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THE DEGENERATE J-FLOW AND THE MABUCHI ENERGY ON MINIMAL SURFACES OF GENERAL TYPE

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Abstract. We prove existence, uniqueness and convergence of solutions of the degenerate J-flow on Kähler surfaces. As an application, we establish the properness of the Mabuchi energy for Kähler classes in a certain subcone of the Kähler cone on minimal surfaces of general type.

1. Introduction. Let M be a compact Kähler surface with two Kähler metrics ω_0 and χ_0. Let $\mathcal{P}_{\chi_0}$ be the space of smooth functions φ with $\chi_\varphi := \chi_0 + dd^c\varphi > 0$. The *J-flow* is a parabolic flow defined on $\mathcal{P}_{\chi_0}$ by

$$\frac{\partial}{\partial t}\varphi = c_0 - \frac{2\chi_\varphi \wedge \omega_0}{\chi_\varphi^2}, \quad \varphi|_{t=0} = \varphi_0 \in \mathcal{P}_{\chi_0}, \tag{1.1}$$

where c_0 is defined by

$$c_0 = \frac{2[\chi_0]\cdot[\omega_0]}{[\chi_0]^2}.$$

The J-flow was introduced by Donaldson [**7**] in the setting of moment maps and by Chen [**2**] as the gradient flow of the $\mathcal{J}$-functional which appears in his formula for the Mabuchi energy. Smooth solutions to (1.1) exist for all time and are unique [**3**].

Under the assumption

$$c_0[\chi_0] - [\omega_0] > 0, \tag{1.2}$$

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it was shown in [**32**] that the solution to the J-flow converges smoothly to φ_∞ solving the critical equation

$$2\chi_{\varphi_\infty} \wedge \omega_0 = c_0 \chi_{\varphi_\infty}^2. \tag{1.3}$$

The fact that smooth solutions to (1.3) exist under the condition (1.2) was conjectured by Donaldson [**7**] and proved by Chen by reducing the equation to the complex Monge–Ampère equation solved by Yau [**34**]. In higher dimensions, it was shown in [**33**] that the flow converges under the cohomological assumption $c_0[\chi] - (n-1)[\omega_0] > 0$. Necessary and sufficient conditions for convergence of the J-flow in terms of $[\chi_0]$ and ω_0 were found in [**26**].

In [**9, 11**], convergence results were proved for generalizations of the J-flow known as inverse σ_k-flows. In [**10**], Fang–Lai analyzed the behavior of the inverse σ_k-flow on general Kähler classes for metrics with Calabi symmetry. The J-flow has been investigated on Hermitian manifolds by Y. Li [**20**], and the critical equation on Hermitian manifolds with boundary by Guan–Li [**13**]. An equation bearing strong similarities to the critical equation for the J-flow is the *complex Hessian equation* (see for example [**1, 5, 14, 15, 19**]); it has been studied intensely in the last few years, and the existence of solutions on compact Kähler manifolds was recently established by Dinew–Kołodziej [**6**].

We now return to the discussion of the J-flow. In complex dimension two, the behavior of the J-flow was investigated [**12**] in the case where $c_0[\chi_0] - [\omega_0] \geqslant 0$, where $\beta \geqslant 0$ means that the cohomology class β admits a smooth nonnegative representative. A uniform L^∞ estimate for φ was established, and it was shown that the J-flow converges smoothly to a singular Kähler metric away from a finite number of curves of negative self-intersection.

In this paper, we generalize the result of [**32**] in a different direction. We consider the case where ω_0 is no longer a Kähler metric, but a closed (1,1) form satisfying a certain nonnegativity condition. More precisely, assume that we have a background Kähler metric $\hat{\omega}$ and an effective divisor D on M with associated line bundle $[D]$. Let H be a fixed Hermitian metric on the line bundle $[D]$, and let s be a holomorphic section of $[D]$ which vanishes exactly along D. Our assumption on ω_0 is:

$$\begin{gathered} \omega_0 \text{ is a smooth closed (1,1) form} \\ \text{with } \omega_0 \geqslant \frac{1}{C_0}|s|_H^{2\beta}\hat{\omega} \text{ and } \omega_0 - \rho R_H \geqslant \frac{1}{C_0}\hat{\omega}, \end{gathered} \tag{1.4}$$

for some positive constants C_0, β, ρ. Here, $R_H = -dd^c \log H$ denotes the curvature form of the Hermitian metric H.

Clearly (1.4) holds if ω_0 is Kähler. It is not uncommon for non-Kähler cohomology classes to admit a closed form ω_0 satisfying (1.4). Indeed, we will see below that if M is a minimal surface of general type then the canonical class, if it is not ample, is such an example. Also, it is well-known that such classes can be found on the boundary of the Kähler cone of blow-ups of Kähler manifolds, as discussed in [**27**] for example.

We call the equation (1.1) with ω_0 satisfying (1.4) the *degenerate J-flow.* Now (1.1) is no longer a parabolic equation in general and we cannot expect to obtain smooth solutions. The main result of this paper is that there exists a unique "weak" solution to this degenerate parabolic equation, which converges to a "weak" solution of the critical equation. More precisely, define a space

$$\mathcal{P}^{\text{weak}}_{\chi_0} := \{\varphi \in C^\infty(M \setminus D) \cap \text{PSH}_{\chi_0}(M) \cap L^\infty(M) \mid \chi_0 + dd^c\varphi > 0 \text{ on } M \setminus D\}, \tag{1.5}$$

where we are writing D also for the subset of M defined by the divisor D. Here $\text{PSH}_{\chi_0}(M)$ consists of upper semicontinuous functions $\varphi : M \to [-\infty, \infty)$ such that $\varphi + \psi_0$ is plurisubharmonic, where ψ_0 is a (smooth) local Kähler potential for χ_0.

We have the following result.

Theorem 1.1. *Let M be a compact Kähler surface, with Kähler metrics χ_0 and $\hat{\omega}$. Assume that ω_0 satisfies (1.4) and assume that*

$$c_0[\chi_0] - [\omega_0] > 0, \quad \textit{for} \quad c_0 = \frac{2[\chi_0]\cdot[\omega_0]}{[\chi_0]^2}. \tag{1.6}$$

For any smooth $\varphi_0 \in \mathcal{P}_{\chi_0}$, there exists a unique solution $\varphi = \varphi(t) \in \mathcal{P}^{\text{weak}}_{\chi_0}$ of the degenerate J-flow

$$\frac{\partial}{\partial t}\varphi = c_0 - \frac{2\chi_\varphi \wedge \omega_0}{\chi_\varphi^2} \quad \textit{on } M \setminus D, \quad \varphi|_{t=0} = \varphi_0, \tag{1.7}$$

with $\sup_{M\setminus D}\left|\frac{\partial \varphi}{\partial t}\right|$ bounded (independent of t).

As $t \to \infty$,

$$\varphi(t) \overset{C^\infty_{\text{loc}}(M\setminus D)}{\longrightarrow} \varphi_\infty,$$

where $\varphi_\infty \in \mathcal{P}^{\text{weak}}_{\chi_0}$ solves the critical equation

$$2\chi_{\varphi_\infty} \wedge \omega_0 = c_0\chi^2_{\varphi_\infty} \quad \textit{on} \quad M \setminus D. \tag{1.8}$$

Moreover, $\varphi_\infty \in \mathcal{P}^{\text{weak}}_{\chi_0}$ is the unique solution of the critical equation up to the addition of a constant.

Note that φ_∞ coincides with the pluripotential solution of (1.8) on M in the sense of Bedford–Taylor (see [**18**], for example).

We recall now the $\mathcal{J}$-functional of Chen [**2**]. Given a Kähler form χ_0 and a closed $(1,1)$ form ω_0, define the $\mathcal{J}$-functional by

$$\mathcal{J}_{\omega_0,\chi_0}(\varphi) = \int_0^1 \int_M \dot{\varphi}_s(2\chi_{\varphi_s} \wedge \omega_0 - c_0\chi^2_{\varphi_s})ds, \quad \text{for } \varphi \in \mathcal{P}_{\chi_0}, \tag{1.9}$$

where φ_s is a smooth path in $\mathcal{P}_{\chi_0}$ between 0 and φ. In the case where ω_0 is Kähler, the J-flow is the gradient flow of the $\mathcal{J}$-functional. A consequence of our main result is that the $\mathcal{J}$-functional is uniformly bounded from below on the space $\mathcal{P}_{\chi_0}$.

