ON THE RATIONAL REAL JACOBIAN CONJECTURE

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Abstract. Jacobian conjectures (that nonsingular implies a global inverse) for rational everywhere defined maps of $\mathbb{R}^n$ to itself are considered, with no requirement for a constant Jacobian determinant or a rational inverse. The birational case is proved and the Galois case clarified. Two known special cases of the Strong Real Jacobian Conjecture (SRJC) are generalized to the rational map context. For an invertible map, the associated extension of rational function fields must be of odd degree and must have no nontrivial automorphisms. That disqualifies the Pinchuk counterexamples to the SRJC as candidates for invertibility.

1. Introduction and summary of results. The Jacobian Conjecture (JC) [**1, 9**] asserts that a polynomial map $F : k^n \to k^n$, where k is a field of characteristic zero, has a polynomial inverse if it is a Keller map [**14**], which means that its Jacobian determinant, $j(F)$, is a nonzero element of k. The JC is still not settled for any $n > 1$ and any specific field k of characteristic zero.

For $k = \mathbb{R}$, the Strong Real Jacobian Conjecture (SRJC) asserts that a polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$, has a real analytic inverse if it is nonsingular, meaning that $j(F)$, whether constant or not, vanishes nowhere on $\mathbb{R}^n$. However, Sergey Pinchuk exhibited a family of counterexamples for $n = 2$ [**20**], so the SRJC holds only in special cases.

The Rational Real Jacobian Conjecture (RRJC) is considered here. It is the extension of the SRJC to everywhere defined rational maps, as well as polynomial ones. Everywhere defined means that each component of the map can be expressed as the quotient of two polynomials with a nowhere vanishing denominator. That rules out rational functions such as $(x^4+y^4)/(x^2+y^2)$, which is not defined at the origin, even though it has a unique continuous extension to all

of $\mathbb{R}^2$. That requirement is crucial, as $F = (x^2y^6 + 2xy^2, xy^3 + 1/y)$ is Keller and maps $(1,1)$ and $(-3,-1)$ to the same point [**22**]. Assume $F : \mathbb{R}^n \to \mathbb{R}^n$ is such a map and is nonsingular. Then all its fibers are finite of size at most the degree of the associated finite algebraic extension of rational function fields. If F also has a (necessarily real analytic) inverse, then the function field extension is of odd degree and has a trivial automorphism group. The extension degree and the maximum fiber size are of the same parity. If odd maximum fiber size is added as an additional hypothesis to the RRJC or SRJC, it disqualifies the Pinchuk counterexamples, for all of which that size is 2 [**7**]. If the extension degree is 1 (the birational case), then F has an inverse that is also an everywhere defined birational nonsingular map. If the extension is Galois, then F has an inverse if, and only if, F is birational.

If the automorphism group condition is added as an additional hypothesis, the RRJC and SRJC are true in the Galois case. Thus if both necessary conditions are assumed, the resulting modified RRJC and SRJC conjectures are true in the birational and Galois cases and have no obvious counterexamples.

Finally, two known special cases of the SRJC are generalized to the RRJC context. They show that F is invertible if $A(F)$, the set of points in the codomain over which F is not locally a trivial fibration, either is of codimension greater than 2, or does not intersect the image of F.

2. Basic properties. Both the Jacobian hypothesis and the conclusion of the RRJC can be restated in various equivalent ways. Principally, the former is equivalent to the assertion that F is locally diffeomorphic or locally real bianalytic, and the latter to the assertion that F is injective or bijective or a homeomorphism or a diffeomorphism. These are all obvious, except for the key result that injectivity, also called univalence, implies bijectivity for maps of $\mathbb{R}^n$ to itself that are polynomial or, more generally, rational and defined on all of $\mathbb{R}^n$ [**2**]. That result does not generalize to semi-algebraic maps of $\mathbb{R}^n$ to itself [**16**]. Clearly any global univalence theorems [**17**] for local diffeomorphisms can yield special cases of the conjecture. Properness suffices, and related topological considerations play a role below. But the focus of this article is on results or conjectures that require the polynomial or rational character of a map and involve properties of the associated extension of rational function fields.

