function. If f is non-degenerate on every \mathcal{S} , we say that f is (Kouchnirenko) non-degenerate. In the opposite case, we say f is degenerate.

Let us recall the following simple properties.

PROPOSITION 1. Let $\mathcal{S} \in \mathcal{N}_0(f)$. Then:

- i. $f_{\mathcal{S}}$ is quasi-homogeneous,
- ii. if $f_{\mathcal{S}}$ has two terms only, then f is non-degenerate on \mathcal{S} ,
- iii. if $f_{\mathcal{S}}$ has a multiple factor that is not a monomial, then f is degenerate on \mathcal{S} ,
- iv. for $n = 2$ the converse of (iii) also holds.

PROOF. Items (i)–(iii) are straightforward. The item (iv) follows from Euler’s formula for quasi-homogeneous polynomials. \square

We cite the Kouchnirenko example, which shows that, when $n \geq 3$, the above-defined nondegeneracy condition is actually too strong for the conclusion of (2) in Theorem 1 to be true.

EXAMPLE 1. Let $f = (x_1 + x_2)^2 + x_1 x_3 + x_3^2$. Note that f is convenient and quasi-homogeneous. By Proposition 1 we conclude that f is degenerate on the face $\mathcal{S} = \overline{(1, 0, 0)(0, 1, 0)}$, because $f_{\mathcal{S}} = (x_1 + x_2)^2$. A simple calculation shows that $\nabla f = (2(x_1 + x_2) + x_3, 2(x_1 + x_2), x_1 + 2x_3)$ generates the ideal (x_1, x_2, x_3) in \mathcal{O}^3 and so there is $\mu(f) = 1$, by the definition of the Milnor number. On the other hand, it is easy to see that $\nu(f) = 3! \cdot \frac{4}{3} - 2! \cdot 6 + 1! \cdot 6 - 1 = 1$, so $\nu(f) = \mu(f)$, although f is degenerate.

In the case of $n = 2$, we recall a notion of non-degeneracy for pairs of holomorphic germs. We say that a pair $(f, g) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ is *non-degenerate* (see [8]) if for every $\mathcal{S} \in \mathcal{N}_0(f), \mathcal{T} \in \mathcal{N}_0(g)$ one of the following conditions holds

- a) $\mathcal{S} \nparallel \mathcal{T}$,
- b) $\mathcal{S} \parallel \mathcal{T}$ and the system $f_{\mathcal{S}} = g_{\mathcal{T}} = 0$ has no solutions in $(\mathbb{C}^*)^2$.

Then the following is true (see [8, Theorem 1.2]).

THEOREM 3. *For any non-degenerate pair $(f, g) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ of convenient germs the intersection multiplicity $(f, g)_0$ of f and g depends on the pair $(\mathcal{N}_0(f), \mathcal{N}_0(g))$ only.*

Finally, the following formula holds (it is an immediate consequence of [4, Proposition 4] and [7, Theorem 10.5]; see also [2, Lemma 3.32] and [2, Proposition 3.35], respectively).

THEOREM 4. *If $f, g : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ and fg is an isolated singularity then*

$$\mu(fg) = \mu(f) + 2(f, g)_0 + \mu(g) - 1.$$

3. Degenerate singularities and their Milnor numbers. First we give an example showing that item (iii) of Introduction cannot be generalised in the form analogical to the degenerate case, even for $n = 2$.

