UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA
doi: 10.4467/20843828AM.12.009.1127 FASCICULUS L, 2012

BLOWUP BEHAVIOR OF THE KÄHLER–RICCI FLOW ON FANO MANIFOLDS

by Valentino Tosatti

Abstract. We study the blowup behavior at infinity of the normalized Kähler–Ricci flow on a Fano manifold which does not admit Kähler–Einstein metrics. We prove an estimate for the Kähler potential away from a multiplier ideal subscheme, which implies that the volume forms along the flow converge to zero locally uniformly away from the same set. Similar results are also proved for Aubin's continuity method.

1. Introduction. Let X be a Fano manifold of complex dimension n, which is a compact complex manifold with positive first Chern class $c_1(X)$, and let ω_0 be a Kähler metric on X with $[\omega_0] = c_1(X) > 0$. Consider the normalized Kähler–Ricci flow, which is a flow of Kähler metrics ω_t in $c_1(X)$ which evolve by

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) + \omega_t \tag{1.1}$$

with initial condition ω_0. Its fixed points are Kähler–Einstein (KE) metrics ω_{KE} which satisfy $\mathrm{Ric}(\omega_{\mathrm{KE}}) = \omega_{\mathrm{KE}}$, and it is known [**7, 16, 20, 28**] that if X admits a KE metric then the flow (1.1) converges smoothly to a (possibly different) KE metric. On the other hand not every Fano manifold admits a KE metric, and a celebrated conjecture of Yau [**31**] predicts that this happens precisely when (X, K_X^{-1}) is stable in a suitable algebro-geometric sense. The precise notion of stability is K-stability, introduced by Tian [**25**] and refined by Donaldson [**9**]. Solutions of this conjecture by Chen–Donaldson–Sun [**3**] and Tian [**26**] have appeared very recently.

2010 *Mathematics Subject Classification.* 53C44, 32Q20, 32Q15.
Key words and phrases. Kähler–Ricci flow, Kähler–Einstein metric, multiplier ideal sheaf, continuity method.
Supported in part by a Sloan Research Fellowship and NSF grant DMS-1236969.

We will also consider a different family of Kähler metrics $\tilde{\omega}_t$ in $c_1(X)$ which solve Aubin's continuity method [**1**]

$$\mathrm{Ric}(\tilde{\omega}_t) = t\tilde{\omega}_t + (1-t)\omega_0, \tag{1.2}$$

with t ranging in an interval inside $[0,1]$. We have that $\tilde{\omega}_0 = \omega_0$, and if (1.2) is solvable up to $t=1$, then $\tilde{\omega}_1$ is KE. On the other hand if no KE exists then (1.2) has a solution defined on a maximal interval $[0, R(X))$ where $R(X) \leqslant 1$ is an invariant of X (independent of ω_0) characterized by Székelyhidi [**22**] as the greatest lower bound for the Ricci curvature of metrics in $c_1(X)$.

In this note we consider a Fano manifold X which does not admit a KE metric, and investigate the question of the behavior in this case of the Kähler–Ricci flow (1.1) as $t \to \infty$ or of the continuity method (1.2) as $t \to R(X)$. In several recent works this question has been studied by reparametrizing the evolving metrics by diffeomorphisms and studying the geometric limiting space [**12, 18, 21, 27, 29**]. The key point of this note is that we do not modify the evolving metrics by diffeomorphisms, but instead we want to understand the way in which they degenerate as tensors on the fixed complex manifold X.

To state our main result, let us introduce some notation. The Kähler–Ricci flow (1.1) is equivalent to a flow of Kähler potentials in the following way. We have that $\omega_t = \omega_0 + \sqrt{-1}\partial\overline{\partial}\,\varphi_t$ where the functions φ_t evolve by

$$\frac{\partial \varphi_t}{\partial t} = \log \frac{\omega_t^n}{\omega_0^n} + \varphi_t - h_0, \quad \varphi_0 = c_0, \tag{1.3}$$

where c_0 is a suitable constant (defined in [**16**, (2.10)]) and where h_0 is the Ricci potential of ω_0 (i.e. it satisfies $\mathrm{Ric}(\omega_0) - \omega_0 = \sqrt{-1}\partial\overline{\partial} h_0$ and $\int_X (e^{h_0}-1)\omega_0^n = 0$). Let us rewrite (1.3) as the following complex Monge–Ampère equation

