

## A REMARK ON FEJÉR AND MITTAG-LEFFLER THEOREMS

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**Abstract.** We discuss some generalizations of the classical Fejér and Mittag-Leffler theorems to the case of several complex variables with applications to the Shilov and Bergman boundaries.

**1. Introduction.** For a bounded domain  $D \subset \mathbb{C}^N$  let  $\mathcal{A}(D)$  (resp.  $\mathcal{O}(\overline{D})$ ) denote the space of all continuous functions  $f : \overline{D} \rightarrow \mathbb{C}$  such that  $f|_D$  is holomorphic (resp.  $f$  extends holomorphically to a neighborhood of  $\overline{D}$ ). Let  $\partial_S D$  (resp.  $\partial_B D$ ) be the *Shilov* (resp. *Bergman*) *boundary* of  $D$ , i.e. the minimal compact set  $K \subset \overline{D}$  such that  $\max_K |f| = \max_{\overline{D}} |f|$  for every  $f \in \mathcal{A}(D)$  (resp.  $f \in \mathcal{O}(\overline{D})$ ). Obviously,  $\mathcal{O}(\overline{D}) \subset \mathcal{A}(D)$  and hence  $\partial_B D \subset \partial_S D \subset \partial D$ . Notice that, in general,  $\partial_B D \subsetneq \partial_S D$ , e.g. for the domain  $D := \{(z, w) \in \mathbb{C}^2 : 0 < |z| < 1, |w| < |z|^{-\log |z|}\}$  (cf. [6], § 16).

The algebra  $\mathcal{A}(D)$  endowed with the supremum norm is a Banach algebra. Then  $\partial_S D$  coincides with the Shilov boundary of  $\mathcal{A}(D)$  in the sense of uniform algebras (cf. [7], Chap. I, Sec. H). Moreover, the Bergman boundary  $\partial_B D$  coincides with the Shilov boundary of the uniform algebra  $\mathcal{B}(D)$  defined as the uniform closure in  $\mathcal{A}(D)$  of  $\mathcal{O}(\overline{D})|_{\overline{D}}$ .

Assume that the envelope of holomorphy  $\tilde{D}$  of  $D$  is univalent. We are interested in characterizations of those domains  $D$  for which  $\partial_S D = \partial_S \tilde{D}$  (resp.  $\partial_B D = \partial_B \tilde{D}$ ) (cf. [9]).

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REMARK 1.1. (a) It is well known (cf. [8], Remark 1.4.5(h)) that

$$(*) \quad \sup_{\tilde{D}} |g| = \sup_D |g|, \quad g \in \mathcal{O}(\tilde{D}).$$

In particular, if  $D$  is bounded, then so is  $\tilde{D}$ .

- (b) It is also well known that if  $D \subset \mathbb{C}^N$  is Reinhardt (resp. balanced, resp. starlike) domain, then  $\tilde{D}$  is univalent and  $\tilde{D}$  is Reinhardt (resp. balanced, resp. starlike) (cf. [8], Remark 1.9.6(c,e,f), Corollary 1.9.18).
- (c) Let  $D \subset G \subset \mathbb{C}^N$  be domains such that  $G$  is a domain of holomorphy and  $(D, G)$  is a *Runge pair*, i.e. the space  $\mathcal{O}(G)|_D$  is dense in  $\mathcal{O}(D)$ . Then the envelope of holomorphy of  $D$  is univalent and  $(\tilde{D}, G)$  is a Runge pair (cf. [3], see also [8], Proposition 3.1.22). In particular, the result applies if  $G = \mathbb{C}^N$  and polynomials are dense in  $\mathcal{O}(D)$ .

First, let us recall some known results.

REMARK 1.2. (a) Since  $\mathcal{A}(\tilde{D})|_{\overline{D}} \subset \mathcal{A}(D)$  (resp.  $\mathcal{O}(\tilde{D})|_{\overline{D}} \subset \mathcal{O}(\overline{D})$ ), we get  $\partial_S \tilde{D} \subset \partial_S D$  (resp.  $\partial_B \tilde{D} \subset \partial_B D$ ).