Corollary 1.2. *As in Theorem 1.1, let M be a compact Kähler surface with Kähler metrics χ_0 and $\hat{\omega}$. Assume that ω_0 satisfies (1.4) and that (1.6) holds. Then there exists a constant K depending only on the fixed data M, ω_0, χ_0 such that*

$$\mathcal{J}_{\omega_0,\chi_0}(\varphi) \geqslant K, \quad \textit{for all } \varphi \in \mathcal{P}_{\chi_0}. \tag{1.10}$$

This result has an immediate application to the *Mabuchi energy functional*, which we now explain. A well-known open problem in Kähler geometry is to determine which Kähler classes on M admit Kähler metrics of constant scalar curvature. The Yau–Tian–Donaldson conjecture relates this to a notion of stability in the sense of geometric invariant theory [**8, 28, 35**]. A related question is to ask instead for which Kähler classes is the Mabuchi energy functional *proper* (see Section 3 below for the definition). Indeed, according to a conjecture of Tian [**29**], these questions are essentially equivalent, modulo some issues which arise if M admits holomorphic vector fields.

It was shown by Chen [**2**] that if the canonical bundle K_M of M satisfies $K_M > 0$ then the Mabuchi energy is bounded below on all Kähler classes $[\chi_0]$ satisfying

$$\left(\frac{2[\chi_0]\cdot K_M}{[\chi_0]^2}\right)[\chi_0] - K_M > 0. \tag{1.11}$$

An alternative proof of this was given by the second-named author using the J-flow [**32**]. Later, the authors observed [**26**] that under the same assumption (1.11), it follows from a result of Tian [**29**] that in fact the Mabuchi energy is not just bounded below but proper. Moreover, we proved analogous results on manifolds M of any dimension with $K_M > 0$.

Fang–Lai–Song–Weinkove [**12**] recently showed that the assumption (1.11) can be weakened to

$$\left(\frac{2[\chi_0]\cdot K_M}{[\chi_0]^2}\right)[\chi_0] - K_M \geqslant 0, \tag{1.12}$$

where $\sigma \geqslant 0$ means that the cohomology class σ admits a smooth nonnegative representative.

In this paper, we instead allow K_M to satisfy a more general nonnegativity condition than being ample. We assume that K_M is big and nef, meaning that $K_M^2 > 0$ and $K_M \cdot C \geqslant 0$ for all curves C on M. This is equivalent to saying that M is a minimal surface of general type.

Corollary 1.3. *Let M be a minimal surface of general type. Then the Mabuchi energy is proper on all Kähler classes $[\chi_0]$ satisfying*

$$\left(\frac{2[\chi_0] \cdot K_M}{[\chi_0]^2}\right)[\chi_0] - K_M > 0. \tag{1.13}$$

Thus one would expect constant scalar curvature Kähler metrics to exist in these classes. It is also expected that such classes are *K-stable* in the sense of [**8, 28**]. In fact, it follows immediately from results of Panov–Ross (see the argument of [**23**, Example 5.9]) that algebraic classes satisfying (1.13) are *slope stable* in the sense of Ross–Thomas [**24**].

The main technical results are contained in Section 2. The idea is to replace the degenerate $(1,1)$-form ω_0 with a Kähler form ω_ε for $\varepsilon > 0$ and obtain estimates for the J-flow away from D which are independent of ε. As $\varepsilon \to 0$, we have $\omega_\varepsilon \to \omega_0$ and we obtain a solution of the degenerate J-flow (cf. results of Song–Tian [**25**] in the case of the Kähler–Ricci flow). The key estimates are contained in Proposition 2.1. In Section 3, we prove Theorem 1.1 and its corollaries.

Finally, some words about notation. When we are given a (1,1) form β, we will often define a tensor with components $\beta_{i\bar{j}}$ by $\beta = \sqrt{-1}\beta_{i\bar{j}}dz^i \wedge \overline{dz^j}$. An exception to this notation is that for a Kähler form ω we write $\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge \overline{dz^j}$, and similarly for ω_0, g_0, etc. Given a positive definite $(1,1)$ form α and a $(1,1)$ form β, we write $\mathrm{tr}_\alpha \beta$ for $\alpha^{\bar{j}i}\beta_{i\bar{j}}$, where $(\alpha^{\bar{j}i})$ is the inverse of $(\alpha_{i\bar{j}})$. We will often denote uniform constants by $C, C_0, C_1, C', C'', \ldots$ etc., which may differ from line to line.

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2. Estimates for solutions of the J-flow. The degenerate J-flow (1.7) defined in the introduction is not a parabolic equation. We peturb the equation to make it parabolic.

Assume we are in the setting of Theorem 1.1. Write $\omega_\varepsilon = \omega_0 + \varepsilon\hat{\omega} > 0$ and define

$$c_\varepsilon := 2\frac{[\chi_0] \cdot [\omega_\varepsilon]}{[\chi_0]^2}. \tag{2.1}$$

As $\varepsilon \to 0$, $c_\varepsilon \to c_0$. From (1.6), we may choose $\varepsilon_0 > 0$ sufficiently small so that for $\varepsilon \in [0, \varepsilon_0]$ we have

$$c_\varepsilon[\chi_0] - [\omega_\varepsilon] > 0. \tag{2.2}$$

Then consider the family of J-flows

$$\frac{\partial}{\partial t}\varphi_\varepsilon = c_\varepsilon - \frac{2\chi_\varphi \wedge \omega_\varepsilon}{\chi_\varphi^2}, \qquad \varphi_\varepsilon|_{t=0} = \varphi_0, \tag{2.3}$$

parametrized by $\varepsilon \in (0, \varepsilon_0]$. By Chen's result [**3**], we know that there exists a solution to (2.3) on $M \times [0, \infty)$. The main result of this section is that we have uniform (independent of ε) L^∞ bounds for φ_ε, $\dot{\varphi}_\varepsilon$ and C^∞ estimates for φ_ε away from D.

PROPOSITION 2.1. *In the setting described above, there exists a uniform constant C such that for all t and $\varepsilon \in (0, \varepsilon_0]$,*

$$\|\varphi_\varepsilon\|_{L^\infty(M)} \leqslant C \quad \textit{and} \quad \|\dot{\varphi}_\varepsilon\|_{L^\infty(M)} \leqslant C. \tag{2.4}$$

Moreover, for any compact set Ω of $M \setminus D$ and $k > 0$, there exists $C_{\Omega,k} > 0$ such that

$$||\varphi_\varepsilon||_{C^k(\Omega,\chi_0)} \leqslant C_{\Omega,k}. \tag{2.5}$$

In order to prove the proposition, it suffices to establish the uniform L^∞ estimates on M and a second order estimate on $M \setminus D$.

LEMMA 2.2. *There exists a uniform constant C such that for all t and $\varepsilon \in (0, \varepsilon_0]$,*

$$\|\varphi_\varepsilon\|_{L^\infty(M)} \leqslant C. \tag{2.6}$$

PROOF. Put

$$\alpha_\varepsilon := c_\varepsilon \chi_0 - \omega_\varepsilon > 0. \tag{2.7}$$

We are allowed to assume without loss of generality that α_ε is Kähler and in addition, there exists $\delta > 0$ such that $\alpha_\varepsilon > \delta\chi_0$ for all $\varepsilon \in [0, \varepsilon_0]$. This is possible since the condition (1.6) implies that we can find a smooth function η with $c_0\chi_0 - \omega_0 + dd^c\eta > 2\delta\chi_0$ for a small $\delta > 0$. Shrinking ε_0 if necessary, we may assume that $c_\varepsilon\chi_0 - \omega_\varepsilon + dd^c\eta > \delta\chi_0$ for all $\varepsilon \in [0, \varepsilon_0]$, and we can estimate $\varphi_\varepsilon - \eta$ instead of φ_ε.

We begin by proving an L^∞ estimate for φ_ε which is independent of ε. We follow an argument similar to that in [**12**]. It uses the trick of [**2**] of rewriting the critical equation as a complex Monge–Ampère equation, together with Yau's L^∞ estimate.

