UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA
doi: 10.4467/20843828AM.12.003.0455 FASCICULUS XLIX, 2011

ON THE GRADIENT OF QUASI-HOMOGENEOUS POLYNOMIALS

by Alain Haraux and Tien Son Phạm

Résumé. Soit $\mathbb{K}$ le corps des réels ou des complexes et $f\colon \mathbb{K}^n \to \mathbb{K}$ un polynôme quasi-homogène de poids $w := (w_1, w_2, \ldots, w_n)$ et de degré d tel que $\nabla f(0) = 0$. L'inégalité bien connue dite du gradient de Łojasiewicz montre qu'il existe un voisinage ouvert U de l'origine dans $\mathbb{K}^n$ et deux constantes positives c et $\rho < 1$ telles que pour tout $x \in U$ on ait $\|\nabla f(x)\| \geq c|f(x)|^\rho$. On montre que si l'ensemble $\widetilde{K}_\infty(f)$ des points où la condition de Fedoryuk est en dfaut est fini, l' inégalité du gradient de Łojasiewicz est vérifiée avec $\rho = 1 - \min_j \frac{w_j}{d}$. On montre de plus que si $n = 2$, alors $\widetilde{K}_\infty(f)$ est soit vide, soit réduit $\{0\}$.

Abstract. Let $\mathbb{K}$ be the real or the complex field, and let $f\colon \mathbb{K}^n \to \mathbb{K}$ be a quasi-homogeneous polynomial with weight $w := (w_1, w_2, \ldots, w_n)$ and degree d. Assume that $\nabla f(0) = 0$. Łojasiewicz's well known gradient inequality states that there exists an open neighbourhood U of the origin in $\mathbb{K}^n$ and two positive constants c and $\rho < 1$ such that for any $x \in U$ we have $\|\nabla f(x)\| \geq c|f(x)|^\rho$. We prove that if the set $\widetilde{K}_\infty(f)$ of points where the Fedoryuk condition fails to hold is finite, then the gradient inequality holds true with $\rho = 1 - \min_j \frac{w_j}{d}$. It is also shown that if $n = 2$, then $\widetilde{K}_\infty(f)$ is either empty or reduced to $\{0\}$.

1. Introduction and statement of main results. Let $\mathbb{K}$ be the real or the complex field, and let $f\colon \mathbb{K}^n \to \mathbb{K}$ be a polynomial function with $f(0) = 0$ and $\nabla f(0) = 0$. According to Łojasiewicz's well known gradient inequality (see [**14**]), there exists an open neighbourhood U of the origin in $\mathbb{K}^n$ and two positive constants c and $\rho < 1$ such that for any $x \in U$ we have

$$\|\nabla f(x)\| \geq c|f(x)|^\rho. \tag{1.1}$$

2010 *Mathematics Subject Classification.* 14P10, 14P15, 32S05, 34A26.

Key words and phrases. Quasi-homogeneous, Łojasiewicz's gradient inequality, Łojasiewicz exponent, Fedoryuk's condition.

The Łojasiewicz gradient exponent of f at the origin, denoted by $\rho(f)$, is the infimum of the exponents satisfying the Łojasiewicz gradient inequality. J. Bochnak and J. J. Risler (cf. [**2**]) proved that $\rho(f)$ is a rational number, cf. also [**22**]. Moreover, Inequality (1.1) holds with exponent $\rho(f)$ and some constant $c > 0$. It is also known that (see, for example, [**1, 6**]) $\rho(f)$ can be bounded by some rational number < 1 depending on n and the degree of f only.

As is often the case, a general estimate of the Łojasiewicz gradient exponent can be replaced by a much simpler one in the weighted quasi-homogeneous case. So let f be a quasi-homogeneous polynomial with weight $w := (w_1, w_2, \ldots, w_n) \in (\mathbb{N} - \{0\})^n$ and degree $d \in \mathbb{N} - \{0\}$; that is

(1.2)
$$f(t^{w_1}x_1, t^{w_2}x_2, \ldots, t^{w_n}x_n) = t^d f(x_1, x_2, \ldots, x_n) \quad \text{for all } x \in \mathbb{K}^n \text{ and } t > 0.$$

Let $w^* := \max_{j=1,2,\ldots,n} w_j$ and $w_* := \min_{j=1,2,\ldots,n} w_j$. Assume that $\nabla f(0) = 0$. It was proven in [**8, 9**] that $\rho(f) \geq 1 - \frac{w^*}{d}$ and in the case $n = 2$ and $\mathbb{K} = \mathbb{R}$ we have $\rho(f) \leq 1 - \frac{w_*}{d}$. In particular, if f is a homogeneous polynomial in two real variables then $\rho(f) = 1 - \frac{1}{d}$.

