A NOTE ON RELATION BETWEEN AZUKAWA ISOMETRIES AT ONE POINT AND GLOBAL BIHOLOMORPHISMS

by Małgorzata Zajęcka

Abstract. We prove that under certain assumptions holomorphic functions which are Azukawa isometries at one point are in fact biholomorphisms.

1. Introduction. The holomorphic contractibility of Carathéodory–Reiffen and Kobayashi–Royden pseudometrics have put much interest in the relation of global biholomorphicity and Carathéodory or Kobayashi isometricity at one point. While from the mentioned property it immediately follows that a biholomorphism must be a Carathéodory and Kobayashi isometry, the oposite statement is obviously not true in the general case. In 1984 Jean-Pierre Vigué proved the following result.

THEOREM (see [**9**]). *Let Ω be a bounded convex domain in $\mathbb{C}^n$ and let M be a complex manifold on which a Carathéodory–Reiffen pseudodistance is a distance. Suppose $F : \Omega \to M$ is a holomorphic mapping which is a Carathéodory–Reiffen isometry at a point $p \in \Omega$. Then F is a biholomorphism.*

A few years later Ian Graham proved analogous theorem for a Kobayashi––Royden isometry.

THEOREM (see [**2**]). *Suppose M is a taut complex manifold of dimension n. Suppose Ω is a strictly convex bounded domain in $\mathbb{C}^n$. Suppose $F : M \to \Omega$ is a holomorphic mapping which is a Kobayashi–Royden isometry at a point $p \in M$. Then F is a biholomorphism.*

In this paper we switch our interest to another holomorphically contractible pseudometric: the Azukawa pseudometric A_G. We obtain the following main result.

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THEOREM 1. *Let $G_1, G_2 \subset \mathbb{C}^n$ be domains. Let $a \in G_1$ and let $F : G_1 \to G_2$ be such that:*

(1) $F \in \mathcal{O}(G_1, G_2)$*;*

(2) G_1 *is taut;*

(3) G_2 *is bounded;*

(4) *for any $z \in G_1$ we have $g_{G_1}(a, z) = l^*_{G_1}(a, z)$;*

(5) *for any $X \in \mathbb{C}^n$ we have $A_{G_2}(F(a); F'(a)X) = A_{G_1}(a; X)$.*

Then F is a biholomorphism.

Proof of this result is given in the last section of this paper.

The following example shows a material difference between our result and theorems of Vigué and Graham.

EXAMPLE 2. Observe that if G is a taut balanced pseudoconvex domain, then assumption (4) in Theorem 1 with $G_1 = G$ is satisfied (see [**3**, Proposition 4.2.7]). We can also use Theorem 1 with $G_2 = G$ (provided G_1 is such that (5) is satisfied). However, taut balanced pseudoconvex domains do not need to be convex, thus in many cases we may apply neither Vigué's nor Graham's theorems with G in the role of Ω. A good example of that situation is the set $G := \{(z, w) \in \mathbb{C}^n : |z| < 1, |w| < 1, |zw| < \alpha\}$, for a fixed $0 < \alpha < 1$.

2. Preliminaries. In this section we recall the definitions and basic properties of the Green function and Azukawa pseudometric. For the detailed proofs, more interesting facts about these objects and their connections to Kobayashi and Carathéodory pseudodistances and pseudometrics see for instance [**1**, **3**–**5**], and [**6**].

Let G be a domain in $\mathbb{C}^n$. To simplify the definitions, for $a \in G$ let $\exp \mathcal{L}_a$ denote the family of functions $u : G \to [0, 1)$ such that $\log u$ is plurisubharmonic on G and there exists a positive constant M such that $u(z) \le M\|z - a\|$, $z \in G$.