There exists a smooth solution ψ_ε of the equation

$$(\alpha_\varepsilon + c_\varepsilon dd^c\psi_\varepsilon)^2 = \omega_\varepsilon^2, \quad \alpha_\varepsilon + c_\varepsilon dd^c\psi_\varepsilon > 0, \quad \sup_M \psi_\varepsilon = 0. \tag{2.8}$$

Indeed, since

$$[\alpha_\varepsilon]^2 = c_\varepsilon^2[\chi_0]^2 + [\omega_\varepsilon]^2 - 2c_\varepsilon[\chi_0] \cdot [\omega_\varepsilon] = [\omega_\varepsilon]^2,$$

this follows from Yau's theorem [**34**]. Moreover,

$$\|\psi_\varepsilon\|_{L^\infty} \leqslant C,$$

for C independent of ε. This follows from Yau's original proof using Moser's iteration, since α_ε is a smooth family of Kähler metrics which satisfy $\alpha_\varepsilon > \delta\chi_0$

and hence $\omega_\varepsilon^2/\alpha_\varepsilon^2$ is uniformly bounded from above. Here we are using the fact that the Sobolev constant, used in Yau's iteration argument, remains bounded on any set of Riemannian metrics which is compact in the C^0 topology and has a uniform lower bound.

Note that Yau's C^∞ estimates for ψ_ε may indeed depend on ε, but in what follows we only need uniformity in the L^∞ estimate. Observe that

$$\chi_{\psi_\varepsilon} = \chi_0 + dd^c\psi_\varepsilon > 0,$$

since $\chi_{\psi_\varepsilon} = \frac{1}{c_\varepsilon}(\alpha_\varepsilon + c_\varepsilon dd^c\psi_\varepsilon + \omega_\varepsilon)$ and $\alpha_\varepsilon + c_\varepsilon dd^c\psi_\varepsilon > 0$.

We have

$$\omega_\varepsilon^2 = (c_\varepsilon\chi_{\psi_\varepsilon} - \omega_\varepsilon)^2 = c_\varepsilon^2\chi_{\psi_\varepsilon}^2 - 2c_\varepsilon\chi_{\psi_\varepsilon}\wedge\omega_\varepsilon + \omega_\varepsilon^2,$$

and hence χ_{ψ_ε} satisfies the critical equation $c_\varepsilon\chi_{\psi_\varepsilon}^2 = 2\chi_{\psi_\varepsilon}\wedge\omega_\varepsilon$.

Now put $\theta_\varepsilon = \varphi_\varepsilon - \psi_\varepsilon$ and compute

$$\frac{\partial}{\partial t}\theta_\varepsilon = \frac{\partial}{\partial t}\varphi_\varepsilon = \frac{2\chi_{\psi_\varepsilon}\wedge\omega_\varepsilon}{\chi_{\psi_\varepsilon}^2} - \frac{2\chi_{\varphi_\varepsilon}\wedge\omega_\varepsilon}{\chi_{\varphi_\varepsilon}^2} = \int_0^1 \frac{d}{dv}\left(\frac{2\eta_v\wedge\omega_\varepsilon}{\eta_v^2}\right)dv, \tag{2.9}$$

where $\eta_v = v\chi_{\psi_\varepsilon} + (1-v)\chi_{\varphi_\varepsilon}$ for $v\in[0,1]$. Define $\tau_v^{\bar\ell k} = \eta_v^{\bar j k}\eta_v^{\bar\ell i}(g_\varepsilon)_{i\bar j}$, which is positive definite, and compute

$$\begin{aligned}
\frac{\partial}{\partial t}\theta_\varepsilon &= \int_0^1\left(\frac{d}{dv}\eta_v^{\bar j i}(g_\varepsilon)_{i\bar j}\right)dv\\
&= -\int_0^1 \eta_v^{\bar j k}\eta_v^{\bar\ell i}\left(\frac{d}{dv}\eta_v\right)_{k\bar\ell}(g_\varepsilon)_{i\bar j}dv\\
&= -\int_0^1 \tau_v^{\bar\ell k}(\chi_{\psi_\varepsilon} - \chi_{\varphi_\varepsilon})_{k\bar\ell}dv\\
&= \left(\int_0^1 \tau_v^{\bar\ell k}dv\right)\partial_k\partial_{\bar\ell}\theta_\varepsilon.
\end{aligned} \tag{2.10}$$

Since $\left(\int_0^1 \tau_v^{\bar\ell k}dv\right)$ is a positive definite tensor, we apply the maximum principle to see that θ_ε is uniformly bounded by $\sup_M|\theta_\varepsilon|$ at $t=0$. Since $\varphi_\varepsilon|_{t=0} = \varphi_0$ and ψ_ε is uniformly bounded it follows that θ_ε is uniformly bounded independent of ε. Hence φ_ε is uniformly bounded independent of ε. □

Next we estimate the time derivative of φ_ε. First some notation: define an operator $\tilde\Delta_\varepsilon := h_\varepsilon^{\bar j i}\partial_i\partial_{\bar j}$, where $h_\varepsilon^{\bar j i} := \chi_{\varphi_\varepsilon}^{\bar j p}\chi_{\varphi_\varepsilon}^{\bar q i}(g_\varepsilon)_{p\bar q}$. Then:

Lemma 2.3. *There exists a uniform constant C such that for all t and $\varepsilon\in(0,\varepsilon_0]$,*

$$\|\dot\varphi_\varepsilon\|_{L^\infty(M)} \leqslant C.$$

Hence we have

$$\chi_{\varphi_\varepsilon} \geqslant \frac{1}{C'}\omega_\varepsilon = \frac{1}{C'}(\omega_0 + \varepsilon\hat{\omega}), \tag{2.11}$$

for a uniform $C' > 0$.

Proof. This follows immediately from the maximum principle as in [**3**]. Indeed, differentiating (2.3) we obtain

$$\frac{\partial}{\partial t}\dot{\varphi}_\varepsilon = \tilde{\Delta}_\varepsilon \dot{\varphi}_\varepsilon.$$

Then by the maximum principle, $\dot{\varphi}_\varepsilon$ is bounded uniformly in time. Moreover, the bound is independent of ε. In particular, $\mathrm{tr}_{\chi_{\varphi_\varepsilon}}\omega_\varepsilon \leqslant C'$, and this gives (2.11). □

It is important to note Lemma 2.3 does not give a uniform bound for $\chi_{\varphi_\varepsilon}$ away from zero which is independent of ε. In particular, we have no *a priori* upper bounds for $\mathrm{tr}_{\chi_{\varphi_\varepsilon}}\hat{\omega}$ or $\mathrm{tr}_{h_\varepsilon}\hat{g} := h_\varepsilon^{\bar{j}i}(\hat{g})_{i\bar{j}}$.

Next we wish to prove an estimate for $\chi_{\varphi_\varepsilon}$. For ease of notation, we drop all subscripts ε and write χ for χ_φ. Write $u = \mathrm{tr}_{\hat{\omega}}\chi$. We denote by $\hat{R}_{k\bar{\ell}i\bar{j}}$ the curvature of $\hat{g}$, and raise indices using $\hat{g}$.

We first derive an evolution equation and differential inequality for $\log u$.

Lemma 2.4. *The evolution equation of* $\log u$ *is given by*

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \tilde{\Delta}\right)\log u = \frac{1}{u}\Big(&-h^{\bar{\ell}k}\hat{R}_{k\bar{\ell}}{}^{\bar{j}i}\chi_{i\bar{j}} - \hat{g}^{\bar{j}i}h^{\bar{s}p}\chi^{\bar{q}r}\hat{\nabla}_i\chi_{r\bar{s}}\hat{\nabla}_{\bar{j}}\chi_{p\bar{q}} \\ &- \hat{g}^{\bar{j}i}h^{\bar{q}r}\chi^{\bar{s}p}\hat{\nabla}_i\chi_{r\bar{s}}\hat{\nabla}_{\bar{j}}\chi_{p\bar{q}} + 2\mathrm{Re}\left(\hat{g}^{\bar{j}i}\chi^{\bar{q}k}\chi^{\bar{\ell}p}\hat{\nabla}_{\bar{j}}\chi_{p\bar{q}}\hat{\nabla}_i g_{k\bar{\ell}}\right) \\ &-\hat{g}^{\bar{j}i}\chi^{\bar{\ell}k}\hat{\nabla}_i\hat{\nabla}_{\bar{j}}g_{k\bar{\ell}} + \chi^{\bar{\ell}k}\hat{R}^{\bar{q}}{}_{\bar{\ell}}g_{k\bar{q}} + \frac{|\partial u|_h^2}{u}\Big).\end{aligned} \tag{2.12}$$