The extension of function fields exists, and is algebraic of finite degree, for any dominant rational $F : \mathbb{R}^n \to \mathbb{R}^n$, whether defined everywhere or not. F is dominant if, and only if, $j(F)$ is not identically zero. The extension is the inclusion of the subfield generated over $\mathbb{R}$ by the (algebraically independent) components of F in the rational function field on the domain of F, and will be written as $\mathbb{R}(F) \subseteq \mathbb{R}(X)$ or $\mathbb{R}(X)/\mathbb{R}(F)$. The degree d of the extension is called the extension degree of F. If F is generically N-to-one for a positive

integer N, then N is called the geometric degree of F. In general, let $t \in \mathbb{R}(X)$ be a primitive element for the extension, meaning that $\mathbb{R}(F)(t) = \mathbb{R}(X)$. For generic y in the codomain, inverse images x of y correspond bijectively to real roots $r = t(x)$ at y of the monic minimal polynomial of t over $\mathbb{R}(F)$. So a generic fiber of F is finite and either empty or of positive size at most d, but F need not have a geometric degree.

By definition, an automorphism of the extension is a field automorphism of $\mathbb{R}(X)$ that fixes every element of $\mathbb{R}(F)$.

PROPOSITION 1. *If the geometric degree of F is 1, then the extension has odd degree and trivial automorphism group.*

PROOF. The nonreal roots occur in complex conjugate pairs, and the degree of the monic minimal polynomial for a primitive element is d. If $G : \mathbb{R}^n \to \mathbb{R}^n$ is the geometric realization of an automorphism of $\mathbb{R}(X)$ as a rational map and every element of $\mathbb{R}(F)$ is fixed by the automorphism, then $F \circ G = F$. For a generic x, G is defined at x, and F is defined and locally diffeomorphic at both x and $x' = G(x)$. Since the geometric degree of F is 1, G is the identity on an open set and therefore, because it is rational, the identity map. So also the automorphism is the identity. □

A map $F : \mathbb{R}^n \to \mathbb{R}^n$ will be called a rational nonsingular (nondegenerate) map if it is an everywhere defined rational map and $j(F)$ vanishes nowhere (resp., does not vanish identically). In either case, both of Proposition 1 conclusions become necessary conditions for the existence of an inverse. The Pinchuk counterexamples [20] to the SRJC (and hence to the RRJC) are nonsingular polynomial maps of $\mathbb{R}^2$ to $\mathbb{R}^2$ with no inverse. All these Pinchuk maps have the same nonconstant, everywhere positive Jacobian determinant, geometric degree 2, no point with more than 2 inverse images, exactly 2 points omitted in the image plane, and the same extension of degree 6 with the trivial automorphism group [7, 8].

All three conjectures discussed are true in the dimension $n = 1$ case $f : \mathbb{R} \to \mathbb{R}$. In the JC case, f is of degree 1. In the SRJC case, f is proper, since any nonconstant polynomial becomes infinite when its argument does. In the RRJC case, f is monotone increasing or decreasing, hence injective, thus surjective, so unbounded above and below, and therefore proper.

In the RRJC context, the distinction between nonzero constant and nowhere vanishing Jacobian determinants is not as critical as it may seem. If $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the hypotheses, let $x \in \mathbb{R}^n, z \in \mathbb{R}$ and define $F^+ : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by $F^+(x, z) = (F(x), z/(j(F)(x)))$. Then F^+ also satisfies the hypotheses, $j(F^+) = 1$, and F^+ is injective if, and only if, F is injective. As pointed out in [6], choosing Pinchuk maps for F yields Keller counterexamples to the RRJC in dimension $n = 3$.

A Samuelson map is a map with a square Jacobian matrix, all of whose leading principal minors, including its determinant, vanish nowhere. A rational Samuelson map defined on all of $\mathbb{R}^n$ has an inverse [5], which is necessarily Nash (semi-algebraic and real analytic), but is rational if, and only if, the function field extension is birational (cf. section 3). The well known real analytic example $(e^x - y^2 + 3, 4ye^x - y^3)$ in [10] shows that a Samuelson map need not be globally injective (consider $(0,2)$ and $(0,-2)$). The variation $F(x,y) = (h - y^2 + 3, 4yh - y^3)$ in [6], where h is the function $h(x) = x + \sqrt{1 + x^2}$ (positive square root intended) has the same properties and is Nash as well. So does F^+, which is also Keller.