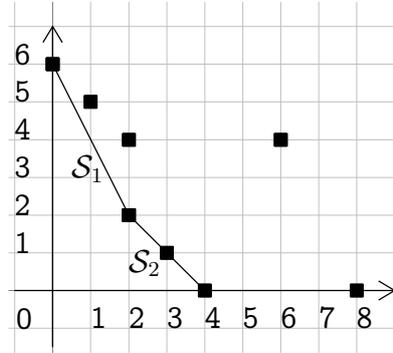
EXAMPLE 2. Consider three germs $f_1, f_2, f_3 \in \mathcal{O}^2$:

$$\begin{aligned} f_1 &:= y - x - x^3, \\ f_2 &:= y - x + x^3, \\ f_3 &:= x^2 + y^4. \end{aligned}$$

We define

$$\begin{aligned} f &:= f_1 f_2 f_3 \\ &= (y^6 + x^2 y^2 - 2x^3 y + x^4) - 2xy^5 + (x^2 - x^6)y^4 - x^8. \end{aligned}$$

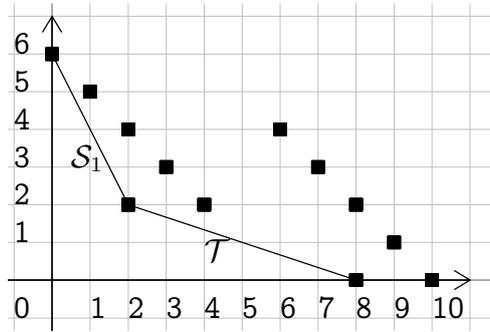
The Newton diagram of f is of the following form.



From the definition of f it follows that $f_{\mathcal{S}_1} = y^6 + x^2y^2 = y^2(y^4 + x^2)$ and $f_{\mathcal{S}_2} = x^2y^2 - 2x^3y + x^4 = x^2(y - x)^2$, and so by Proposition 1, f is non-degenerate on \mathcal{S}_1 and degenerate on \mathcal{S}_2 . It is easily seen that $\nu(f) = 11$. In order to compute $\mu(f)$ we change the coordinates: $x \mapsto x, y \mapsto x + y$. Then f takes the form

$$\begin{aligned} \hat{f} := f(x, x + y) &= (y^6 + x^2y^2 - x^8) + (4xy^5 + 6x^2y^4 + 4x^3y^3 + x^4y^2) \\ &\quad + (-x^6y^4 - 4x^7y^3 - 6x^8y^2 - 4x^9y - x^{10}) \end{aligned}$$

and the Newton diagram of \hat{f} is



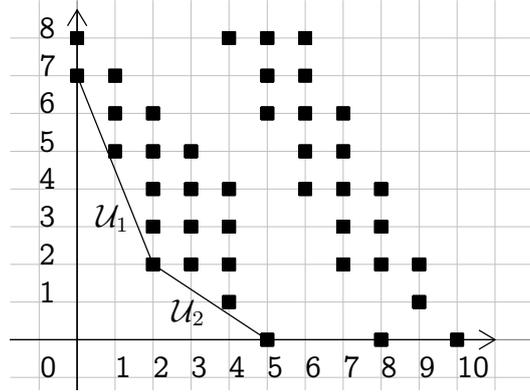
Now it is $\hat{f}_{\mathcal{S}_1} = y^6 + x^2y^2$ and $\hat{f}_{\mathcal{T}} = x^2y^2 - x^8$, so again by Proposition 1 we conclude that \hat{f} is non-degenerate. Since the Milnor number is an analytic invariant of a singularity then by Kouchnirenko Theorem, $\mu(f) = \mu(\hat{f}) = \nu(\hat{f}) = 15$.

The next step is to ‘degenerate’ f on \mathcal{S}_1 . Namely, we consider $g := f + 2xy^4$. It is obvious that from the point of view of the Newton boundary, the only difference between f and g is on the segment \mathcal{S}_1 : $g_{\mathcal{S}_1} = f_{\mathcal{S}_1} + 2xy^4 = y^2(y^2 + x)^2$ and $g_{\mathcal{S}_2} = f_{\mathcal{S}_2}$. We conclude that g is degenerate on the both of its segments – on \mathcal{S}_1 and \mathcal{S}_2 . Since $\mathcal{N}_0(g) = \mathcal{N}_0(f)$ then $\nu(g) = \nu(f) = 11$. To compute $\mu(g)$

we change the coordinates once again: $x \mapsto x - y^2, y \mapsto x + y$. We obtain

$$\hat{g} := g(x - y^2, x + y) = (4y^7 + x^2y^2 + 2x^5) - 8xy^5 - 5x^2y^4 - 4x^3y^3 + 10x^2y^3 - x^4y^2 + 12x^3y^2 + 8x^4y + \text{other terms of degree } \geq 7.$$