$$(\omega_0 + \sqrt{-1}\partial\overline{\partial}\,\varphi_t)^n = e^{h_0 - \varphi_t + \dot{\varphi}_t}\omega_0^n, \tag{1.4}$$

where here and henceforth, we'll write $\dot{\varphi}_t = \frac{\partial \varphi_t}{\partial t}$. The flow (1.3) has a global solution φ_t for all $t \geqslant 0$ [**2**], and since X does not admit KE metrics, we must have $\sup_{X\times[0,\infty)} \varphi_t = \infty$ (see e.g. [**16**]). From now on we fix a sequence of times $t_i \to \infty$ such that

$$\sup_{X\times[0,t_i]} \varphi_t = \sup_X \varphi_{t_i} \to \infty. \tag{1.5}$$

For simplicity we'll write $\varphi_i = \varphi_{t_i}$ and $\omega_i = \omega_{t_i}$.

On the other hand if we write $\tilde{\omega}_t = \omega_0 + \sqrt{-1}\partial\overline{\partial}\,\tilde{\varphi}_t$, then the continuity method (1.2) is equivalent to the complex Monge–Ampère equation

$$(\omega_0 + \sqrt{-1}\partial\overline{\partial}\,\tilde{\varphi}_t)^n = e^{h_0 - t\tilde{\varphi}_t}\omega_0^n. \tag{1.6}$$

If X does not admit KE metrics then a solution $\tilde{\varphi}_t$ exists for $t \in [0, R(X))$ with $0 < R(X) \leqslant 1$, and $\sup_X \tilde{\varphi}_t \to \infty$ as t approaches $R(X)$. We then fix a sequence $t_i \in [0, R(X))$ with $t_i \to R(X)$ and write $\tilde{\varphi}_i = \tilde{\varphi}_{t_i}$ and $\tilde{\omega}_i = \tilde{\omega}_{t_i}$.

In [**14**], Nadel proved that there is a proper analytic subvariety $S \subset X$ (a suitable multiplier ideal subscheme [**13**]) such that the measures $\tilde{\omega}_i^n$ converge (as measures) to zero on compact subsets of $X \backslash S$. More recently, the same statement was proved for the measures ω_i^n along the Kähler–Ricci flow by Clarke–Rubinstein [**5**, Lemma 6.5] (see also [**15**] for a weaker statement). It is natural to ask whether this convergence can be improved. In this note, we show that away from a possibly larger proper analytic subvariety the measures ω_i^n and $\tilde{\omega}_i^n$ converge to zero uniformly on compact sets. More precisely, we have:

THEOREM 1.1. *Assume that X is a Fano manifold that does not admit a Kähler–Einstein metric, and let φ_i, ω_i be defined as above. Then for any $\varepsilon > 0$ there is a proper nonempty analytic subvariety $S_\varepsilon \subset X$ and a subsequence of φ_i (still denoted by φ_i) such that given any compact set $K \subset X \backslash S_\varepsilon$ there is a constant C that depends only on K, ε, ω_0 such that for all $x \in K$ and for all i we have*

$$- \varphi_i(x) + (1-\varepsilon) \sup_X \varphi_i \leqslant C. \tag{1.7}$$

In particular, φ_i goes to plus infinity locally uniformly outside S_ε, and the volume forms ω_i^n converge to zero in the same sense. Finally, the same properties hold for $\tilde{\varphi}_i$ and $\tilde{\omega}_i$ which solve Aubin's continuity method.

Moreover we can identify the subvariety S_ε as follows: from weak compactness of currents, there exists ψ an L^1 function on X which is ω_0-plurisubharmonic, such that a subsequence of $\varphi_i - \sup_M \varphi_i$ converges to ψ in L^1. Then we have that

$$S_\varepsilon = V\left(\mathcal{I}\left(\frac{C}{\varepsilon}\psi\right)\right),$$

where C is a constant that depends only on ω_0, and $\mathcal{I}$ denotes the multiplier ideal sheaf. We note here that in the results of [**14**] and [**5**] the multiplier ideal sheaf that enters is $\mathcal{I}(\gamma\psi)$, with $\frac{n}{n+1} < \gamma < 1$, which gives a smaller subvariety.