In the present note we generalize this result to quasi-homogeneous polynomial functions $f \colon \mathbb{K}^n \to \mathbb{K}$ with the property that the set $\widetilde{K}_\infty(f)$ of points where the Fedoryuk condition fails to hold is finite. More precisely, for any polynomial $f \colon \mathbb{K}^n \to \mathbb{K}$, we let

$$\widetilde{K}_\infty(f) \quad := \quad \{\lambda \in \mathbb{K} \mid \exists x^k \to \infty, f(x^k) \to \lambda \ \text{ and } \ \|\nabla f(x^k)\| \to 0\}.$$

If $\lambda \notin \widetilde{K}_\infty(f)$, then we say that f satisfies *Fedoryuk's condition* at λ. We see that this condition restricts the asymptotic behavior of $\nabla f(x)$ as $\|x\| \to \infty$ and $f(x) \to \lambda$. The set $\widetilde{K}_\infty(f)$ has been studied by many authors; see, for instance, [**3, 4, 7, 10, 11, 13, 15, 16, 18, 19, 20, 21**].

Our main result is

THEOREM 1.1. *Let $f \colon \mathbb{K}^n \to \mathbb{K}$ be a quasi-homogeneous polynomial with weight $w := (w_1, w_2, \ldots, w_n)$ and degree $d > 1$. If the set $\widetilde{K}_\infty(f)$ is finite, then $\rho(f) \leq 1 - \frac{w_*}{d}$.*

REMARK 1.2.
(i) Let us note that [**10**] for $n = 1$ and $n = 2$ the set $\widetilde{K}_\infty(f)$ is always finite (see also Section 3 below).
(ii) In [**11**], Z. Jelonek showed that the number of points of the set $K_0(f) \cup \widetilde{K}_\infty(f)$ is less than or equal to $(\deg f - 1)^n$ provided that $\#\widetilde{K}_\infty(f) < \infty$, where $K_0(f)$ denotes the *set of critical values* of f.
(iii) As we will see in the next example, the converse of Theorem 1.1 does not hold: There exist quasi-homogeneous polynomials for which $\rho(f) \leq 1 - \frac{w_*}{d}$ and the set $\widetilde{K}_\infty(f)$ is infinite.

Example 1.3. Let $f(x,y,z) := x^2y - xz \in \mathbb{K}[x,y,z]$. Then f is a quasi-homogeneous polynomial with weight $w := (1,1,2)$ and degree $d := 3$. Define the curve

$$\varphi\colon (0,1) \to \mathbb{C}^3, \quad \tau \mapsto (\tau, \tau^{-2}, 2\tau^{-1}).$$

We have then

$$\lim_{\tau\to 0} \|\varphi(\tau)\| = \infty, \quad \lim_{\tau\to 0} f(\varphi(\tau)) = -1, \quad \lim_{\tau\to 0} \|\nabla f(\varphi(\tau))\| = 0.$$

Hence, $-1 \in \widetilde{K}_\infty(f)$. By virtue of the quasi-homogeneity of the polynomial f, we find that $\widetilde{K}_\infty(f) = \mathbb{K}$.

On the other hand, by the definition,

$$\nabla f(x,y,z) = (2xy - z, x^2, -x).$$

Hence with $c = 2^{-1/2} > 0$ we have

$$\|\nabla f(x,y,z)\| \geq c(|2xy - z| + |x|),$$

while

$$|f(x,y,z)| = |x^2y - xz| \leq |2x^2y - xz| + |x|^2 \leq |2xy - z|^2 + 2|x|^2,$$

whenever $|y| \leq 1$. Thus

$$|f(x,y,z)| \;\leq\; 2(|2xy - z| + |x|)^2 \;\leq\; (2/c^2)\|\nabla f(x,y,z)\|^2, \tag{1.3}$$

whenever $|y| \leq 1$. In particular, $\rho(f) \leq \frac{1}{2}$.

On the other hand

$$f(x,0,x) = -x^2, \quad \nabla f(x,0,x) = (-x, x^2, -x).$$

Hence Inequality (1.3) is sharp; so the polynomial f satisfies the Łojasiewicz gradient inequality for the exponent $\rho(f) = \frac{1}{2} < 1 - \frac{1}{d}$.

However, we have

Corollary 1.4. *Let $f\colon \mathbb{K}^n \to \mathbb{K}$ be a homogeneous polynomial of degree $d > 1$. Then the following conditions are equivalent*

(i) $\widetilde{K}_\infty(f)$ *is either empty or reduced to* $\{0\}$.
(ii) $\widetilde{K}_\infty(f)$ *is finite.*
(iii) $\rho(f) = 1 - \frac{1}{d}$.
(iv) There exists a positive constant c such that

$$\|\nabla f(x)\| \geq c\|f(x)\|^{1-\frac{1}{d}} \quad \textit{for all} \quad x \in \mathbb{K}^n.$$

(v) The polynomial $f(x)$ is bounded on the set $\{x \in \mathbb{K}^n \mid \|\nabla f(x)\| \leq 1\}$.
(vi) ∇f and f are separated at infinity, which means that there exist $c, R > 0$ and $q \in \mathbb{R}$ such that if $|f(x)| \geq R$ then $\|\nabla f(x)\| \geq c|f(x)|^q$.