The (truncated above 8) Newton diagram of \hat{g} is of the form



and we can see that $\hat{g}_{\mathcal{U}_1} = 4y^7 + x^2y^2$ and $\hat{g}_{\mathcal{U}_2} = x^2y^2 + 2x^5$. It means that \hat{g} is nondegenerate and we can apply Theorem 1 to compute its Milnor number. We obtain $\mu(\hat{g}) = \nu(\hat{g}) = 13$. By the invariancy of the Milnor number, we conclude that

$$\mu(g) = \mu(\hat{g}) = 13 < 15 = \mu(f).$$

Summing up, we have found f and g such that:

1. $\text{Supp } g = \text{Supp } f \cup \{\text{single point}\}$,
2. $\mathcal{N}_0(f) = \mathcal{N}_0(g) = \{\mathcal{S}_1, \mathcal{S}_2\}$,
3. f is non-degenerate on \mathcal{S}_1 and degenerate on \mathcal{S}_2 ,
4. g is degenerate on \mathcal{S}_1 and \mathcal{S}_2 ,
5. $\mu(f) > \mu(g)$.

The above shows that the Milnor number is not ‘monotonic with respect to degeneracy,’ in general.

A positive result concerning the problem can also be given. Let f be an isolated and convenient singularity with $\#\mathcal{N}_0(f) \geq 2$, and let $\mathcal{S}_0 \in \mathcal{N}_0(f)$. Let g be the factor of f associated to this segment. It means that there exists a decomposition of f of the form $f = gh$ such that:

- i. $\mathcal{N}_0(g) = \{\mathcal{S}_1\}$ and $\mathcal{S}_1 \parallel \mathcal{S}_0$,
- ii. $\bigwedge_{\mathcal{T} \in \mathcal{N}_0(h)} \mathcal{T} \not\parallel \mathcal{S}_0$

(see e.g. [12, Lemma 2.44]). The following holds.

PROPOSITION 2. *Assume that f is non-degenerate on \mathcal{S}_0 and that there exists an isolated singularity \tilde{g} such that $\mathcal{N}_0(\tilde{g}) = \{\mathcal{S}_1\}$ and \tilde{g} is degenerate. Define $\tilde{f} := \tilde{g}h$. Then:*

1. $\mathcal{N}_0(f) = \mathcal{N}_0(\tilde{f})$,
2. \tilde{f} is degenerate on \mathcal{S}_0 ,
3. $\mu(f) < \mu(\tilde{f})$.

PROOF. By assumption $\mathcal{N}_0(g) = \mathcal{N}_0(\tilde{g})$. Since f is convenient, g and \tilde{g} are convenient, too. This implies that $\mathcal{N}(g) = \mathcal{N}(\tilde{g})$. Now the first assertion follows from the known properties of Newton diagrams (see [10, Section 3.6] or [1])

$$\mathcal{N}(f) = \mathcal{N}(gh) = \mathcal{N}(g) + \mathcal{N}(h) = \mathcal{N}(\tilde{g}) + \mathcal{N}(h) = \mathcal{N}(\tilde{g}h) = \mathcal{N}(\tilde{f}).$$

We claim that conditions (i) and (ii) of the assumption imply that the following is true:

- (1) f is non-degenerate on $\mathcal{S}_0 \Leftrightarrow g$ is non-degenerate on \mathcal{S}_1

and similarly

- (2) \tilde{f} is degenerate on $\mathcal{S}_0 \Leftrightarrow \tilde{g}$ is degenerate on \mathcal{S}_1 .