Finally let us remark that we expect Theorem 1.1 to hold also when $\varepsilon = 0$, but our arguments below can only prove this when $n = 1$ (in which case the theorem is empty because there is just one Fano manifold, $\mathbb{CP}^1$, which does admit a KE metric). In fact more should be true: Tian's conjectural "partial C^0 estimate" for the continuity method [**26**] roughly says that $-\tilde{\varphi}_i + \sup_X \tilde{\varphi}_i$ should blow up at most logarithmically as we approach a subvariety. The partial C^0 estimate was proved by Tian [**24**] for Kähler–Einstein Fano surfaces and more recently by Chen–Wang [**4**] for the Kähler–Ricci flow on Fano surfaces. Very recently it was proved by Donaldson–Sun [**10**] for Kähler–Einstein metrics on Fano manifolds, by Chen–Donaldson–Sun [**3**] and Tian [**26**] for conic Kähler–Einstein metrics and by Phong–Song–Sturm [**17**] for shrinking Kähler–Ricci solitons.

Acknowledgements. Most of this work was carried out while the author was visiting the Morningside Center of Mathematics in Beijing in 2007, which he would like to thank for the hospitality. He is also grateful to S.-T. Yau for many discussions, to D. H. Phong for support and encouragement, and to B. Weinkove for useful comments.

2. Proof of the main theorem. Before we start the proof of the main theorem, we need to recall a few estimates which are known to hold along the Kähler–Ricci flow on Fano manifolds. The first one is the bound

$$|\dot{\varphi}_t| \leqslant C, \tag{2.8}$$

which holds for all $t \geqslant 0$, and was proved by Perelman (see [**20**]). It uses crucially the choice of c_0 in (1.3) given by [**16**, (2.10)]. We will also need the following uniform Sobolev inequality [**32, 33**]

$$\left(\int_X |f|^{\frac{2n}{n-1}}\omega_t^n\right)^{\frac{n-1}{n}} \leqslant C_S\left(\int_X |\nabla f|^2_{\omega_t}\omega_t^n + \int_X |f|^2\omega_t^n\right), \tag{2.9}$$

which holds for all $t \geqslant 0$ and for all $f \in C^\infty(X)$, for a constant C_S that depends only on ω_0. The following Harnack inequality [**19**] will also be used

$$-\inf_X \varphi_t \leqslant C + n\sup_X \varphi_t, \tag{2.10}$$

which again holds for all $t \geqslant 0$. Finally, we will use the following basic result:

Proposition 2.1 (Tian [**23**]). *Let (X,ω) be a compact Kähler manifold. For any fixed $\lambda > 0$ and for any sequence φ_i of Kähler potentials for ω, there exists a subsequence, still denoted by φ_i, and a proper subvariety $S \subset X$ such that for any $p \in X\backslash S$ there exists $r, C > 0$ that depend only on λ, ω and p, such that*

$$\int_{B_\omega(p,r)} e^{-\lambda(\varphi_i - \sup_X \varphi_i)}\omega^n \leqslant C. \tag{2.11}$$

Also, the constants r, C are uniform when p ranges in a compact set of $X\backslash S$.

Proof of Theorem 1.1. The starting point is the parabolic analogue of the Aubin–Yau's C^2 estimate (see e.g. [**2, 30**]), which says that there exists a constant C that depends only on ω_0 such that for all $t \geqslant 0$ we have

$$\mathrm{tr}_{\omega_0}\omega_t \leqslant Ce^{C\varphi_t - (C+1)\inf_{X\times[0,t]}\varphi_s}. \tag{2.12}$$

Here and in the following we will denote by C a uniform positive constant which might change from line to line. Combining (2.12) with (2.10) and (1.5) we get

$$\mathrm{tr}_{\omega_0}\omega_i \leqslant Ce^{C\sup_X \varphi_i + C\sup_{X\times[0,t_i]}\varphi_s} = Ce^{C\sup_X \varphi_i}.$$

Notice that for any two Kähler metrics η, χ we always have that

$$\mathrm{tr}_{\eta}\chi \leqslant \frac{1}{(n-1)!}(\mathrm{tr}_{\chi}\eta)^{n-1}\frac{\chi^n}{\eta^n},$$

and so in our case

$$\mathrm{tr}_{\omega_i}\omega_0 \leqslant Ce^{C\sup_X \varphi_i}\frac{\omega_0^n}{\omega_i^n}.$$

Using the Monge–Ampère equation (1.4) and the estimate (2.8) we get

$$\mathrm{tr}_{\omega_i}\omega_0 \leqslant Ce^{C\sup_X \varphi_i + \varphi_i} \leqslant C_0 e^{D\sup_X \varphi_i}, \tag{2.13}$$

where C_0 and D are uniform constants. An alternative derivation of (2.13) can be obtained by evolving the quantity $\log \mathrm{tr}_{\omega_t}\omega_0 - A\varphi_t$, with A large, to obtain

$$\mathrm{tr}_{\omega_t}\omega_0 \leqslant Ce^{A(\varphi_t - \inf_{X\times[0,t]}\varphi_s)},$$

and using again (2.10). As an aside, note that when $n = 1$ we can actually choose $D = 1$.