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a rational nonsingular map. It is a local diffeomorphism, hence an open map. Let $x \in \mathbb{R}^n$ and $y = F(x) \in \mathbb{R}^n$ and define $m(x)$ to be the number of inverse images of y under F, potentially allowing $+\infty$ as a possible value. Since F is open, $m(x') \geq m(x)$ for $x' \in \mathbb{R}^n$ in a neighborhood of x. So if $A \subseteq \mathbb{R}^n$, the maximum value of m on A is also the maximum value of m on its topological closure $\bar{A}$. So all fibers of F, not just generic ones, are finite of size at most d, where d is the extension degree of F. The fiber size maximum, N, is attained on an open subset of the codomain, which must contain a point where N is the number of real roots of a polynomial of degree d with real coefficients. Thus N and d have the same parity. Note that if N and d are odd, then a generic fiber is nonempty because a real polynomial of odd degree has at least one real root, and so $F(\mathbb{R}^n)$ is a connected dense open semi-algebraic subset of the codomain. All subsets of $\mathbb{R}^n$ that can be described in the first order logic of ordered fields are semi-algebraic. The description may include real constant symbols (coefficients, values, etc.) and quantification over real variables (but not over subsets, functions or natural numbers); results for any dimension $n > 0$ and involving polynomials of arbitrary degrees follow from schemas specifying first order descriptions for any fixed choice of the natural number parameters. As a first application of that principle, the N subsets of the domain $\mathbb{R}^n$ on which $m(x)$ has a specified numeric value in the range $1, \ldots, N$, and the $N + 1$ subsets of the codomain $\mathbb{R}^n$ on which y has a specified number of inverse images in the range $0, \ldots, N$, are all semi-algebraic. By definition, F is proper at a point y in its codomain if y has an open neighborhood U, such that any compact subset of U has a compact inverse image under F. The set of points y in the codomain at which F is proper is readily verified to be the open set of points at which the number of inverse images of y is locally constant. That set contains all points with N inverse images and has an ϵ-ball first order description. Its complement $A(F)$, the asymptotic variety of F, is therefore closed semi-algebraic and the inclusion $A(F) \subset \mathbb{R}^n$ is strict. $A(F)$ is the union for $i = 0, \ldots, N - 1$ of the semi-algebraic sets consisting of points y in the codomain at which F is not proper and for which y has

exactly i inverse images. At an interior point y of one of these sets F would be proper, contradicting $y \in A(F)$. Thus each such set has empty interior, hence is of dimension less than n. Consequently $\dim A(F) < n$. It follows that the complement of $A(F)$ is a finite union of disjoint connected open semi-algebraic subsets of $\mathbb{R}^n$ on each of which the number of inverse images of points is a constant, with possibly differing constants for different connected components. If U is any such connected component that intersects $F(\mathbb{R}^n)$, then $F^{-1}(U)$ is nonempty, open and semi-algebraic. Let V be one of its finitely many connected components. Since V is an open and closed subset of $F^{-1}(U)$, the map $V \to U$ induced by F is a proper local homeomorphism of connected, locally compact, and locally arcwise connected spaces and hence it is a covering map. Such a map is surjective, so all of U is contained in $F(\mathbb{R}^n)$. V must be exactly one of the finitely many connected components of the open semi-algebraic set $\mathbb{R}^n \setminus F^{-1}(A(F))$, since it is closed in that subset as one element of a finite cover by disjoint total spaces of covering maps. Speaking informally, this presents a view of F as a finite collection of n-dimensional covering maps, of possibly different degrees, glued together along semi-algebraic sets of positive codimension to form $\mathbb{R}^n$ at the total space level, whose base spaces, which may sometimes coincide for different total spaces, are similarly glued together to form $F(\mathbb{R}^n)$. $F(\mathbb{R}^n) \cap A(F)$ is in general neither empty nor all of $A(F)$, a behavior exhibited by any Pinchuk map F, since then $A(F)$ is a polynomial curve and exactly two of its points are not in the image of F.