DEFINITION 3. Define

$$g_G(a, z) := \sup\{u(z) : \ u \in \exp \mathcal{L}_a\}, \ a \in G, \ z \in G;$$

$$A_G(a; X) := \sup\{\limsup_{0 \ne \lambda \to 0} \frac{u(a + \lambda X)}{|\lambda|} : \ u \in \exp \mathcal{L}_a\}, \ a \in G, \ X \in \mathbb{C}^n.$$

The function g_G is called *a pluricomplex Green function with a pole at point a* and A_G is called *an Azukawa pseudometric*.

Both g_G and A_G are holomorphically contractible, i.e. for a domain $D \subset \mathbb{C}^m$ and a holomorphic function $F : G \to D$ we have

$$g_D(F(a), F(z)) \le g_G(a, z), \ a, z \in G,$$

and

$$A_D(F(a); F'(a)X) \le A_G(a; X), \ a \in G, \ X \in \mathbb{C}^n.$$

Obviously, if F is a biholomorphism, we get equalities: this property is called biholomorphic invariance. Moreover

$$g_{\mathbb{D}}(\lambda', \lambda'') = m(\lambda', \lambda'') := |(\lambda' - \lambda'')/(1 - \lambda'\overline{\lambda''})|, \ \lambda', \lambda'' \in \mathbb{D},$$

and

$$A_{\mathbb{D}}(\lambda; X) = |X|\gamma(\lambda) := |X|/(1 - |\lambda|^2), \ \lambda \in \mathbb{D}, \ X \in \mathbb{C},$$

where by $\mathbb{D}$ we denote a unit disc in $\mathbb{C}$.

One can show that $g_G(a, \cdot) \in \exp \mathcal{L}_a$, $a \in G$, (see [**5**]). Consequently we obtain an equivalent and much more useful definition of Azukawa pseudometric

$$A_G(a; X) = \limsup_{0 \neq \lambda \to 0} \frac{g_G(a, a + \lambda X)}{|\lambda|}, \ a \in G, \ X \in \mathbb{C}^n.$$

Now let us introduce the following notation. By l_G we denote the Lempert function, by k_G the Kobayashi pseudodistance and by K_G the Kobayashi––Royden pseudometric. We also use the convention: for a function f let f^* denote $\tanh f$.

DEFINITION 4. We say that a domain $G \subset \mathbb{C}^n$ is *k-hyperbolic* (resp. *K-hyperbolic*), if for any $z, w \in G$, $z \neq w$, we have $k_G(z, w) > 0$ (resp. for any $z \in G$, $X \in \mathbb{C}^n$, $X \neq 0$, we have $K_G(z; X) > 0$).

For a domain in $\mathbb{C}^n$ k-hyperbolicity is equivalent to K-hyperbolicity (see for instance [**3**, Theorem 7.2.2]). In the proof of Theorem 1 we will use the following property.

LEMMA 5 (see [**3**, Corollary 7.2.4]). *Any taut domain in $\mathbb{C}^n$ is k-hyperbolic.*

3. Proof of the main theorem. Before we proceed to the proof, we need one easy but interesting lemma.

LEMMA 6. *Let $G \subset \mathbb{C}^n$ be a domain such that $0 \in G$ and let $\varphi \in \mathcal{O}(\mathbb{D}, G)$ be such that $\varphi(0) = 0$. Then the following conditions are equivalent:*

(i) *there exists a $\lambda' \in \mathbb{D} \setminus \{0\}$ such that $g_G(0, \varphi(\lambda')) = |\lambda'|$;*
(ii) *$g_G(0, \varphi(\lambda)) = |\lambda|$ for all $\lambda \in \mathbb{D}$;*
(iii) *$A_G(0; \varphi'(0)) = 1$.*

PROOF. The proof is similar to the proof of an analogous theorem for complex Carathéodory and Carathéodory–Reiffen geodesics (see Proposition 8.1.3 in [**3**]). □

PROOF OF THEOREM 1. Using biholomorphic invariance of g_G and A_G we may assume $a = 0$ and $F(a) = 0$. Observe that $A_{G_1}(0; X) > 0$ for $X \neq 0$.