Moreover, there exists a constant C depending only on $\hat{g}$ and $\|g_0\|_{C^2(M,\hat{g})}$ such that

$$\left(\frac{\partial}{\partial t} - \tilde{\Delta}\right)\log u \leqslant C\mathrm{tr}_h\hat{g} + \frac{C}{u}(\mathrm{tr}_\chi\hat{\omega})(\mathrm{tr}_\omega\hat{\omega}) + 2\mathrm{Re}\left(\chi^{\bar{s}k}\left(\frac{\partial_k u}{u^2}\right)\partial_{\bar{s}}\mathrm{tr}_{\hat{\omega}}\omega\right).$$

Proof. For any fixed $p \in M$, we choose a holomorphic coordinate system centered at p with the property that $(\partial_k \hat{g}_{i\bar{j}})|_p = 0$ for all i, j, k. Compute at p,

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{tr}_{\hat{\omega}}\chi = \frac{\partial}{\partial t}\left(\hat{g}^{\bar{j}i}\partial_i\partial_{\bar{j}}\varphi\right) &= -\hat{g}^{\bar{j}i}\partial_i\partial_{\bar{j}}(\chi^{\bar{\ell}k}g_{k\bar{\ell}}) \\
&= -\hat{g}^{\bar{j}i}\partial_i(-\chi^{\bar{q}k}\chi^{\bar{\ell}p}(\partial_{\bar{j}}\chi_{p\bar{q}})g_{k\bar{\ell}} + \chi^{\bar{\ell}k}\partial_{\bar{j}}g_{k\bar{\ell}}) \\
&= \hat{g}^{\bar{j}i}\{g_{k\bar{\ell}}\chi^{\bar{q}k}\chi^{\bar{\ell}p}\partial_i\partial_{\bar{j}}\chi_{p\bar{q}} - \chi^{\bar{s}k}\chi^{\bar{q}r}(\partial_i\chi_{r\bar{s}})\chi^{\bar{\ell}p}(\partial_{\bar{j}}\chi_{p\bar{q}})g_{k\bar{\ell}} \\
&\quad - \chi^{\bar{q}k}\chi^{\bar{s}p}\chi^{\bar{\ell}r}(\partial_i\chi_{r\bar{s}})(\partial_{\bar{j}}\chi_{p\bar{q}})g_{k\bar{\ell}} + \chi^{\bar{q}k}\chi^{\bar{\ell}p}(\partial_{\bar{j}}\chi_{p\bar{q}})(\partial_i g_{k\bar{\ell}}) \\
&\quad + \chi^{\bar{s}k}\chi^{\bar{\ell}r}(\partial_i\chi_{r\bar{s}})(\partial_{\bar{j}}g_{k\bar{\ell}}) - \chi^{\bar{\ell}k}\partial_i\partial_{\bar{j}}g_{k\bar{\ell}}\} \\
&= \hat{g}^{\bar{j}i}h^{\bar{q}p}\partial_i\partial_{\bar{j}}\chi_{p\bar{q}} - \hat{g}^{\bar{j}i}h^{\bar{s}p}\chi^{\bar{q}r}(\partial_i\chi_{r\bar{s}})(\partial_{\bar{j}}\chi_{p\bar{q}}) - \hat{g}^{\bar{j}i}h^{\bar{q}r}\chi^{\bar{s}p}(\partial_i\chi_{r\bar{s}})(\partial_{\bar{j}}\chi_{p\bar{q}}) \\
&\quad + 2\mathrm{Re}\left(\hat{g}^{\bar{j}i}\chi^{\bar{q}k}\chi^{\bar{\ell}p}(\partial_{\bar{j}}\chi_{p\bar{q}})(\partial_i g_{k\bar{\ell}})\right) - \hat{g}^{\bar{j}i}\chi^{\bar{\ell}k}\partial_i\partial_{\bar{j}}g_{k\bar{\ell}}.
\end{aligned}$$

And

$$\begin{aligned}
\tilde{\Delta}\log u = \frac{\tilde{\Delta}u}{u} - \frac{|\partial u|_h^2}{u^2} &= \frac{1}{u}\left(h^{\bar{\ell}k}\partial_k\partial_{\bar{\ell}}(\hat{g}^{\bar{j}i}\chi_{i\bar{j}}) - \frac{|\partial u|_h^2}{u}\right) \\
&= \frac{1}{u}\left(h^{\bar{\ell}k}\hat{R}_{k\bar{\ell}}{}^{\bar{j}i}\chi_{i\bar{j}} + h^{\bar{\ell}k}\hat{g}^{\bar{j}i}\partial_k\partial_{\bar{\ell}}\chi_{i\bar{j}} - \frac{|\partial u|_h^2}{u}\right),
\end{aligned}$$

where $|\partial u|_h^2 := h^{\bar{j}i}\partial_i u\partial_{\bar{j}}u$. Then (2.12) follows from these two equations, the Kähler condition for χ and the fact that in our coordinate system we have

$$\hat{g}^{\bar{j}i}\chi^{\bar{\ell}k}\partial_i\partial_{\bar{j}}g_{k\bar{\ell}} = \hat{g}^{\bar{j}i}\chi^{\bar{\ell}k}\hat{\nabla}_i\hat{\nabla}_{\bar{j}}g_{k\bar{\ell}} - \hat{g}^{\bar{j}i}\chi^{\bar{\ell}k}\hat{R}_{i\bar{j}}{}^{\bar{q}}{}_{\bar{\ell}}g_{k\bar{q}} = \hat{g}^{\bar{j}i}\chi^{\bar{\ell}k}\hat{\nabla}_i\hat{\nabla}_{\bar{j}}g_{k\bar{\ell}} - \chi^{\bar{\ell}k}\hat{R}^{\bar{q}}{}_{\bar{\ell}}g_{k\bar{q}}.$$

To deal with the terms involving one derivative of χ we use a completing the square argument, which is formally similar to that of Cherrier [**4**] (see also [**30**, Proposition 3.1]). Compute

$$K = \hat{g}^{\bar{\ell}i}\chi^{\bar{j}p}h^{\bar{q}k}B_{i\bar{j}k}\overline{B_{\ell\bar{p}q}} \geqslant 0,$$

where

$$B_{i\bar{j}k} = \hat{\nabla}_i\chi_{k\bar{j}} - \chi_{i\bar{j}}\frac{\partial_k u}{u} - g^{\bar{b}a}\chi_{k\bar{b}}\hat{\nabla}_i g_{a\bar{j}}.$$

We compute

$$\begin{aligned}
K = {}& \hat{g}^{\bar{\ell}i}\chi^{\bar{j}p}h^{\bar{q}k}\hat{\nabla}_i\chi_{k\bar{j}}\hat{\nabla}_{\bar{\ell}}\chi_{p\bar{q}} + \hat{g}^{\bar{\ell}i}\chi^{\bar{j}p}h^{\bar{q}k}\chi_{i\bar{j}}\left(\frac{\partial_k u}{u}\right)\chi_{p\bar{\ell}}\left(\frac{\partial_{\bar{q}}u}{u}\right) \\
& + \hat{g}^{\bar{\ell}i}\chi^{\bar{j}p}h^{\bar{q}k}g^{\bar{b}a}\chi_{k\bar{b}}(\hat{\nabla}_i g_{a\bar{j}})g^{\bar{s}r}\chi_{r\bar{q}}\hat{\nabla}_{\bar{\ell}}g_{p\bar{s}} - 2\mathrm{Re}\left(\hat{g}^{\bar{\ell}i}\chi^{\bar{j}p}h^{\bar{q}k}(\hat{\nabla}_i\chi_{k\bar{j}})\chi_{p\bar{\ell}}\left(\frac{\partial_{\bar{q}}u}{u}\right)\right) \\
& - 2\mathrm{Re}\left(\hat{g}^{\bar{\ell}i}\chi^{\bar{j}p}h^{\bar{q}k}(\hat{\nabla}_i\chi_{k\bar{j}})g^{\bar{b}a}\chi_{a\bar{q}}\hat{\nabla}_{\bar{\ell}}g_{p\bar{b}}\right) \\
& + 2\mathrm{Re}\left(\hat{g}^{\bar{\ell}i}\chi^{\bar{j}p}h^{\bar{q}k}\chi_{i\bar{j}}\left(\frac{\partial_k u}{u}\right)g^{\bar{s}r}\chi_{r\bar{q}}\hat{\nabla}_{\bar{\ell}}g_{p\bar{s}}\right).
\end{aligned}$$