Remark 1.5. Let $\mathbb{K} := \mathbb{C}$. It follows from the results of J. Gwoździewicz and A. Płoski [**5**] that each of conditions (i)–(vi) is equivalent to the following condition:

(vii) The polynomial f is integral over the algebra $\mathbb{C}[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}]$, which means that there exits a polynomial $P \in \mathbb{C}[y_1, y_2, \ldots, y_{n+1}]$ monic with respect to y_1 such that

$$P\left(f(x), \frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)\right) \equiv 0.$$

It should be noticed that, in his paper [**21**], S. Spodzieja proved that one can take $q = -d(d-1)^{n-1}$ in condition (vi). On the other hand, by Corollary 1.4 (iv), we may put $q = 1 - \frac{1}{d}$, which is the best (largest) possible value of q in the condition of separation at infinity (vi). One may consult, for example, [**17**] for more details about the problem of separation at infinity of arbitrary complex polynomial mappings.

Remark 1.6. Let $f \colon \mathbb{C}^n \to \mathbb{C}$ be a homogeneous polynomial such that the hypersurface $f(x) = 0$ in the projective space $\mathbb{CP}^{n-1}$ has ordinary singularities only (see [**5**] for exact definitions). Then, by the results of J. Gwoździewicz and A. Płoski [**5**], the set $\widetilde{K}_\infty(f)$ is finite. On the other hand, in general, as we will see in the next example, there exist homogeneous polynomials for which the set $\widetilde{K}_\infty(f)$ is infinite.

Example 1.7. [**8**, Remark 2.4] Consider the homogeneous polynomial $f(x, y, z) := x^2y - xz^2 \in \mathbb{K}[x, y, z]$ and define the curve

$$\varphi \colon (0, 1) \to \mathbb{C}^3, \quad \tau \mapsto (\tau^2, \frac{1}{2}\tau^{-4}, \tau^{-1}).$$

We have then

$$\lim_{\tau \to 0} \|\varphi(\tau)\| = \infty, \quad \lim_{\tau \to 0} f(\varphi(\tau)) = -\frac{1}{2}, \quad \lim_{\tau \to 0} \|\nabla f(\varphi(\tau))\| = 0.$$

Hence, $-\frac{1}{2} \in \widetilde{K}_\infty(f)$. By virtue of the homogeneity of the polynomial f, we find that $\widetilde{K}_\infty(f) = \mathbb{K}$. Together with Corollary 1.4, this implies that the polynomial f does not satisfy the Łojasiewicz gradient inequality for the exponent $1 - \frac{1}{d}$.

The paper is organized as follows. The proof of the results mentioned above will occupy Section 2. In Section 3 we present a simple elementary proof of the following result: *If $f \colon \mathbb{K}^2 \to \mathbb{K}$ is a quasi-homogeneous polynomial then the set $\widetilde{K}_\infty(f)$ is either empty or reduced to* $\{0\}$.

2. Proof of the main result. In the sequel for $t > 0$, for any $w := (w_1, w_2, \dots, w_n) \in (\mathbb{N} - \{0\})^n$ and $x := (x_1, x_2, \dots, x_n) \in \mathbb{K}^n$ we denote

$$t \bullet x \quad := \quad (t^{w_1} x_1, t^{w_2} x_2, \dots, t^{w_n} x_n).$$

Let $f\colon \mathbb{K}^n \to \mathbb{K}, x \mapsto f(x)$, be a quasi-homogeneous polynomial function with weight w and degree $d > 1$. Consider the polynomial function $g\colon \mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}, (x, y) \mapsto f(x)$. Then it follows from definitions that for $m > 0$

$$\rho(f) = \rho(g), \quad K_0(f) = K_0(g), \quad \text{and} \quad \widetilde{K}_\infty(f) \cup K_0(f) = \widetilde{K}_\infty(g).$$

Hence, in the sequel, we may without loss of generality assume that the function f really depends on all the variables. In this case, it is easy to check that d is uniquely defined by (1.2), and, in particular, we have $d \geq w^*$.

Proposition 2.1. *Under the above conventions, the set $K_0(f)$ of critical values of f is either empty or reduced to $\{0\}$. Moreover, the set $\widetilde{K}_\infty(f)$ is finite if and only if it is either empty or reduced to $\{0\}$.*

Proof. By the assumption, we have

$$f(t \bullet x) = t^d f(x) \quad \text{for all } x \in \mathbb{K}^n \text{ and for } t > 0.$$

Differentiating $f(t \bullet x)$ with respect to the variable t yields

$$dt^{d-1} f(x) \quad = \quad \sum_{j=1}^n w_j t^{w_j - 1} x_j \frac{\partial f}{\partial x_j}(t \bullet x).$$

In particular, we have the generalized Euler identity

$$df(x) \quad = \quad \sum_{j=1}^n w_j x_j \frac{\partial f}{\partial x_j}(x).$$

As an immediate corollary, the first assertion follows easily.