Indeed, using (4), Proposition 3.18 from [**7**], and Lemma 5 we obtain

$$A_{G_1}(0; X) = \limsup_{0 \neq \lambda \to 0} \frac{g_{G_1}(0, \lambda X)}{|\lambda|} = \limsup_{0 \neq \lambda \to 0} \frac{l_{G_1}^*(0, \lambda X)}{|\lambda|}$$
$$= \limsup_{0 \neq \lambda \to 0} \frac{l_{G_1}(0, \lambda X)}{|\lambda|} = K_{G_1}(0; X) > 0.$$

Now, from (5) we get $A_{G_2}(0; F'(0)X) > 0$, $X \neq 0$. Thus, $F'(0)$ is an isomorphism and so F is injective in a neighborhood U of zero. From the tautness of G_1 we get the equality between the Euclidean topology of G_1 and its Kobayashi topology (see [**3**, Proposition 3.3.4]). Using (5) we may assume $U \supset B_{g_{G_1}}(r_0) := \{z \in G_1 : g_{G_1}(0, z) < r_0\}$ for some $r_0 \in (0, 1)$.

We show that $g_{G_2}(0, F(z)) = g_{G_1}(0, z)$ for $z \in G_1$. Fix a $z_0 \in G_1 \setminus \{0\}$. Since G_1 is taut, there exists an extremal disc $\varphi \in \mathcal{O}(\mathbb{D}, G_1)$ for the pair $(0, z_0)$, i.e. $\varphi(0) = 0$, $\varphi(\lambda_0) = z_0$, and $l_{G_1}^*(0, \varphi(\lambda_0)) = |\lambda_0|$ for some $\lambda_0 \in (0, 1)$. Thus $g_{G_1}(0, \varphi(\lambda_0)) = |\lambda_0|$ and from Lemma 6 we obtain $A_{G_1}(0; \varphi'(0)) = 1$. From (5) we get $A_{G_2}((F \circ \varphi)(0); (F \circ \varphi)'(0)) = 1$. Thus once again from Lemma 6 we get $g_{G_2}((F \circ \varphi)(0), (F \circ \varphi)(\lambda)) = |\lambda|$ for any $\lambda \in \mathbb{D}$. Hence

$$g_{G_2}(0, F(z_0)) = |\lambda_0| = g_{G_1}(0, z_0).$$

Consequently, $F^{-1}(B_{g_{G_2}}(r)) \subset B_{g_{G_1}}(r)$ and $F(B_{g_{G_1}}(r)) \subset B_{g_{G_2}}(r)$, $r \in (0, 1)$. Therefore F is a proper holomorphic map $G_1 \to G_2$. In particular, F is surjective. Consequently, $F(B_{g_{G_1}}(r)) = B_{g_{G_2}}(r)$ and $F^{-1}(B_{g_{G_2}}(r)) = B_{g_{G_1}}(r)$, $r \in (0, 1)$[1].

Define $V := \{z \in G_1 : J_F(z) = 0\}$. Then (see [**8**, Chapter 15.1])

- $F(V)$ is an analytic subset of G_2;
- there exists an $m \in \mathbb{N}$ such that $\sharp F^{-1}(w) = m$ for $w \notin F(V)$ and $\sharp F^{-1}(w) < m$ for $w \in F(V)$;
- $F : G_1 \setminus F^{-1}(F(V)) \to G_2 \setminus F(V)$ is a holomorphic covering map.

Since $F : B_{g_{G_1}}(r_0) \to B_{g_{G_2}}(r_0)$ is biholomorphic, we conclude that $m = 1$. Thus F is a biholomorphism. $\qquad\square$

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[1]Observe that we could not get the equalities without surjectivity of F.

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Institute of Mathematics
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland

Institute of Mathematics
Cracow University of Technology
Warszawska 24
31-155 Kraków, Poland
e-mail: `malgorzata.zajecka@gmail.com`