Let $A > 0$ be a constant, to be determined later, and compute

$$\Delta_{\omega_i} e^{-A\varphi_i} \geqslant -Ae^{-A\varphi_i}\Delta_{\omega_i}\varphi_i \geqslant -nAe^{-A\varphi_i}. \tag{2.14}$$

Proposition 2.2. *For any $x \in X$ and $0 < r < \mathrm{diam}(X, \omega_i)/2$ there is a constant C that depends only on A, ω_0 such that*

$$\sup_{B_{\omega_i}(x,r/2)} e^{-A\varphi_i} \leqslant \frac{C}{r^{2n}}\int_{B_{\omega_i}(x,r)} e^{-A\varphi_i}\omega_i^n, \tag{2.15}$$

holds for all i.

Proof. We apply the method of Moser iteration to the inequality (2.14). The method is standard, except for the fact that r could be bigger than the injectivity radius of ω_i and so the balls $B_{\omega_i}(x, r)$ need not be diffeomorphic to Euclidean balls, and in fact might not even be smooth domains. From now on let i be fixed, and fix two positive numbers $\rho < R < \mathrm{diam}(X, \omega_i)/2$. Then let η be a cutoff function of the form $\eta(y) = \psi(\mathrm{dist}_{\omega_i}(y, x))$ where ψ is a smooth nonincreasing function from $\mathbb{R}$ to $\mathbb{R}$ such that $\psi(y) = 1$ for $y \leqslant \rho$, $\psi(y) = 0$ for $y \geqslant (R + \rho)/2$ and

$$\sup_{\mathbb{R}} |\psi'| \leqslant \frac{4}{R-\rho}.$$

Then η is Lipschitz, equal to 1 on $B_{\omega_i}(x,\rho)$, supported inside $B_{\omega_i}(x,R)$ and satisfies

$$|\nabla\eta|_{\omega_i} \leqslant \frac{4}{R-\rho}$$

almost everywhere. For simplicity of notation we will let $f = e^{-A\varphi_i}$ and suppress all references to the metric ω_i. So we can write (2.14) as

$$\Delta f \geqslant -nAf. \tag{2.16}$$

Then for any $p \geqslant 2$ we compute

$$\int_{B(x,R)} \eta^2|\nabla(f^{p/2})|^2 = \frac{p^2}{4(p-1)}\int_{B(x,R)} \eta^2\langle\nabla(f^{p-1}),\nabla f\rangle.$$

At this point we want to integrate by parts, and we can do this because of the following argument: we can exhaust $B(x,R)$ with an increasing sequence of subdomains B_j, $j=1,2,\dots$, that have smooth boundary. Then we can apply Stokes' Theorem to each B_j, and when j is sufficiently large η will vanish on ∂B_j so we get

$$\int_{B_j} \eta^2\langle\nabla(f^{p-1}),\nabla f\rangle = -2\int_{B_j} \eta f^{p-1}\langle\nabla\eta,\nabla f\rangle - \int_{B_j}\eta^2 f^{p-1}\Delta f.$$

Then we can let j go to infinity and by dominated convergence the integrals on B_j converge to the same integrals on $B(x,R)$. Thus

$$\int_{B(x,R)} \eta^2|\nabla(f^{p/2})|^2 = -\frac{p^2}{4(p-1)}\int_{B(x,R)} \eta f^{p-1}(2\langle\nabla\eta,\nabla f\rangle + \eta\Delta f).$$

Using (2.16) and the Cauchy–Schwarz and Young inequalities we have that

$$\begin{aligned}\int_{B(x,R)} \eta^2|\nabla(f^{p/2})|^2 &\leqslant Cp\int_{B(x,R)} \eta^2 f^p + p\int_{B(x,R)} f^p|\nabla\eta|^2\\ &\quad + \frac{p}{4}\int_{B(x,R)}\eta^2 f^{p-2}|\nabla f|^2,\end{aligned} \tag{2.17}$$

where C depends only on A, n.