- (b) If  $\mathcal{A}(D) \subset \mathcal{A}(\tilde{D})|_{\overline{D}}$  (resp.  $\mathcal{O}(\overline{D}) \subset \mathcal{O}(\tilde{D})|_{\overline{D}}$ ), then  $\partial_S D = \partial_S \tilde{D}$  (resp.  $\partial_B D = \partial_B \tilde{D}$ ).
- (c) There exists a bounded Reinhardt domain  $D \subset \mathbb{C}_* \times \mathbb{C}$  such that  $\mathcal{A}(D) \not\subset \mathcal{A}(\tilde{D})|_{\overline{D}}$  (cf. [11], Example 6.1). On the other hand, if  $D \subset \mathbb{C}^N$  is a bounded Reinhardt domain, then  $\partial_S D = \partial_S \tilde{D}$  (cf. [11], Corollary 6.2).
- (d) If  $\overline{D}$  has a neighborhood basis consisting of domains with univalent envelopes of holomorphy, then  $\mathcal{O}(\overline{D}) \subset \mathcal{O}(\tilde{D})|_{\overline{D}}$  and hence  $\partial_B D = \partial_B \tilde{D}$ .

In fact, by (b), we only need to prove that if  $\overline{D} \subset G \subset \mathbb{C}^N$ , where  $G$  is a bounded domain with a univalent envelope of holomorphy  $\tilde{G}$ , then  $\overline{\tilde{D}} \subset \tilde{G}$ . For every  $g \in \mathcal{O}(\tilde{G})$ , using (\*), we get  $\sup_{\tilde{D}} |g| = \sup_D |g| = \max_{\overline{D}} |g|$ . Thus  $\tilde{D} \subset \widehat{\overline{D}}_{\mathcal{O}(\tilde{G})} \subset \tilde{G}$ , where  $\widehat{K}_{\mathcal{O}(\Omega)} := \{z \in \Omega : \forall_{g \in \mathcal{O}(\Omega)} : |g(z)| \leq \max_K |g|\}$  (cf. [8], Theorem 1.10.4).

- (e) Let  $\mathbb{B}(a, r)$  denote the Euclidean ball centered at  $a \in \mathbb{C}^N$  with radius  $r > 0$ ;  $\mathbb{B}(r) := \mathbb{B}(0, r)$ . If  $D \subset \mathbb{C}^N$  is a bounded balanced (resp. starlike) domain, then  $\{\overline{D} + \mathbb{B}(\varepsilon)\}_{\varepsilon > 0}$  gives a neighborhood basis of  $\overline{D}$  consisting of bounded balanced (resp. starlike) domains. Hence  $\partial_B D = \partial_B \tilde{D}$ .
- (f) If  $\partial_B D = \partial_B \tilde{D}$  and  $\mathcal{A}(D) = \mathcal{B}(D)$ , then  $\partial_S D = \partial_S \tilde{D}$ .

Indeed,  $\partial_S D = \partial_B D = \partial_B \tilde{D} \subset \partial_S \tilde{D} \subset \partial_S D$ .

- (g) There exists a bounded Hartogs domain  $D \subset \mathbb{C}^2$  with a univalent envelope of holomorphy  $\tilde{D}$  such that  $\partial_S D \neq \partial_S \tilde{D}$ ,  $\partial_B D \neq \partial_B \tilde{D}$ , and  $\mathcal{O}(\overline{D}) \not\subset \mathcal{A}(\tilde{D})|_{\overline{D}}$  (cf. [9]).

The paper is organized as follows:

— first, we prove two general theorems on polynomial approximation in balanced and starlike domains (Theorems 2.1, 3.1); these results are of independent interest;

— next, we prove the following result.

- THEOREM 1.3. (a) *If  $D$  is a bounded balanced domain, then  $\partial_S D = \partial_S \tilde{D}$  and  $\partial_B D = \partial_B \tilde{D}$ .*  
 (b) *If  $D$  is a bounded strictly starlike domain, then  $\partial_S D = \partial_S \tilde{D}$  and  $\partial_B D = \partial_B \tilde{D}$ .*

Notice that (a) answers an open problem formulated in [9].

Recall that a bounded starlike domain  $D \subset \mathbb{C}^n$  is said to be *strictly starlike with respect to the origin* if  $\overline{D} \subset (1 + \varepsilon)D$  for every  $\varepsilon > 0$ . The equality  $\partial_S D = \partial_S \tilde{D}$  for an arbitrary bounded starlike domain  $D$  seems to be an open problem.

## 2. Fejér theorem for holomorphic functions.

THEOREM 2.1 (Fejér theorem). *Let  $D \subset \mathbb{C}^N$  be a bounded balanced domain, let  $f \in \mathcal{A}(D)$ , and let*

$$f(z) = \sum_{j=0}^{\infty} Q_j(z), \quad z \in D,$$

*be the Taylor series development of  $f$  in  $D$  ( $Q_j$  is a homogeneous polynomial of degree  $j$ ). Put*

$$s_n := \sum_{j=0}^n Q_j, \quad \sigma_n := \frac{s_0 + \cdots + s_{n-1}}{n}, \quad n \in \mathbb{N}.$$