Moreover, it is worth noting that the polynomial $\frac{\partial f}{\partial x_j}$ is quasi-homogeneous with weight w and degree $d - w_j$. Together with the assumption, this proves the second assertion. □

Proof of Theorem 1.1. It follows from the assumptions and Proposition 2.1 that each of the the sets $K_0(f)$ and $\widetilde{K}_\infty(f)$ is either empty or reduced to $\{0\}$. As a corollary,

$$K_0(f) \cap \mathbb{S} = \emptyset \quad \text{and} \quad \widetilde{K}_\infty(f) \cap \mathbb{S} = \emptyset,$$

here $\mathbb{S} := \{\lambda \in \mathbb{K} \mid |\lambda| = 1\}$.

Put

$$c := \inf_{x \in f^{-1}(\mathbb{S})} \|\nabla f(x)\| < \infty.$$

We first show that $c > 0$. Indeed, by contradiction, assume that $c = 0$. This means that there is a sequence of points $x^k \in \mathbb{K}^n$ such that $f(x^k) \in \mathbb{S}$ and $\|\nabla f(x^k)\| \to 0$.

If a sequence x^k is bounded, then there is a subsequence $x^{k_j} \to x^0$. We have

$$f(x^0) \ \in \ \mathbb{S} \quad \text{and} \quad \|\nabla f(x^0)\| \ = \ 0.$$

This implies that $f(x^0) \in K_0(f) \cap \mathbb{S}$, which is a contradiction.

If a sequence x^k is unbounded, then there is subsequence $x^{k_j} \to \infty$ such that $f(x^{k_j}) \to \lambda \in \mathbb{S}$. Since $\|\nabla f(x^{k_j})\| \to 0$, the value λ belongs to $\widetilde{K}_\infty(f)$, which is a contradiction. Therefore $c > 0$.

On the other hand, there exists a positive number δ such that $\max\limits_{\|x\| \le \delta} |f(x)| < 1$ because $f(0) = 0$. We shall prove

$$\|\nabla f(x)\| \ge c|f(x)|^{1-\frac{w_*}{d}} \quad \text{for all } \|x\| \le \delta.$$

Indeed, let $x \in \mathbb{K}^n$ be such that $\|x\| \le \delta$ and $f(x) \neq 0$. Then

$$0 < |f(x)| < 1.$$

Consequently, $|f(t \bullet x)| = 1$, where $t := |f(x)|^{-\frac{1}{d}} > 1$. Hence, by the definition of c,

$$c \le \|\nabla f(t \bullet x)\|.$$

Since the polynomial $\frac{\partial f}{\partial x_j}$ is quasi-homogeneous with weight w and degree $d - w_j$, this gives

$$\begin{aligned} c &\le \max_{j=1,2,\ldots,n} \left| t^{d-w_j} \frac{\partial f}{\partial x_j}(x) \right| \\ &\le \max_{j=1,2,\ldots,n} t^{d-w_j} \max_{j=1,2,\ldots,n} \left| \frac{\partial f}{\partial x_j}(x) \right| = t^{d-w_*} \|\nabla f(x)\|. \end{aligned}$$

(The second inequality follows from $t > 1$ and $w_* = \min_{j=1,2,\ldots,n} w_j$.)

We obtain

$$\|\nabla f(x)\| \ge c t^{-d+w_*} = c|f(x)|^{1-\frac{w_*}{d}}.$$

It is clear that the above inequality also holds for all x such that $f(x) = 0$. Hence, by the definition of $\rho(f)$, we get $\rho(f) \le 1 - \frac{w_*}{d}$. The proof is complete. □

Proof of Corollary 1.4. It is trivial that (iv) ⇒ (i) ⇒ (ii). Clearly, (v) ⇒ (iv) ⇒ (vi). By a similar argument as in [**5**], we get (iv) ⇒ (v). On the other hand, it follows from Theorem 4.1 in [**9**] that $\rho(f) \ge 1 - \frac{w^*}{d}$. But $w^* = w_* = 1$ because f is homogeneous. Hence, in view of Theorem 1.1, we obtain the implication (ii) $\Rightarrow$ (iii). We shall show the implications (iii) $\Rightarrow$ (iv) and (vi) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (iv): Indeed, by definition of $\rho(f)$ and [**2**], there exist two positive constants c, r such that

$$\|\nabla f(x)\| \geq c|f(x)|^{1-\frac{1}{d}} \quad \text{for all } \|x\| \leq r.$$

Let x be an element of $\mathbb{K}^n, x \neq 0$. Let $t := \frac{r}{\|x\|}$. Then it is easy to check that $\|(t \bullet x)\| = r$. Hence

$$\|\nabla f(t \bullet x)\| \geq c|f(t \bullet x)|^{1-\frac{1}{d}}.$$

This gives

$$\|t^{d-1}\nabla f(x)\| \geq c|t^d f(x)|^{1-\frac{1}{d}}.$$

Consequently,

$$\|\nabla f(x)\| \geq c|f(x)|^{1-\frac{1}{d}},$$

which proves (iv).