Using the definition of $h^{\bar{j}i}$ and the Kähler condition for χ,

$$
\begin{aligned}
K &= \hat{g}^{\bar{\ell}i}\chi^{\bar{j}p}h^{\bar{q}k}\hat{\nabla}_i\chi_{k\bar{j}}\hat{\nabla}_{\bar{\ell}}\chi_{p\bar{q}} + \frac{|\partial u|_h^2}{u} \\
&\quad + \hat{g}^{\bar{\ell}i}\chi^{\bar{j}p}\chi^{\bar{q}c}\chi^{\bar{d}k}g_{c\bar{d}}g^{\bar{b}a}\chi_{k\bar{b}}g^{\bar{s}r}\chi_{r\bar{q}}(\hat{\nabla}_i g_{a\bar{j}})(\hat{\nabla}_{\bar{\ell}}g_{p\bar{s}}) - 2\mathrm{Re}\left(\partial_k\left(\hat{g}^{\bar{\ell}i}\chi_{i\bar{\ell}}\right)h^{\bar{q}k}\frac{\partial_{\bar{q}}u}{u}\right) \\
&\quad - 2\mathrm{Re}\left(\hat{g}^{\bar{\ell}i}\chi^{\bar{j}p}\chi^{\bar{q}c}\chi^{\bar{d}k}g_{c\bar{d}}(\hat{\nabla}_i\chi_{k\bar{j}})g^{\bar{b}a}\chi_{a\bar{q}}(\hat{\nabla}_{\bar{\ell}}g_{p\bar{b}})\right) \\
&\quad + 2\mathrm{Re}\left(\hat{g}^{\bar{\ell}i}\chi^{\bar{j}p}\chi^{\bar{q}c}\chi^{\bar{d}k}g_{c\bar{d}}\chi_{i\bar{j}}\left(\frac{\partial_k u}{u}\right)g^{\bar{s}r}\chi_{r\bar{q}}(\hat{\nabla}_{\bar{\ell}}g_{p\bar{s}})\right) \\
&= \hat{g}^{\bar{\ell}i}\chi^{\bar{j}p}h^{\bar{q}k}\hat{\nabla}_i\chi_{k\bar{j}}\hat{\nabla}_{\bar{\ell}}\chi_{p\bar{q}} - \frac{|\partial u|_h^2}{u} + \hat{g}^{\bar{\ell}i}\chi^{\bar{j}p}g^{\bar{s}a}(\hat{\nabla}_i g_{a\bar{j}})(\hat{\nabla}_{\bar{\ell}}g_{p\bar{s}}) \\
&\quad - 2\mathrm{Re}\left(\hat{g}^{\bar{\ell}i}\chi^{\bar{j}p}\chi^{\bar{b}k}(\hat{\nabla}_i\chi_{k\bar{j}})(\hat{\nabla}_{\bar{\ell}}g_{p\bar{b}})\right) + 2\mathrm{Re}\left(\chi^{\bar{s}k}\hat{g}^{\bar{\ell}p}\left(\frac{\partial_k u}{u}\right)(\hat{\nabla}_{\bar{\ell}}g_{p\bar{s}})\right).
\end{aligned}
$$

Combining this with (2.12) gives,

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \tilde{\Delta}\right)\log u &= \frac{1}{u}\Big\{-h^{\bar{\ell}k}\hat{R}_{k\bar{\ell}}{}^{\bar{j}i}\chi_{i\bar{j}} - \hat{g}^{\bar{j}i}h^{\bar{s}p}\chi^{\bar{q}r}\hat{\nabla}_i\chi_{r\bar{s}}\hat{\nabla}_{\bar{j}}\chi_{p\bar{q}} - K \\
&\qquad + \hat{g}^{\bar{\ell}i}\chi^{\bar{j}p}g^{\bar{s}a}(\hat{\nabla}_i g_{a\bar{j}})(\hat{\nabla}_{\bar{\ell}}g_{p\bar{s}}) + 2\mathrm{Re}\left(\chi^{\bar{s}k}\left(\frac{\partial_k u}{u}\right)\hat{g}^{\bar{\ell}p}(\hat{\nabla}_{\bar{s}}g_{p\bar{\ell}})\right) \\
&\qquad - \hat{g}^{\bar{j}i}\chi^{\bar{\ell}k}\hat{\nabla}_i\hat{\nabla}_{\bar{j}}g_{k\bar{\ell}} + \chi^{\bar{\ell}k}\hat{R}^{\bar{q}}{}_{\bar{\ell}}g_{k\bar{q}}\Big\} \\
&\leqslant C\mathrm{tr}_h\hat{g} + \frac{C}{u}(\mathrm{tr}_\chi\hat{\omega})(\mathrm{tr}_\omega\hat{\omega}) + 2\mathrm{Re}\left(\chi^{\bar{s}k}\left(\frac{\partial_k u}{u^2}\right)\partial_{\bar{s}}\mathrm{tr}_{\hat{\omega}}\omega\right).
\end{aligned}
$$

Indeed, to see the last inequality, we estimate

$$\frac{1}{u}|h^{\bar{\ell}k}\hat{R}_{k\bar{\ell}}{}^{\bar{j}i}\chi_{i\bar{j}}| \leqslant C\mathrm{tr}_h\hat{g},$$

and

$$
\begin{aligned}
\frac{1}{u}\left|\hat{g}^{\bar{j}i}\chi^{\bar{\ell}k}\hat{\nabla}_i\hat{\nabla}_{\bar{j}}g_{k\bar{\ell}}\right| + \frac{1}{u}\left|\chi^{\bar{\ell}k}\hat{R}^{\bar{q}}{}_{\bar{\ell}}g_{k\bar{q}}\right| &+ \frac{1}{u}\hat{g}^{\bar{\ell}i}\chi^{\bar{j}p}g^{\bar{s}a}(\hat{\nabla}_i g_{a\bar{j}})(\hat{\nabla}_{\bar{\ell}}g_{p\bar{s}}) \\
&\leqslant \frac{C}{u}(\mathrm{tr}_\chi\hat{\omega})(\mathrm{tr}_\omega\hat{\omega}),
\end{aligned}
$$

for a constant C depending only on $\hat{g}$ and $\|g_0\|_{C^2(M,\hat{g})}$. Note that $\mathrm{tr}_g\hat{g}$ is uniformly bounded from below away from zero, but may blow up along D. This completes the proof of the lemma. □

Next we prove the estimate on χ.

LEMMA 2.5. *There exist uniform constants C, γ, independent of ε, such that*

$$u = \mathrm{tr}_{\hat{\omega}}\chi \leqslant \frac{C}{|s|_H^{2\gamma}}.$$

PROOF. From the condition (2.7), we may choose uniform positive constants η, δ and σ to be sufficiently small so that

$$c\chi_0 - \omega - c\delta R_H > 3\eta\omega, \tag{2.13}$$

and

$$\chi_0 - \delta R_H \geqslant \sigma\hat{\omega}, \tag{2.14}$$

where we write $c = c_\varepsilon = 2[\chi_0] \cdot [\omega_\varepsilon]/[\chi_0]^2$.

Define (cf. [**31**])

$$\tilde{\varphi} = \varphi - \delta \log |s|_H^2.$$

Note that $\tilde{\varphi}$ is bounded from below, and tends to infinity along D. Let $A > 1$ be a large constant to be determined later. Consider the evolution of the quantity

$$Q = \log u - A\tilde{\varphi} + \frac{1}{\tilde{\varphi} + C_0},$$

where we choose the uniform constant C_0 so that

$$0 \leqslant \frac{1}{\tilde{\varphi} + C_0} \leqslant 1.$$

This type of quantity was used by Phong–Sturm [**22**] in their study of the degenerate complex Monge–Ampère equation (for its later use in a parabolic setting, see [**30**]). Note that Q achieves a maximum at each time t away from D. Since φ is uniformly bounded, it suffices to bound Q from above at its maximum, as long as A is chosen uniformly.