PROPOSITION 2. *If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a rational nonsingular map and F is generically injective, then F is invertible and its inverse is a nonsingular real analytic map defined on all of $\mathbb{R}^n$.*

PROOF. Suppose F is injective on a nonempty Zariski open set $U \subset \mathbb{R}^n$. Let V be the complement of U. Since V is algebraic and $\dim V < n$, $F(V)$ is semi-algebraic of maximum dimension at most $n - 1$. So $F(V)$ is not Zariski dense and therefore the open set of points of maximum fiber size N contains a point with inverse images only in U. It follows that $N = 1$, that F is injective, and hence that F is surjective [**2**]. F is locally real bianalytic, and so its global inverse is a nonsingular real analytic map. $\square$

REMARK. The asymptotic variety was defined by Ronen Peretz as the set of finite limits of a map along curves that tend to infinity [**18, 19**]. For real polynomial maps, it can fail to be Zariski closed, and therefore not technically a variety [**11**]. In that context it has been extensively studied by Zbigniew Jelonek as the set of points at which a map is not proper [**12,13**]. As one result, he shows that for a nonconstant polynomial map $F : \mathbb{R}^n \to \mathbb{R}^m$, where n and m are any positive integers and no other conditions are imposed, the set $A(F)$ is $\mathbb{R}$-uniruled. By that he means that for any $a \in A(F)$ there is a nonconstant

polynomial map $g : \mathbb{R} \to \mathbb{R}^m$ (a polynomial curve) such that $g(0) = a$ and $g(t) \in A(F)$ for all $t \in \mathbb{R}$. That in turn implies that every connected component of $A(F)$ is unbounded and has positive dimension. These results do not hold for everywhere defined rational maps, as shown by $y = 1/(1 + x^2)$, which is proper except at $y = 0$.

3. The birational and Galois cases.

THEOREM 1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a birational nonsingular map. Then F has a global inverse, which is also a birational nonsingular map.*

PROOF. $\mathbb{R}(F) = \mathbb{R}(X)$, so the extension degree is 1. As it bounds the size of all fibers, F is injective, hence invertible. Thus the rational inverse of F extends to a real analytic map on all of $\mathbb{R}^n$. Let $g = a/b$ be a component of the inverse, where a and b are polynomials with no nonconstant common factor and suppose $b(x) = 0$ for some $x \in \mathbb{R}^n$. Let U be an open neighborhood of x in $\mathbb{C}^n$, such that g extends to a complex analytic function $\tilde{g}$ on U satisfying $b\tilde{g} = A$. Let c be an irreducible complex polynomial factor of b satisfying $c(x) = 0$. Then a vanishes on the irreducible hypersurface $c = 0$ in $\mathbb{C}^n$, because it does so in U. So c is also an irreducible factor of a. Using complex conjugation, it follows easily that a^2 and b^2 have a nonconstant common factor in the real polynomial ring. But then so do a and b, by unique factorization. This contradiction shows that b vanishes nowhere. So all components of the inverse are everywhere defined rational functions. That makes the inverse an everywhere defined rational map, and it is clearly nonsingular and birational. □

If F is defined over a subfield $k \subset \mathbb{R}$, then so is its inverse, since extension degree is preserved by a faithfully flat extension of the coefficients. In that case, F induces a birational bijection of k^n onto k^n. Note that $y = x + x^3$ is polynomial, nonsingular, invertible, and defined over $\mathbb{Q}$, but the induced map from $\mathbb{Q}$ to $\mathbb{Q}$ is not surjective.

REMARK. In [15], polynomial maps $F : \mathbb{R}^n \to \mathbb{R}^n$ that map $\mathbb{R}^n$ bijectively onto $\mathbb{R}^n$ are considered, and the question is raised of when the inverse is rational. If so, the inverse is everywhere defined on $\mathbb{R}^n$ and F is called a polynomial-rational bijection (PRB) of $\mathbb{R}^n$. A key technical result is that a polynomial bijection is a PRB if its natural extension to a polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ maps only real points to real points. A PRB F has a nowhere vanishing Jacobian determinant $j(F)$. Conversely, it is shown that a nowhere vanishing $j(F)$ alone suffices to establish that a polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$ of degree two is a bijection and a PRB. A related but stronger condition is defined and shown to be sufficient, but not necessary, for polynomial maps of degree greater than two.