(vi) $\Rightarrow$ (i): Let x^k be a sequence of points in $\mathbb{C}^n$ such that

$$x^k \to \infty, f(x^k) \to \lambda \ \text{ and } \ \|\nabla f(x^k)\| \to 0. \tag{2.1}$$

Let t be a positive number such that $t^d|\lambda| > R$. Then $|f(t \bullet x^k)| = t^d|f(x^k)| \geq R$ for k large enough. Hence, condition (vi) implies that

$$\|\nabla f(t \bullet x^k)\| \geq c|f(t \bullet x^k)|^q = ct^{dq}|f(x^k)|^q.$$

Let us note that the polynomial $\frac{\partial f}{\partial x_j}$ is a quasi-homogeneous polynomial with weight w and degree $d - w_j$. This, together with (2.1), implies that $\lim\limits_{k\to\infty} \|\nabla f(t \bullet x^k)\| = 0$. Therefore,

$$0 \geq ct^{dq}|\lambda|^q,$$

which yields $\lambda = 0$. This proves condition (i). $\square$

3. The Fedoryuk condition for quasi-homogeneous polynomials in two variables. The main result of this section is the following:

PROPOSITION 3.1. *Let $f\colon \mathbb{K}^2 \to \mathbb{K}$ be a quasi-homogeneous polynomial. Then $\widetilde{K}_\infty(f)$ is either empty or reduced to $\{0\}$.*

In the first step, it suffices to study the proposition for the case $\mathbb{K} = \mathbb{C}$.

LEMMA 3.2. *If Proposition 3.1 holds in the case $\mathbb{K} = \mathbb{C}$ then it also holds in the case $\mathbb{K} = \mathbb{R}$.*

PROOF. Indeed, let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a quasi-homogeneous polynomial. Let $f_{\mathbb{C}}\colon \mathbb{C}^2 \to \mathbb{C}$ be the complexification of the polynomial f. Then it follows from definitions that

$$\widetilde{K}_\infty(f) \subset \widetilde{K}_\infty(f_{\mathbb{C}}).$$

This relation proves the statement. $\square$

We next prove Proposition 3.1 in a special case.

LEMMA 3.3. *Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be a homogeneous polynomial of degree d. Then $\widetilde{K}_\infty(f)$ is either empty or reduced to $\{0\}$.*

PROOF. Suppose that $\widetilde{K}_\infty(f) \neq \emptyset$. Let $\lambda \in \widetilde{K}_\infty(f)$. We shall show that $\lambda = 0$. By definition, there exists a sequence of points $(x_k, y_k) \in \mathbb{C}^2$ such that

$$\lim_{k\to\infty} \|(x_k, y_k)\| = \infty, \ \lim_{k\to\infty} f(x_k, y_k) = \lambda, \ \lim_{k\to\infty} \|\nabla f(x_k, y_k)\| = 0.$$

Without loss of generality we may assume that $x_k \to \infty$.

There are two cases to be considered.

CASE 1: *The sequence y_k is bounded.* In this case, we have

$$\lim_{k\to\infty} \frac{y_k}{x_k} = 0.$$

On the other hand, since f is a homogeneous polynomial of degree d, we may write

$$f(x, y) = y^l (a_1 x - b_1 y)(a_2 x - b_2 y) \cdots (a_{d-l} x - b_{d-l} y),$$

where $l \in \mathbb{N}, a_i, b_i \in \mathbb{C}$ and $a_i \neq 0$ for $i = 1, 2, \ldots, d - l$.

If $l = 0$ then

$$f(x_k, y_k) = x_k^{d-l}\left(a_1 - b_1\frac{y_k}{x_k}\right)\left(a_2 - b_2\frac{y_k}{x_k}\right)\cdots\left(a_{d-l} - b_{d-l}\frac{y_k}{x_k}\right) \to \infty \text{ as } k \to \infty,$$

which contradicts the fact that $f(x_k, y_k) \to \lambda \in \mathbb{C}$. Thus $l > 0$.

On the other hand, it is easy to see that we may also expand

$$\frac{\partial f}{\partial y}(x, y) = y^{l-1}(\alpha_1 x - \beta_1 y)(\alpha_2 x - \beta_2 y) \cdots (\alpha_{d-l} x - \beta_{d-l} y),$$

where $\alpha_i, \beta_i \in \mathbb{C}$ and $\alpha_i \neq 0$ for $i = 1, 2, \ldots, d - l$.

We may then rewrite, for $x \neq 0$,

$$\begin{aligned} f(x, y) &= y^l x^{d-l}\left(a_1 - b_1\frac{y}{x}\right)\left(a_2 - b_2\frac{y}{x}\right)\cdots\left(a_{d-l} - b_{d-l}\frac{y}{x}\right), \\ \frac{\partial f}{\partial y}(x, y) &= y^{l-1} x^{d-l}\left(\alpha_1 - \beta_1\frac{y}{x}\right)\left(\alpha_2 - \beta_2\frac{y}{x}\right)\cdots\left(\alpha_{d-l} - \beta_{d-l}\frac{y}{x}\right). \end{aligned}$$