Indeed, if $v \perp \mathcal{S}_0$ is a vector with positive integer coefficients and we consider the v -gradation on \mathcal{O}^2 , then denoting by in_v the initial form operator with respect to this gradation, we see that

$$f_{\mathcal{S}_0} = \text{in}_v f = \text{in}_v g \cdot \text{in}_v h = g_{\mathcal{S}_1} \cdot (\text{a monomial}).$$

Using Proposition 1 (iii)–(iv) we arrive at (1). The argument for \tilde{f} runs in the same way so also (2) holds, and in particular the second assertion is proved.

Now note that by assumption (ii) the pair (g, h) is non-degenerate, and since $\mathcal{N}_0(\tilde{g}) = \{\mathcal{S}_1\}$ the same is true for the pair (\tilde{g}, h) . By Theorem 3, it means that $(\tilde{g}, h)_0 = (g, h)_0$. On the other hand, since \tilde{g} is degenerate (by assumption) and g is not (by (1)), Theorem 2 asserts that $\mu(\tilde{g}) > \mu(g)$. Summing up

$$\begin{aligned} \mu(\tilde{f}) &= \mu(\tilde{g}h) = \mu(\tilde{g}) + 2(\tilde{g}, h)_0 + \mu(h) - 1 \\ &> \mu(g) + 2(g, h)_0 + \mu(h) - 1 = \mu(gh) = \mu(f). \end{aligned}$$

□

We illustrate the proposition with the following example.

EXAMPLE 3. Consider f of Example 2. We can take $g = f_3 = x^2 + y^4$ and $h = f_1 f_2$. Then $\mathcal{N}_0(g) = \{\mathcal{S}_0\} = \{\overline{(0, 4)}, \overline{(2, 0)}\}$ and $\mathcal{S}_0 \parallel \overline{(0, 6)}, \overline{(2, 2)} = \mathcal{S}_1 \in \mathcal{N}_0(g)$. We consider a degenerated g , e.g. $\tilde{g} := (x + y^2)^2 + x^3$ and define $\tilde{f} := \tilde{g}h = f + (2xy^2 + x^3)f_1 f_2$. It is easy to see that $\mu(\tilde{f}) = 17 > 15 = \mu(f)$ (one can use Theorem 4 to perform the calculation).

REMARK 1. It is possible to give a parametric version of Example 2. Namely, one can consider $f_1 := y - x - sx^3$, $f_2 := y - x + sx^3$, $f_3 := y^4 + s^2x^2$ and ${}^s f := f_1 f_2 f_3 + x^7$, where $|s| \ll 1$. Additionally, let ${}^{st} f := {}^s f + 2txy^4$, for $|s|, |t| \ll 1$. Then ${}^{st} f$ is a holomorphic unfolding of $f_0 := (y - x)^2 y^4 + x^7$. It is easy to see that $\mathcal{N}_0({}^{st} f) = \mathcal{N}_0({}^s f) = \{\mathcal{S}_1, \mathcal{S}_2\}$, where $\mathcal{S}_1, \mathcal{S}_2$ are the segments as in Example 2. One can check that ${}^{st} f_{\mathcal{S}_1} = y^2(y^4 + 2txy^2 + s^2x^2)$ and ${}^{st} f_{\mathcal{S}_2} = s^2x^2(y - x)^2$. It means that ${}^{st} f$ is non-degenerate on \mathcal{S}_1 for $s \neq t$ and degenerate on \mathcal{S}_2 . Here $\mu({}^{st} f) = 12$ for $s \neq t$ different from 0. In accordance with property (i) of Introduction, $\mu({}^{st} f|_{t=0}) = \mu({}^s f) = 14 > 12$ and $\mu({}^{st} f|_{t=s}) = 13 > 12$. However, ${}^{st} f|_{t=s} = {}^s f + 2sxy^4$ is degenerate on \mathcal{S}_1 , while ${}^s f$ is not, and yet $\mu({}^{st} f|_{t=s}) < \mu({}^s f)$. The skipped calculations are similar to those of Example 2.

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