The last term in (2.17) is equal to $\frac{1}{p}\int_{B(x,R)}\eta^2|\nabla(f^{p/2})|^2$ and so can be absorbed in the left hand side. We thus get

$$\begin{aligned}\int_{B(x,R)} \eta^2|\nabla(f^{p/2})|^2 &\leqslant Cp\int_{B(x,R)} \eta^2 f^p + \frac{Cp}{(R-\rho)^2}\int_{B(x,R)} f^p\\ &\leqslant \frac{Cp}{(R-\rho)^2}\int_{B(x,R)} f^p,\end{aligned} \tag{2.18}$$

as long as $R-\rho$ is small. Now we use the Cauchy–Schwarz and Young inequalities again to bound

$$\begin{aligned}\int_{B(x,R)} |\nabla(\eta f^{p/2})|^2 &\leqslant 2\int_{B(x,R)} \eta^2|\nabla(f^{p/2})|^2 + 2\int_{B(x,R)} f^p|\nabla\eta|^2 \\ &\leqslant 2\int_{B(x,R)} \eta^2|\nabla(f^{p/2})|^2 + \frac{C}{(R-\rho)^2}\int_{B(x,R)} f^p.\end{aligned} \tag{2.19}$$

Combining (2.18) and (2.19) we have

$$\int_{B(x,R)} |\nabla(\eta f^{p/2})|^2 \leqslant \frac{Cp}{(R-\rho)^2}\int_{B(x,R)} f^p.$$

This together with the Sobolev inequality (2.9) gives

$$\begin{aligned}\left(\int_{B(x,R)} \eta^{2\beta} f^{p\beta}\right)^{1/\beta} &\leqslant C\int_{B(x,R)} |\nabla(\eta f^{p/2})|^2 + C\int_{B(x,R)} \eta^2 f^p \\ &\leqslant \frac{Cp}{(R-\rho)^2}\int_{B(x,R)} f^p,\end{aligned} \tag{2.20}$$

where we write $\beta = n/(n-1) > 1$. Raising this to the $1/p$ gives

$$\left(\int_{B(x,\rho)} f^{p\beta}\right)^{1/p\beta} \leqslant \frac{C^{1/p}p^{1/p}}{(R-\rho)^{2/p}}\left(\int_{B(x,R)} f^p\right)^{1/p}. \tag{2.21}$$

For each $j \geqslant 0$ we now set $p_j = 2\beta^j$ and $R_j = \rho + \frac{R-\rho}{2^j}$. Setting $p = p_j$, $R = R_j$, $\rho = R_{j+1}$ in (2.21) and iterating (notice that $(R_{j+1} - R_j) = (R-\rho)2^{-j-1}$ is small) we easily get

$$\begin{aligned}\sup_{B(x,\rho)} f &\leqslant \frac{C}{(R-\rho)^n}\left(\int_{B(x,R)} f^2\right)^{1/2} \\ &\leqslant \frac{C}{(R-\rho)^n}\left(\sup_{B(x,R)} f\right)^{1/2}\left(\int_{B(x,R)} f\right)^{1/2}.\end{aligned}$$

Using Young's inequality we see that

$$\sup_{B(x,\rho)} f \leqslant \frac{1}{2}\sup_{B(x,R)} f + \frac{C}{(R-\rho)^{2n}}\int_{B(x,R)} f.$$

From here a standard iteration argument (see e.g. [**11**, Lemma 3.4]) implies that

$$\sup_{B(x,\rho)} f \leqslant \frac{C}{(R-\rho)^{2n}}\int_{B(x,R)} f,$$

and finally setting $\rho = r/2$, $R = r$ we get (2.15). □

We now apply Proposition 2.1 with $\lambda = A+1$ and get an analytic subvariety S with the property that given any compact set $K \subset X\backslash S$ there exists $r_0 > 0$ such that for any $x \in K$ we have that $B_{\omega_0}(x, r_0) \Subset X\backslash S$ and

$$\int_{B_{\omega_0}(x,r_0)} e^{-(A+1)(\varphi_i - \sup_X \varphi_i)}\omega_0^n \leqslant C \tag{2.22}$$

holds for all i. We now let $r_i = r_0(C_0 e^{D \sup_X \varphi_i})^{-1/2}$, so that (2.13) implies that