This implies that

$$f(x_k, y_k) = y_k \frac{\partial f}{\partial y}(x_k, y_k) \frac{\left(a_1 - b_1 \frac{y_k}{x_k}\right)\left(a_2 - b_2 \frac{y_k}{x_k}\right) \cdots \left(a_{d-l} - b_{d-l} \frac{y_k}{x_k}\right)}{\left(\alpha_1 - \beta_1 \frac{y_k}{x_k}\right)\left(\alpha_2 - \beta_2 \frac{y_k}{x_k}\right) \cdots \left(\alpha_{d-l} - \beta_{d-l} \frac{y_k}{x_k}\right)}.$$

An immediate consequence of this representation is

$$\lambda = \lim_{k\to\infty} f(x_k, y_k) = 0$$

because y_k is bounded and

$$\lim_{k\to\infty} \frac{\partial f}{\partial y}(x_k, y_k) = 0,$$

$$\lim_{k\to\infty} \frac{\left(a_1 - b_1 \frac{y_k}{x_k}\right)\left(a_2 - b_2 \frac{y_k}{x_k}\right) \cdots \left(a_{d-l} - b_{d-l} \frac{y_k}{x_k}\right)}{\left(\alpha_1 - \beta_1 \frac{y_k}{x_k}\right)\left(\alpha_2 - \beta_2 \frac{y_k}{x_k}\right) \cdots \left(\alpha_{d-l} - \beta_{d-l} \frac{y_k}{x_k}\right)} = \frac{a_1 a_2 \ldots a_{d-l}}{\alpha_1 \alpha_2 \ldots \alpha_{d-l}}.$$

Case 2: *The sequence y_k is unbounded.*
Having selected a subsequence, we may assume that $\lim_{k\to\infty} y_k = \infty$.
Since $\frac{\partial f}{\partial y}$ is homogeneous polynomial of degree $d-1$, we may write

$$\frac{\partial f}{\partial y}(x, y) = (\alpha_1 x - \beta_1 y)(\alpha_2 x - \beta_2 y) \cdots (\alpha_{d-1} x - \beta_{d-1} y),$$

where $\alpha_i, \beta_i \in \mathbb{C}$ and $(\alpha_i, \beta_i) \neq (0,0)$ for $i = 1, 2, \ldots, d-1$.

Since $\lim_{k\to\infty} \frac{\partial f}{\partial y}(x_k, y_k) = 0$, there exists $i_0 \in \{1, 2, \ldots, d-1\}$ such that

$$\lim_{k\to\infty} \alpha_{i_0} x_k - \beta_{i_0} y_k = 0.$$

In particular, we have that $\beta_{i_0} \neq 0$ because $\lim_{k\to\infty} x_k = \infty$.

We change the coordinates in the following way:

$$x = x, \quad u = \alpha_{i_0} x - \beta_{i_0} y.$$

Let

$$\tilde{f}(x, u) := f\left(x, \frac{\alpha_{i_0} x - u}{\beta_{i_0}}\right).$$

Then $\tilde{f}$ is homogeneous polynomial of degree d. Moreover, it is easy to check that the following conditions hold

(i) $\lim_{k\to\infty} x_k = \infty$ and $\lim_{k\to\infty} u_k = 0$, where $u_k := \alpha_{i_0} x_k - \beta_{i_0} y_k$;
(ii) $\lim_{k\to\infty} \tilde{f}(x_k, u_k) = \lim_{k\to\infty} f(x_k, y_k) = \lambda$;
(iii) $\lim_{k\to\infty} \frac{\partial \tilde{f}}{\partial x}(x_k, u_k) = \lim_{k\to\infty} \frac{\partial f}{\partial x}(x_k, y_k) = 0$; and
(iv) $\lim_{k\to\infty} \frac{\partial \tilde{f}}{\partial u}(x_k, u_k) = \lim_{k\to\infty} \left[-\frac{1}{\beta_{i_0}} \frac{\partial f}{\partial y}(x_k, y_k)\right] = 0$.

Hence, by applying Case 1 to the homogeneous polynomial $\tilde{f}$ and the sequence of points $(x_k, u_k) \in \mathbb{C}^2$, we obtain $\lambda = 0$. $\square$

Now we can pass to a proof of Proposition 3.1.

PROOF OF PROPOSITION 3.1. By Lemma 3.2, it suffices to prove the claim in the case $\mathbb{K} = \mathbb{C}$.

Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be a quasi-homogeneous polynomial with weight $w := (w_1, w_2)$ and degree d. If $w_1 = w_2$ (i.e., if the polynomial f is homogeneous), then Lemma 3.3 applies and there is nothing to prove. Thus, with no loss of generality, we may as well assume that $w_1 > w_2$.

Suppose that $\widetilde{K}_\infty(f) \neq \emptyset$. Let $\lambda \in \widetilde{K}_\infty(f)$. We need to show that $\lambda = 0$. By definition, there exists a sequence of points $(x_k, y_k) \in \mathbb{C}^2$ such that

$$(3.1)\ \lim_{k\to\infty} \|(x_k, y_k)\| = \infty, \ \lim_{k\to\infty} f(x_k, y_k) = \lambda, \ \lim_{k\to\infty} \|\nabla f(x_k, y_k)\| = 0.$$

There are two cases to be considered.