Note that at a maximum of Q we have

$$\frac{\partial_k u}{u} = \left(A + \frac{1}{(\tilde{\varphi} + C_0)^2}\right)\partial_k\tilde{\varphi}.$$

Then at a maximum of Q we have, from Lemma 2.4,

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \tilde{\Delta}\right) Q \leqslant{}& C_1 \mathrm{tr}_h \hat{g} + \frac{C(\mathrm{tr}_\chi\hat{\omega})(\mathrm{tr}_\omega\hat{\omega})}{u} \\
&+ \frac{2}{u}\mathrm{Re}\left(\chi^{\bar{s}k}\left(A + \frac{1}{(\tilde{\varphi} + C_0)^2}\right)(\partial_k\tilde{\varphi})\partial_{\bar{s}}\mathrm{tr}_{\hat{\omega}}\omega\right) \\
&- \left(A + \frac{1}{(\tilde{\varphi} + C_0)^2}\right)\left(\frac{\partial}{\partial t} - \tilde{\Delta}\right)\tilde{\varphi} - \frac{2}{(\tilde{\varphi} + C_0)^3}|\partial\tilde{\varphi}|_h^2.
\end{aligned} \tag{2.15}$$

But from (1.4) and (2.11),

$$\frac{C(\mathrm{tr}_\chi\hat{\omega})(\mathrm{tr}_\omega\hat{\omega})}{u} \leqslant \frac{C'(\mathrm{tr}_\chi\omega)(\mathrm{tr}_{\hat{\omega}}\hat{\omega})}{u|s|_H^{4\beta}} \leqslant \frac{C''}{u|s|_H^{4\beta}}. \tag{2.16}$$

Observe that at a maximum of Q we may assume that $\frac{C''}{u|s|_H^{4\beta}} \leqslant 1$. Indeed if not, then assuming that $\delta A \geqslant 2\beta$, we have that $\log u + A\delta \log |s|_H^2 \leqslant C$ and it follows immediately that Q is bounded from above, which is what we need to show.

Next,

$$\frac{2}{u}\mathrm{Re}\left(\chi^{\bar{s}k}\left(A + \frac{1}{(\tilde{\varphi}+C_0)^2}\right)(\partial_k\tilde{\varphi})\partial_{\bar{s}}\mathrm{tr}_{\hat{\omega}}\omega\right) \leqslant \frac{CA}{u}|\partial\tilde{\varphi}|_\chi\,|\partial\mathrm{tr}_{\hat{\omega}}\omega|_\chi.$$

But

$$\frac{1}{u}|\partial\tilde{\varphi}|_\chi^2 = \frac{1}{u}\chi^{\bar{j}i}\partial_i\tilde{\varphi}\partial_{\bar{j}}\tilde{\varphi} \leqslant \chi^{\bar{j}k}\chi^{\bar{\ell}i}\hat{g}_{k\bar{\ell}}\partial_i\tilde{\varphi}\partial_{\bar{j}}\tilde{\varphi} \leqslant \frac{C}{|s|_H^{2\beta}}h^{\bar{j}i}\partial_i\tilde{\varphi}\partial_{\bar{j}}\tilde{\varphi} = \frac{C}{|s|_H^{2\beta}}|\partial\tilde{\varphi}|_h^2,$$

and, using (2.11),

$$|\partial\mathrm{tr}_{\hat{\omega}}\omega|_\chi^2 \leqslant \frac{C}{|s|_H^{2\beta}}.$$

Hence

$$\frac{CA}{u}|\partial\tilde{\varphi}|_\chi\,|\partial\mathrm{tr}_{\hat{\omega}}\omega|_\chi \leqslant \frac{C'A}{\sqrt{u}|s|_H^{2\beta}}|\partial\tilde{\varphi}|_h \leqslant 2\frac{|\partial\tilde{\varphi}|_h^2}{(\tilde{\varphi}+C_0)^3} + \frac{C''A^2(\tilde{\varphi}+C_0)^3}{u|s|_H^{4\beta}}.$$

Putting this together, we obtain

$$\frac{2}{u}\mathrm{Re}\left(\chi^{\bar{s}k}\left(A + \frac{1}{(\tilde{\varphi}+C_0)^2}\right)(\partial_k\tilde{\varphi})\partial_{\bar{s}}\mathrm{tr}_{\hat{\omega}}\omega\right) \leqslant 2\frac{|\partial\tilde{\varphi}|_h^2}{(\tilde{\varphi}+C_0)^3} + 1,$$

since we may assume without loss of generality, by an argument similar to the one given above, that $C''A^2(\tilde{\varphi}+C_0)^3/(u|s|_H^{4\beta}) \leqslant 1$. Combining this with (2.15) and (2.16), we obtain at a maximum point of Q,

$$0 \leqslant \left(\frac{\partial}{\partial t} - \tilde{\Delta}\right)Q \leqslant C_1\mathrm{tr}_h\hat{g} + 2 - \left(A + \frac{1}{(\tilde{\varphi}+C_0)^2}\right)\left(\frac{\partial}{\partial t} - \tilde{\Delta}\right)\tilde{\varphi}. \tag{2.17}$$

Now compute on $M \setminus D$,

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \tilde{\Delta}\right)\tilde{\varphi} &= c - \mathrm{tr}_\chi\omega - h^{\bar{j}i}\partial_i\partial_{\bar{j}}(\varphi - \delta\log|s|_H^2)\\ &= c - 2\chi^{\bar{j}i}g_{i\bar{j}} + h^{\bar{j}i}((\chi_0)_{i\bar{j}} - \delta(R_H)_{i\bar{j}})\\ &= c - 2\chi^{\bar{j}i}g_{i\bar{j}} + \eta h^{\bar{j}i}((\chi_0)_{i\bar{j}} - \delta(R_H)_{i\bar{j}})\\ &\quad + \frac{(1-\eta)}{c}h^{\bar{j}i}(c\,(\chi_0)_{i\bar{j}} - c\delta(R_H)_{i\bar{j}}),\end{aligned} \tag{2.18}$$

for $\eta > 0$ as in (2.13). From (2.14), we choose A (depending only on C_1, η and σ) sufficiently large so that

$$A\eta h^{\bar{j}i}((\chi_0)_{i\bar{j}} - \delta(R_H)_{i\bar{j}}) \geqslant C_1 \mathrm{tr}_h \hat{g}.$$

Then from this together with (2.17) and (2.18), we obtain

$$c - 2\chi^{\bar{j}i} g_{i\bar{j}} + \frac{(1-\eta)}{c} h^{\bar{j}i}(c\,(\chi_0)_{i\bar{j}} - c\delta(R_H)_{i\bar{j}}) \leqslant \frac{2}{A}, \tag{2.19}$$

and hence from (2.13),

$$c - 2\chi^{\bar{j}i} g_{i\bar{j}} + \frac{(1-\eta)(1+3\eta)}{c} h^{\bar{j}i} g_{i\bar{j}} \leqslant \frac{2}{A}. \tag{2.20}$$

This implies that, shrinking η if necessary,

$$c - 2\chi^{\bar{j}i} g_{i\bar{j}} + \frac{(1+\eta)}{c} h^{\bar{j}i} g_{i\bar{j}} \leqslant \frac{2}{A}.$$

Now choosing coordinates for which g is the identity and χ is diagonal with entries λ_1, λ_2, we have

$$c + \frac{(1+\eta)}{c} \sum_{i=1}^{2} \frac{1}{\lambda_i^2} - 2\sum_{i=1}^{2} \frac{1}{\lambda_i} \leqslant \frac{2}{A}.$$

Completing the square as in [**26, 33**], we get

$$\sum_{i=1}^{2} \left(\frac{\sqrt{c}}{\sqrt{1+\eta}} - \frac{\sqrt{1+\eta}}{\sqrt{c}\,\lambda_i} \right)^2 \leqslant \frac{2}{A} - c + \frac{2c}{1+\eta} = \frac{2}{A} + \frac{c(1-\eta)}{1+\eta}.$$

We may assume that A is chosen large enough so that

$$\frac{2}{A} \leqslant \eta \frac{c(1-\eta)}{1+\eta},$$

and thus

$$\sum_{i=1}^{2} \left(\frac{\sqrt{c}}{\sqrt{1+\eta}} - \frac{\sqrt{1+\eta}}{\sqrt{c}\,\lambda_i} \right)^2 \leqslant c(1-\eta).$$

Hence, for $i = 1, 2$,

$$\frac{\sqrt{c}}{\sqrt{1+\eta}} - \frac{\sqrt{1+\eta}}{\sqrt{c}\,\lambda_i} \leqslant \sqrt{c(1-\eta)},$$

which implies that

$$\lambda_i \leqslant \frac{1+\eta}{c(1-\sqrt{1-\eta^2})}.$$

Then

$$\mathrm{tr}_g \chi \leqslant C,$$

at this maximum point of Q. Hence $\mathrm{tr}_{\hat{g}}\chi \leqslant C$ at this point, and we see that Q is bounded from above. This completes the proof. $\square$

The higher order estimates (2.5) follows immediately by applying the standard local parabolic theory (as in [**26**], for example). This completes the proof of Proposition 2.1.