THEOREM 2. *If $F: \mathbb{R}^n \to \mathbb{R}^n$ is a rational nonsingular map and $\mathbb{R}(X)/\mathbb{R}(F)$ is a Galois extension, then F is invertible if, and only if, F is birational.*

PROOF. If F is invertible, then the extension has no nontrivial automorphisms. So it can be Galois only if it is of degree 1. In that (birational) case F does have an inverse. $\square$

If F is defined over a subfield $k \subset \mathbb{R}$ and $k(X)/k(F)$ is Galois, then so is $\mathbb{R}(X)/\mathbb{R}(F)$.

REMARK. The Galois case of the standard JC states that a polynomial Keller map with a Galois field extension has a polynomial inverse. It was first proved for $k = \mathbb{C}$ only [**4**], using methods of the theory of several complex variables. The general characteristic zero case appears in [**21**] and, independently, in [**23**]. The theorem above is manifestly weaker. Of course, the existence of a polynomial inverse implies the triviality of the field extension, so the JC theorem has no concrete examples.

In contrast, in the SRJC and RRJC contexts, the existence of an inverse does not imply the field extension is Galois, much less birational. For instance, if $y = x + x^3$, the field extension $\mathbb{R}(y) \subset \mathbb{R}(x)$ is neither. Even so, a Galois extension of degree $d \neq 1$ would represent a counterexample to the RRJC of a new, and unexpected, type.

4. Promoted SRJC cases. The two theorems below have been proved in the SRJC context and, because of their topological character, they generalize to the RRJC context almost effortlessly. In both theorems, let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a rational nonsingular map. The theorems impose conditions on $A(F)$ that are illusory, in that they conclude that F is invertible, and so $A(F)$ is actually empty. For polynomial F, the first theorem was proved by Zbigniew Jelonek [**13**, Theorem 8.2] and the second by Christopher I. Byrnes and Anders Lindquist [**3**, Remark 2].

THEOREM 3. *If the dimension of $A(F)$ is less than $n - 2$, then F is invertible.*

PROOF. If $A \subset \mathbb{R}^n$ is a closed semi-algebraic set and $\dim A < n - 2$, then $A^c = \mathbb{R}^n \setminus A$ is simply connected [**13**, Lemma 8.1]. This applies to both $A(F)$ and to $B(F) = F^{-1}(A(F))$, which satisfies $\dim B(F) = \dim A(F) \cap F(R^n)$. The induced map from $B(F)^c$ to $A(F)^c$ is proper, hence a covering map, and therefore a homeomorphism. Since $B(F)$ is not Zariski dense, F is generically injective, and so invertible. This proof is that of Jelonek, which simply applies to rational maps as well. $\square$

THEOREM 4. *If $A(F) \cap F(\mathbb{R}^n) = \emptyset$, then F is invertible.*

PROOF. The condition states that every point of the (connected, open) image of F is a point at which F is proper. Equivalently, the induced map $\mathbb{R}^n \to F(\mathbb{R}^n)$ is proper. The main result of [**3**] is that the standard complex JC holds for polynomial maps that are proper as maps onto their image. In Remark 2 at the end of the note, that result is also proved in the SRJC context. Briefly, $\mathbb{R}^n$ is a universal covering space, of finite degree d, of $F(\mathbb{R}^n)$. By well known results of the branch of topology called P. A. Smith theory, there are no fixed point free homeomorphisms of $\mathbb{R}^n$ onto itself of prime period. But the fundamental group $\pi_1(F(\mathbb{R}^n))$ is of order d, and contains an element of prime period unless $d = 1$. So $d = 1$, F is injective, and therefore invertible. The assumption that F is polynomial, rather than just real analytic, is used at only two points in the proof. First, it ensures that the degree of the covering map is finite, and second, that injectivity implies invertibility. Rationality is sufficient in both situations, so this proof works in the RRJC context as well. $\square$

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