CASE 1: *The sequence y_k is bounded.* Note that $f(x, 0) = cx^m$ for some $c \in \mathbb{C}$ and $m \in \mathbb{N}$. Hence, if $y_k \equiv 0$ for k large enough, then from (3.1) it is easily seen that $\lambda = c = 0$ and there is nothing to prove. Thus, with no loss of generality, we may as well assume that $y_k \neq 0$ for large k. Let $(u_k, v_k) \in \mathbb{C}^2$ be such that

$$\begin{aligned} u_k^{w_1} &= x_k, \\ v_k^{w_2} &= y_k. \end{aligned}$$

Then $\lim_{k\to\infty} u_k = \infty$ and the limit $\lim_{k\to\infty} \frac{y_k}{u_k}$ is finite ($= 0$).

Note that

$$df(x, y) = w_1 x \frac{\partial f}{\partial x}(x, y) + w_2 y \frac{\partial f}{\partial y}(x, y).$$

Hence, it follows from (3.1) that

$$\begin{aligned} \lim_{k\to\infty} \frac{x_k}{u_k} \frac{\partial f}{\partial x}(x_k, y_k) &= 0, \\ \lim_{k\to\infty} \frac{y_k}{v_k} \frac{\partial f}{\partial y}(x_k, y_k) &= 0. \end{aligned}$$

We next need to introduce an auxiliary polynomial function $g\colon \mathbb{C}^2 \to \mathbb{C}$ by

$$g(u, v) := f(u^{w_1}, v^{w_2}).$$

Clearly, the polynomial g is homogeneous of degree d and $\lim_{k\to\infty} g(u_k, v_k) = \lambda$. Moreover, it is not hard to show that

$$\lim_{k\to\infty} \frac{\partial g}{\partial u}(u_k, v_k) = \lim_{k\to\infty} w_1 \frac{x_k}{u_k} \frac{\partial f}{\partial x}(x_k, y_k) = 0,$$
$$\lim_{k\to\infty} \frac{\partial g}{\partial v}(u_k, v_k) = \lim_{k\to\infty} w_2 \frac{y_k}{v_k} \frac{\partial f}{\partial y}(x_k, y_k) = 0.$$

In other words, $\lambda \in \widetilde{K}_\infty(g)$. By Lemma 3.3, we get $\lambda = 0$.

CASE 2: *The sequence y_k is unbounded.* Having selected a subsequence, we may assume that $\lim_{k\to\infty} y_k = \infty$. Assume that we have proved:

LEMMA 3.4. *There exists a homogeneous polynomial function $h\colon \mathbb{C}^2 \to \mathbb{C}$ of degree d such that*

$$h(u, v^{w_1}) = [f(u, v^{w_2})]^{w_1}.$$

This, of course, implies that

$$\frac{\partial h}{\partial u}(u, v^{w_1}) = w_1 [f(u, v^{w_2})]^{w_1 - 1} \frac{\partial f}{\partial x}(u, v^{w_2}),$$
$$\frac{\partial h}{\partial v}(u, v^{w_1}) = w_2 [f(u, v^{w_2})]^{w_1 - 1} v^{w_2 - w_1} \frac{\partial f}{\partial y}(u, v^{w_2}).$$

Let $(u_k, v_k) \in \mathbb{C}^2$ be such that

$$u_k = x_k,$$
$$v_k^{w_2} = y_k.$$

Then it is easy to check that $\lim_{k\to\infty} \|(u_k, v_k^{w_1})\| = \infty, \lim_{k\to\infty} h(u_k, v_k^{w_1}) = \lambda^{w_1}$ and

$$\lim_{k\to\infty} \frac{\partial h}{\partial u}(u_k, v_k^{w_1}) = \lim_{k\to\infty} w_1 [f(x_k, y_k)]^{w_1 - 1} \frac{\partial f}{\partial x}(x_k, y_k) = 0,$$
$$\lim_{k\to\infty} \frac{\partial h}{\partial v}(u_k, v_k^{w_1}) = \lim_{k\to\infty} w_2 [f(x_k, y_k)]^{w_1 - 1} v_k^{w_2 - w_1} \frac{\partial f}{\partial y}(x_k, y_k) = 0.$$

(Note that $w_1 > w_2$ and $\lim_{k\to\infty} |v_k| = \lim_{k\to\infty} |y_k|^{\frac{1}{w_2}} = \infty$.) In other words, $\lambda^{w_1} \in \widetilde{K}_\infty(h)$. Therefore, by Lemma 3.3, $\lambda^{w_1} = 0$ and hence $\lambda = 0$. This completes the proof.