3. Proof of the main theorem and corollaries. We can now prove the main results of the paper.

Proof of Theorem 1.1. From Proposition 2.1, we can find a sequence $\varepsilon_j \to 0$ such that φ_{ε_j} converges in C^∞ on compact subsets of $(M \setminus D) \times [0, \infty)$. Define on $M \setminus D$,

$$\varphi = \lim_{j\to\infty} \varphi_{\varepsilon_j}.$$

Then on $(M \setminus D) \times [0, \infty)$, φ is smooth, satisfies $\chi_0 + dd^c\varphi > 0$ and solves the degenerate J-flow equation (1.7). Moreover, again from Proposition 2.1, $\sup_{M\setminus D} |\varphi|$ and $\sup_{M\setminus D} |\dot{\varphi}|$ are uniformly bounded independent of t. It is a standard result in pluripotential theory that a smooth function φ on $M - D$ which satisfies $\chi_0 + dd^c\varphi > 0$ and $\sup_{M-D} |\varphi| \leqslant C$ can be extended uniquely to an element of $\mathcal{P}^{\mathrm{weak}}_{\chi_0}$ (see [**18**] for example). Writing again φ for this function, we obtain the required solution φ to (1.7).

Now recall that the $\mathcal{J}$-functional is defined on $\mathcal{P}_{\chi_0}$ by (1.9). One can also write down an explicit formula:

(3.1)

$$\mathcal{J}_{\omega_0,\chi_0}(\varphi) = \int_M \varphi(\chi_\varphi \wedge \omega_0 + \chi_0 \wedge \omega_0) - \frac{c_0}{3}\int_M \varphi(\chi_\varphi^2 + \chi_\varphi \wedge \chi_0 + \chi_0^2), \quad \varphi \in \mathcal{P}_{\chi_0},$$

and this definition extends to $\varphi \in \mathcal{P}^{\mathrm{weak}}_{\chi_0}$. From the uniform L^∞ bound for the solution $\varphi(t)$ of the degenerate J-flow, as argued in [**12**], we see that

$$\mathcal{J}_{\omega_0,\chi_0}(\varphi(t)) \geqslant -C, \tag{3.2}$$

for a uniform constant C independent of t. In addition, $\mathcal{J}_{\omega_0,\chi_0}(\varphi(t))$ satisfies

$$\frac{d}{dt}\mathcal{J}_{\omega_0,\chi_0}(\varphi(t)) = -\int_{M\setminus D} \dot{\varphi}^2 \chi^2_{\varphi(t)} \leqslant 0. \tag{3.3}$$

Then it follows from the proof of Theorem 1.1 in [**12**] that $\dot{\varphi}$ tends to zero in C^∞ on compact subsets of $M \setminus D$.

Next, define the $\mathcal{I}$-functional on $\mathcal{P}^{\mathrm{weak}}_{\chi_0}$ by

$$\mathcal{I}_{\omega_0,\chi_0}(\varphi) = \frac{1}{3}\int_M \varphi(\chi_\varphi^2 + \chi_\varphi \wedge \chi_0 + \chi_0^2),$$

and we see that $\mathcal{I}_{\omega_0,\chi_0}(\varphi(t)) = \mathcal{I}_{\omega_0,\chi_0}(\varphi_0)$ for all t. It follows from the same argument as in [**12**] that as $t \to \infty$, the solution $\varphi(t)$ to the degenerate J-flow converges in C^∞ on compact subsets of $M \setminus D$ to the unique $\varphi_\infty \in \mathcal{P}^{\mathrm{weak}}_{\chi_0}$ satisfying the critical equation

$$2\chi_{\varphi_\infty} \wedge \omega_0 = c_0 \chi^2_{\varphi_\infty}$$

on $M \setminus D$ subject to the normalization condition $\mathcal{I}_{\omega_0,\chi_0}(\varphi_\infty) = \mathcal{I}_{\omega_0,\chi_0}(\varphi_0)$. Indeed, to see this last uniqueness statement, observe that φ_∞ must satisfy

$$(\alpha_0 + c_0 dd^c \varphi_\infty)^2 = \omega_0^2, \quad \alpha_0 + c_0 dd^c \varphi_\infty > 0 \text{ on } M \setminus D, \tag{3.4}$$

for $\alpha_0 = c_0\chi_0 - \omega_0 > 0$. Moreover, $c_0\varphi_\infty$ lies in $\mathcal{P}^{\mathrm{weak}}_{\alpha_0}$. But such solutions of the complex Monge–Ampère equation (3.4) are unique up to the addition of a constant [**17**, Corollary 4.2].

It remains to prove the uniqueness of the solution $\varphi(t)$ to the degenerate J-flow. We use an argument similar to one given in [**25**]. Suppose there is another solution $\psi(t) \in \mathcal{P}^{\mathrm{weak}}_{\chi_0}$ of (1.7) satisfying $\sup_{M\setminus D} |\dot{\psi}| \leqslant C$. Define $\theta_\delta = \varphi - \psi - \delta \log |s|^2_H$ on $M \setminus D$, which tends to infinity along D. For $v \in [0,1]$, let $\eta_v = v\chi_\varphi + (1-v)\chi_\psi$, $\tau_v^{\bar{\ell}k} = \eta_v^{\bar{j}k}\eta_v^{\bar{\ell}i}(g_0)_{i\bar{j}}$. Computing as in (2.10), we have on $M \setminus D$,

$$\frac{\partial}{\partial t}\theta_\delta = \left(\int_0^1 \tau_v^{\bar{\ell}k} dv\right) \partial_k \partial_{\bar{\ell}} \theta_\delta - \delta \left(\int_0^1 \tau_v^{\bar{\ell}k} dv\right) (R_H)_{k\bar{\ell}}.$$

Fix a time interval $[0,T]$. From the estimates $\sup_{M\setminus D} |\dot{\varphi}| \leqslant C$ and $\sup_{M\setminus D} |\dot{\psi}| \leqslant C$ we have the estimate $\eta_v \geqslant \frac{1}{C}\omega_0$ for a uniform constant $C > 0$. It follows that $(\tau_v^{\bar{\ell}k}) \leqslant C(g_0^{\bar{\ell}k})$. Then

$$\delta \left(\int_0^1 \tau_v^{\bar{\ell}k} dv\right) (R_H)_{k\bar{\ell}} \leqslant C\delta g_0^{\bar{\ell}k} (R_H)_{k\bar{\ell}} \leqslant \frac{2C\delta}{\rho},$$

since, from (1.4) we have $\omega_0 - \rho R_H > 0$ for a uniform $\rho > 0$.

Hence

$$\frac{\partial}{\partial t}\theta_\delta \geqslant \left(\int_0^1 \tau_s^{\bar{\ell}k} ds\right) \partial_k \partial_{\bar{\ell}} \theta_\delta - \frac{2C\delta}{\rho},$$

and so by the maximum principle, we have

$$\theta_\delta \geqslant -A\delta t \geqslant -A\delta T,$$

for a uniform constant A. It follows that $\varphi \geqslant \psi + \delta \log |s|^2_H - A\delta T$ and so $\varphi \geqslant \psi$ after letting $\delta \to 0$. The same argument shows that $\psi \geqslant \varphi$ and so $\varphi = \psi$. □

Proof of Corollary 1.2. Given the discussion above, this is now immediate, since for any $\varphi_0 \in \mathcal{P}_{\chi_0}$, we have $\mathcal{J}_{\omega_0,\chi_0}(\varphi_0) \geqslant \lim_{t\to\infty} \mathcal{J}_{\omega_0,\chi_0}(\varphi(t)) = \mathcal{J}_{\omega_0,\chi_0}(\varphi_\infty)$. For the last equality, we have used Lemma 3.2 in [**12**]. □

Proof of Corollary 1.3. The Mabuchi energy on the Kähler class $[\chi_0]$ is defined by

$$\mathcal{M}_{\chi_0}(\varphi) = -\int_0^1 \int_M \dot{\varphi}_s (R_{\chi_{\varphi_s}} - \underline{R}) \chi_{\varphi_s}^2 ds,$$

where φ_s is a smooth path in $\mathcal{P}_{\chi_0}$ between 0 and φ, and $\underline{R}$ is the average scalar curvature $\underline{R} = \frac{1}{\int_M \chi_0^2} \int_M R_{\chi_0} \chi_0^2$. Let

$$E_{\chi_0}(\varphi) = \sqrt{-1} \int_M \partial\varphi \wedge \overline{\partial}\varphi \wedge (\chi_0 + \chi_\varphi)$$

be the well-known Aubin–Yau functional (often denoted by I_{χ_0}). Then we say the Mabuchi energy is *proper* [**29**] if there exists an increasing function $f : [0, \infty) \to \mathbb{R}$, satisfying $\lim_{x\to\infty} f(x) = \infty$, such that for all $\varphi \in \mathcal{P}_{\chi_0}$,

$$\mathcal{M}_{\chi_0}(\varphi) \geqslant f(E_{\chi_0}(\varphi))).$$

In fact, for the purposes of this corollary, we may take f to be linear (cf. [**21**]).