$$B_{\omega_i}(x, r_i) \subset B_{\omega_0}(x, r_0), \tag{2.23}$$

so in particular $r_i < \mathrm{diam}(X, \omega_i)/2$. Therefore we can apply Proposition 2.2, and combining (2.15), (2.22) and (2.23) we obtain

$$\begin{aligned} e^{-A\varphi_i(x)} &\leqslant \sup_{B_{\omega_i}(x,r_i/2)} e^{-A\varphi_i} \\ &\leqslant \frac{C}{r_i^{2n}} \int_{B_{\omega_i}(x,r_i)} e^{-A\varphi_i}\omega_i^n \\ &\leqslant Ce^{nD\sup_X \varphi_i} \int_{B_{\omega_0}(x,r_0)} e^{-(A+1)\varphi_i}\omega_0^n \\ &= Ce^{(nD-A-1)\sup_X \varphi_i} \int_{B_{\omega_0}(x,r_0)} e^{-(A+1)(\varphi_i - \sup_X \varphi_i)}\omega_0^n \\ &\leqslant Ce^{(nD-A-1)\sup_X \varphi_i}, \end{aligned}$$

where C depends only on given data and A. Taking log gives

$$-A\varphi_i(x) + (A - nD + 1)\sup_X \varphi_i \leqslant C.$$

We now let $A = \frac{nD}{\varepsilon}$ and divide by A (keeping in mind that $\sup_X \varphi_i > 0$), and obtain the desired bound

$$-\varphi_i(x) + (1-\varepsilon)\sup_X \varphi_i \leqslant C,$$

where C does not depend on i or on $x \in K \subset X\backslash S$, and where the subvariety $S = S_\varepsilon$ now depends on ε. This, together with (2.8), immediately implies that the volume form $\omega_i^n = \omega_0^n e^{h_0 - \varphi_i + \dot{\varphi}_{t_i}}$ goes to zero locally uniformly on $X\backslash S$. □

We now identify the subvariety S_ε: from its definition, that is from Proposition 2.1, we see that S_ε is equal to the multiplier ideal subscheme of the sequence φ_i with exponent $(nD/\varepsilon + 1)$ as defined by Nadel in [**13**]. But the main Theorem in [**8**] then shows that this is the same as the multiplier ideal subscheme defined by $(nD/\varepsilon + 1)\psi$ where ψ is a weak limit of $\varphi_i - \sup_M \varphi_i$.

The same proof as above goes through with minimal changes in the case of Aubin's continuity method, that is for the functions $\tilde{\varphi}_i$ and the metrics $\tilde{\omega}_i$. We just need to justify why estimates analogous to (2.9) and (2.10) hold. To see these, note that the metrics $\tilde{\omega}_i$ satisfy $\mathrm{Ric}(\tilde{\omega}_i) \geqslant C^{-1}\tilde{\omega}_i > 0$, so the Bonnet–Myers theorem gives us the estimate $\mathrm{diam}(X, \tilde{\omega}_i) \leqslant C$, which together with the fact that their volume is fixed allows us to apply a result of Croke [**6**] which gives a uniform Sobolev inequality of the form (2.9) for the metrics $\tilde{\omega}_i$. The Harnack inequality (2.10) in this case is proved in [**23**].