So we are left with proving Lemma 3.4. Let us define the polynomial $\tilde{f}\colon \mathbb{C}^2 \to \mathbb{C}, (u, v) \mapsto \tilde{f}(u, v)$, by

$$\tilde{f}(u, v) := [f(u, v^{w_2})]^{w_1}.$$

So $\tilde{f}$ is quasi-homogeneous with weight $(w_1, 1)$ and degree $w_1 d$. Then we may write (cf. Proposition 2.1 in [**9**])

$$\tilde{f}(u,v) = \sum_{w_1 i + j = w_1 d} a_{ij} u^i v^j = \sum a_{ij} u^i v^{w_1(d-i)}.$$

Let $h(u,v) := \sum a_{ij} u^i v^{(d-i)}$. Then the polynomial h satisfies the conditions of the lemma. This completes the proof of the lemma and hence of Proposition 3.1. □

Remark 3.5. Proposition 3.1 is actually a consequence of a result of Hà Huy Vui [**10**] (see also [**3, 7, 12, 18**]). We give the present proof in order to keep our paper self-contained.

Acknowledgments. The second author would like to thank the Abdus Salam International Centre for Theoretical Physics Trieste-Italy for its hospitality and support. The second author was partially supported by NAFOSTED (Vietnam).

References

1. D'Acunto D., Kurdyka K., *Explicit bounds for the Łojasiewicz exponent in the gradient inequality for polynomials*, Ann. Pol. Math., **87** (2005), 51–61.
2. Bochnak J., Risler J. J., *Sur les exposants de Łojasiewicz*, Comment. Math. Helv., **50** (1975), 493–507.
3. Chadzyński J., Krasiński T., *The gradient of a polynomial at infinity*, Kodai Math. J., **26** (2003), 317–339.
4. Fedoryuk M. V., *The asymptotics of a Fourier transform of the exponential function of a polynomial*, Soviet Math. Dokl., **17** (1976), 486–490.
5. Gwoździewicz J., Płoski A., *On the gradient of a homogeneous polynomial*, Univ. Iagel. Acta Math., **32** (1995), 25–28.
6. Gwoździewicz J., *The Łojasiewicz exponent of an analytic function at an isolated zero*, Comment. Math. Helv., **74**, No. **3** (1999), 364–375.
7. Gwoździewicz J., Spodzieja S., *The Łojasiewicz gradient inequality in a neighbourhood of the fibre*, Ann. Polon. Math., **87** (2005), 151–163.
8. Haraux A., *Positively homogeneous functions and the Łojasiewicz gradient inequality*, Ann. Polon. Math., **87** (2005), 165–174.
9. Haraux A., Phạm T. S., *On the Łojasiewicz exponents of quasi-homogeneous functions*, preprints of the Laboratoire Jacques-Louis Lions, 2007, Univ. Pierre et Marie Curie, No. R07041 (`http://www.ann.jussieu.fr/publications/2007/R07041.pdf`).
10. Hà H. V., *Nombres de Łojasiewicz et singularités à l'infini des polynômes de deux variables complexes*, Comptes Rendus de l'Acad. des Sciences. Série I. Mathématique, **311** (1990), 429–432.
11. Jelonek Z., *On bifurcation points of a complex polynomial*, Proceedings of the AMS, **131**, No. **5** (2002), 1361–1367.
12. Kuo T. C., Parusiński A., *Newton Polygon relative to an arc*, in Real and Complex Singularities (São Carlos, 1998), Chapman & Hall Res. Notes Math., **412** (2000), 76–93.

13. Kurdyka K., Orro P., Simon S., *Semialgebraic Sard theorem for generalized critical values*, J. Diff. Geom., **56** (2000), 67–92.
14. Łojasiewicz S., *Ensembles semi-analytiques*, preprint IHES, 1965.
15. Parusiński A., *On the bifurcation set of a complex polynomial with isolated singularities at infinity*, Compositio mathematica, **97** (1995), 369–384.
16. Phạm Frederic, *Vanishing homologies and the n variables saddlepoint method*, Proc. Symp. Pure Math., **40**, (1983) 310–333.
17. Płoski A., Tworzewski P., *A separation condition for polynomial mappings*, Bull. Pol. Acad. Sci., Math., **44**, No. **3** (1996), 327–331.
18. Rabier P. J., *On the Malgrange condition for complex polynomials of two variables*, Manuscripta Math., **109** (2004), 493–509.
19. Rabier P. J., *Ehresmann's firbrations and Palais-Smale conditions for morphisms of Finsler manifolds*, Annals of Math., **146** (1997), 647–691.
20. Rodak T., *The Fedoryuk Condition and the Łojasiewicz exponent near a fibre of a polynomial*, Bull. Acad. Polon. Sci. Math., **52** (2004), 227–229.
21. Spodzieja S., *Łojasiewicz inequality at infinity for the gradient of a polynomial*, Bull. Acad. Polon. Sci. Math., **50** (2002), 273–281.
22. Spodzieja S., *The Łojasiewicz inequality at infinity for the gradient of a polynomial*, Ann. Polon. Math., **87** (2005), 247–263.

Received August 28, 2011

Laboratoire Jacques-Louis Lions
U.M.R C.N.R.S. 7598,
Université Pierre et Marie Curie
Boîte courrier 187, 75252 Paris Cedex 05
France
e-mail: `haraux@ann.jussieu.fr`

Department of Mathematics
Dalat University
1, Phu Dong Thien Vuong, Dalat
Vietnam
e-mail: `sonpt@dlu.edu.vn`