Since K_M is big and nef, it is well-known that there exists a closed nonnegative $(1,1)$ form $\omega_0 \in c_1(K_M)$ satisfying (1.4). Indeed, one can take a Fubini–Study metric induced from the pluricanonical system $|mK_M|$ for sufficiently large m, and divide by m to obtain a smooth closed nonnegative $(1,1)$ form $\omega_0 \in c_1(K_M)$. Note that since ω_0 is the pull-back of a holomorphic (and hence smooth) map from M into projective space, it is smooth everywhere on M. However, it is only positive definite on $M \setminus D$ where D is the base locus. Moreover, $[\omega_0] - \rho c_1([D])$ is Kähler for all $\rho > 0$ sufficiently small. Hence we can find a Hermitian metric H on $[D]$ so that $\omega_0 - \rho R_H \geqslant \frac{1}{C_0}\hat{\omega}$ for some fixed Kähler metric $\hat{\omega}$ and a positive constants C_0, ρ. By definition of ω_0, we have $\omega_0 \geqslant \frac{1}{C_0}|s|_H^{2\beta}\hat{\omega}$, for some positive β, after possibly increasing C_0.

The condition (1.13) implies that

$$c_0[\chi_0] - [\omega_0] > 0, \quad \text{for} \quad c_0 = \frac{[\chi_0]\cdot[\omega_0]}{[\chi_0]^2}. \tag{3.5}$$

Hence we can apply Corollary 1.2 to see that $\mathcal{J}_{\omega_0,\chi_0}$ is uniformly bounded from below. The formula of Chen [**2**] gives

$$\mathcal{M}_{\chi_0} = \mathcal{J}_{\omega_0,\chi_0} + \mathcal{F},$$

for a certain functional $\mathcal{F}$, which is proper on $\mathcal{P}_{\chi_0}$ [**26, 29**]. This completes the proof. □

Remark 3.1. We remark that one can give an alternative proof of these two corollaries by elliptic methods. However, we believe that the degenerate J-flow is interesting in its own right, and may be important in extending these results to higher dimensions.

References

1. Błocki Z., *Weak solutions to the complex Hessian equation*, Ann. Inst. Fourier (Grenoble), **55**, No. **5** (2005), 1735–1756.
2. Chen X., *On the lower bound of the Mabuchi energy and its application*, Int. Math. Res. Notices, **12** (2000), 607–623.
3. Chen X., *A new parabolic flow in Kähler manifolds*, Comm. Anal. Geom., **12**, No. **4** (2004), 837–852.
4. Cherrier P., *Équations de Monge–Ampère sur les variétés hermitiennes compactes*, Bull. Sci. Math. (2), **111**, No. **4** (1987), 343–385.
5. Chinh L. H., *Solutions to degenerate complex Hessian equations*, preprint, `arXiv:1202.2436`.
6. Dinew S., Kołodziej S., *Liouville and Calabi–Yau type theorems for complex Hessian equations*, preprint, `arXiv.org/1203.3995`.
7. Donaldson S. K., *Moment maps and diffeomorphisms*, Asian J. Math., **3**, No. **1** (1999), 1–16.
8. Donaldson S. K., *Scalar curvature and stability of toric varieties*, J. Differential Geom., **62** (2002), 289–349.
9. Fang H., Lai M., *On the geometric flows solving Kählerian inverse σ_k equations*, Pacific J. Math., **258**, No. **2** (2012), 291–304.
10. Fang H., Lai M., *Convergence of general inverse σ_k-flow on Kähler manifolds with Calabi Ansatz*, preprint, `arxiv:1203.5253`.
11. Fang H., Lai M., Ma X., *On a class of fully nonlinear flows in Kähler geomety*, J. Reine Angew. Math., **653** (2011), 189–220.
12. Fang H., Lai M., Song J., Weinkove B., *The J-flow on Kähler surfaces: a boundary case*, preprint, `arXiv:1204.4068`.
13. Guan B., Li Q., *A Monge–Ampere Type Fully Nonlinear Equation on Hermitian Manifolds*, Discrete Contin. Dyn. S., series B, **17**, No. **6** (2012), 1991–1999.
14. Hou Z., *Complex Hessian equation on Kähler manifold*, Int. Math. Res. Not. IMRN 2009, No. **16**, 3098–3111.
15. Hou Z., Ma X. N., Wu D., *A second order estimate for complex Hessian equations on a compact Kähler manifold*, Math. Res. Lett., **17**, No. **3** (2010), 547–561.
16. Kołodziej S., *The complex Monge–Ampère equation*, Acta Math., **180**, No. **1** (1998), 69–117.
17. Kołodziej S., *The Monge–Ampère equation on compact Kähler manifolds*, Indiana Univ. Math. J., **52**, No. **3** (2003), 667–686.
18. Kołodziej S., *The complex Monge–Ampère equation and pluripotential theory*, Mem. Amer. Math. Soc., **178**, No. **840** (2005).
19. Li S.-Y., *On the Dirichlet problems for symmetric function equations of the eigenvalues of the complex Hessian*, Asian J. Math., **8**, No. **1** (2004), 87–106
20. Li Y., *A priori estimates on Donaldson equation over compact Hermitian manifolds*, preprint, `arXiv: 1210.0254`.
21. Phong D. H., Song J., Sturm J., Weinkove B., *The Moser–Trudinger inequality on Kähler–Einstein manifolds*, Amer. J. Math., **130**, No. **4** (2008), 1067–1085.
22. Phong D. H., Sturm J., *The Dirichlet problem for degenerate complex Monge–Ampère equations*, Comm. Anal. Geom., **18**, No. **1** (2010), 145–170.
23. Panov D., Ross J., *Slope stability and exceptional divisors of high genus*, Math. Ann., **343**, No. **1** (2009), 79–101.

24. Ross J., Thomas R., *An obstruction to the existence of constant scalar curvature Kähler metrics*, J. Differential Geom., **72**, No. **3** (2006), 429–466.
25. Song J., Tian G., *The Kähler–Ricci flow through singularities*, preprint, arXiv:0909.4898 [math.DG].
26. Song J., Weinkove B., *The convergence and singularities of the J-flow with applications to the Mabuchi energy*, Comm. Pure Appl. Math., **61**, No. **2** (2008), 210–229.
27. Song J., Weinkove B., *Contracting exceptional divisors by the Kähler–Ricci flow*, Duke Math. J. **162**, No. **2** (2013), 367–415.
28. Tian G., *Kähler–Einstein metrics with positive scalar curvature*, Invent. Math., **130**, No. **1** (1997), 1–37.
29. Tian G., *Canonical metrics in Kähler geometry.* Notes taken by Meike Akveld. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2000.
30. Tosatti V., Weinkove B., *On the evolution of a Hermitian metric by its Chern–Ricci form*, preprint, arXiv:1201.0312.
31. Tsuji H., *Existence and degeneration of Kähler–Einstein metrics on minimal algebraic varieties of general type*, Math. Ann., **281** (1988), 123–133.
32. Weinkove B., *Convergence of the J-flow on Kähler surfaces*, Comm. Anal. Geom., **12**, No. **4** (2004), 949–965.
33. Weinkove B., *On the J-flow in higher dimensions and the lower boundedness of the Mabuchi energy*, J. Differential Geom., **73**, No. **2** (2006), 351–358.
34. Yau S.-T., *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation*, I, Comm. Pure Appl. Math., **31** (1978), 339–411.
35. Yau S.-T., *Open problems in geometry*, Proc. Symposia Pure Math., **54** (1993), 1–28.

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