References

1. Aubin T., *Réduction du cas positif de l'équation de Monge–Ampère sur les variétés kählériennes compactes à la démonstration d'une inégalité*, J. Funct. Anal., **57**, No. **2** (1984), 143–153.
2. Cao H.-D., *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math., **81**, No. **2** (1985), 359–372.
3. Chen X., Donaldson S. K., Sun S., *Kähler–Einstein metrics and stability*, Int. Math. Res. Not. IMRN, **2013**, Art. ID rns279, 7 pp.
4. Chen X., Wang B., *The Kähler Ricci flow on Fano manifolds (I)*, J. Eur. Math. Soc. (JEMS), **14**, No. **6** (2012), 2001–2038.
5. Clarke B., Rubinstein Y., *Ricci flow and the metric completion of the space of Kähler metrics*, to appear in Amer. J. Math., `arXiv:1102.3787`.
6. Croke C. B., *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. Ècole Norm. Sup. (4), **13**, No. **4** (1980), 419–435.
7. Collins T. C., Székelyhidi G., *The twisted Kähler–Ricci flow*, `arXiv:1207.5441`.
8. Demailly J.-P., Kollár J., *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, Ann. Sci. École Norm. Sup. (4), **34**, No. **4** (2001), 525–556.
9. Donaldson S. K., *Scalar curvature and stability of toric varieties*, J. Differential Geom., **62**, No. **2** (2002), 289–349.
10. Donaldson S. K., Sun S., *Gromov–Hausdorff limits of Kähler manifolds and algebraic geometry*, `arXiv:1206.2609`.
11. Han Q., Lin F., *Elliptic partial differential equations*, AMS 1997.
12. Li C., *On the limit behavior of metrics in continuity method to Kähler–Einstein problem in toric Fano case*, to appear in Compos. Math., `arXiv:1012.5229`.
13. Nadel A. M., *Multiplier ideal sheaves and Kähler–Einstein metrics of positive scalar curvature*, Ann. of Math. (2,) **132**, No. **3** (1990), 549–596.
14. Nadel A. M., *Multiplier ideal sheaves and Futaki's invariant*, in *Geometric theory of singular phenomena in partial differential equations (Cortona, 1995)*, 7–16, Cambridge Univ. Press, 1998.
15. Pali N., *Characterization of Einstein–Fano manifolds via the Kähler–Ricci flow*, Indiana Univ. Math. J., **57**, No. **7** (2008), 3241–3274.
16. Phong D. H., Šešum N., Sturm J., *Multiplier ideal sheaves and the Kähler–Ricci flow*, Comm. Anal. Geom., **15**, No. **3** (2007), 613–632.
17. Phong D. H., Song J., Sturm J., *Degeneration of Kähler–Ricci solitons on Fano manifolds*, `arXiv:1211.5849`.
18. Phong D. H., Song J., Sturm J., Weinkove B., *The Kähler–Ricci flow and the $\overline{\partial}$ operator on vector fields*, J. Differential Geom., **81**, No. **3** (2009), 631–647.

19. Rubinstein Y. A., *On the construction of Nadel multiplier ideal sheaves and the limiting behavior of the Ricci flow*, Trans. Amer. Math. Soc., **361**, No. **11** (2009), 5839–5850.
20. Šešum N., Tian G., *Bounding scalar curvature and diameter along the Kähler Ricci flow (after Perelman)*, J. Inst. Math. Jussieu, **7**, No. **3** (2008), 575–587.
21. Shi Y., Zhu X., *An example of a singular metric arising from the blow-up limit in the continuity approach to Kähler–Einstein metrics*, Pacific J. Math., **250**, No. **1** (2011), 191–203.
22. Székelyhidi G., *Greatest lower bounds on the Ricci curvature of Fano manifolds*, Compos. Math., **147**, No. **1** (2011), 319–331.
23. Tian G., *On Kähler–Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$*, Invent. Math., **89**, No. **2** (1987), 225–246.
24. Tian G., *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math., **101**, No. **1** (1990), 101–172.
25. Tian G., *Kähler–Einstein metrics with positive scalar curvature*, Invent. Math., **130**, No. **1** (1997), 1–37.
26. Tian G., *K-stability and Kähler-Einstein metrics*, `arXiv:1211.4669`.
27. Tian G., Wang B., *On the structure of almost Einstein manifolds*, `arXiv:1202.2912`.
28. Tian G., Zhu X., *Convergence of Kähler–Ricci flow*, J. Amer. Math. Soc., **20**, No. **3** (2007), 675–699.
29. Tosatti V., *Kähler–Ricci flow on stable Fano manifolds*, J. Reine Angew. Math., **640** (2010), 67–84.
30. Yau S.-T., *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I*, Comm. Pure Appl. Math., **31** (1978), 339–411.
31. Yau S.-T., *Open problems in geometry*, Proc. Sympos. Pure Math., **54** (1993), 1–28 (problem 65).
32. Ye R., *The logarithmic Sobolev inequality along the Ricci flow*, preprint, `arXiv:math/0707.2424`.
33. Zhang Q. S., *A uniform Sobolev inequality under Ricci flow*, Int. Math. Res. Not. IMRN, **2007**, no. **17**, Art. ID rnm056, 17 pp. and erratum in Int. Math. Res. Not. IMRN, **2007**, No. **19**, Art. ID rnm096, 4 pp.

Received December 16, 2012

Department of Mathematics
Northwestern University
2033 Sheridan Road
Evanston, IL 60201, USA
e-mail: `tosatti@math.northwestern.edu`