*Then  $\sigma_n \longrightarrow f$  uniformly on  $\overline{D}$ .*

PROOF. It is known that for arbitrary  $n \in \mathbb{N}$  and  $z \in D$  we have

$$\sigma_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}z) F_n(t) dt,$$

where

$$F_n(t) := \frac{1}{n} \left( \frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t} \right)^2$$

is the  $n$ -th Fejér kernel. In particular,

$$1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(t) dt, \quad n \in \mathbb{N}.$$

Let  $M := \max_{\overline{D}} |f|$ ,  $\omega(\delta) := \max\{|f(e^{it}z) - f(z)| : z \in \overline{D}, |t| \leq \delta\}$ ,  $\delta > 0$ . Since the function  $\overline{D} \times [-\pi, \pi] \ni (z, t) \mapsto f(e^{it}z)$  is uniformly continuous, we have  $\lim_{\delta \rightarrow 0+} \omega(\delta) = 0$ . The standard reasoning gives

$$\begin{aligned} |\sigma_n(z) - f(z)| &\leq \frac{1}{2\pi} \int_{|t| \leq \delta} |f(e^{it}z) - f(z)| F_n(t) dt \\ &+ \frac{1}{2\pi} \int_{\delta < |t| \leq \pi} |f(e^{it}z) - f(z)| F_n(t) dt \leq \omega(\delta) + \frac{2M}{n \sin^2 \frac{1}{2}\delta}, \quad z \in D, \quad 0 < \delta < \pi. \end{aligned}$$

Consequently, given  $\varepsilon > 0$ , we first find a  $\delta \in (0, \pi)$  such that  $\omega(\delta) \leq \frac{\varepsilon}{2}$  and next we choose an  $n_0 \in \mathbb{N}$  with  $\frac{2M}{n \sin^2 \frac{1}{2}\delta} \leq \frac{\varepsilon}{2}$  for  $n \geq n_0$ . Finally,  $|\sigma_n(z) - f(z)| \leq \varepsilon$  for  $n \geq n_0$  and  $z \in D$ , and therefore for  $z \in \overline{D}$ .  $\square$

### 3. Mittag-Leffler theorem.

**THEOREM 3.1** (Mittag-Leffler theorem, cf. [5]). *There exist numbers  $c_{n,j} \in \mathbb{C}$ ,  $n \in \mathbb{N}$ ,  $j \in \{0, \dots, k_n\}$ , such that for every  $N \in \mathbb{N}$ , for every starlike domain  $D \subset \mathbb{C}^N$ , and for every  $f \in \mathcal{O}(D)$  with the Taylor development*

$$f(z) = \sum_{j=0}^{\infty} Q_j(z)$$

*in a neighborhood of 0, the sequence of polynomials*

$$\sigma_n := \sum_{j=0}^{k_n} c_{n,j} Q_j, \quad n \in \mathbb{N},$$

*converges to  $f$  locally uniformly in  $D$ . In particular, the sequence  $\{\sigma_n\}_{n=1}^{\infty}$  is locally uniformly convergent in the maximal starlike domain  $G_f$  to which  $f$  is analytically continuable ( $G_f$  is called the Mittag-Leffler star).*

- REMARK 3.2.** (a) The case  $N = 1$  is due to Mittag-Leffler (cf. [12]).  
 (b) Our proof of Theorem 3.1 will be based on a method proposed (for  $N = 1$ ) by E. Borel (cf. [2]).  
 (c) Generalizations of the Mittag-Leffler theorem to the case  $N \geq 2$  were studied (using various methods) by several authors (cf. e.g. [1], [10], [5]).  
 (d) Theorem 3.1 implies that polynomials are dense in  $\mathcal{O}(D)$  (see also [4]). In particular, the envelope of holomorphy  $\tilde{D}$  of  $D$  is a starlike Runge domain (cf. Remark 1.1(c)).

*Proof of Theorem 3.1.* By Runge's theorem there exists a sequence  $\{W_n\}_{n=1}^{\infty}$  of polynomials that converges locally uniformly in  $\mathbb{C} \setminus [1, +\infty)$  to

the function  $W(\lambda) := \frac{1}{1-\lambda}$ . Let

$$W_n(\lambda) = \sum_{j=0}^{k_n} c_{n,j} \lambda^j, \quad n \in \mathbb{N}.$$

Fix  $f \in \mathcal{O}(D)$  and  $a \in D$ . Since  $[0, 1] \cdot a$  is a compact subset of  $D$  there exists an  $r > 0$  such that  $\Delta \cdot \overline{\mathbb{B}}(a, r) \subset D$ , where

$$\Delta := \{x + iy \in \mathbb{C} : -r \leq x \leq 1 + r, |y| \leq r\}.$$

For every  $z \in \mathbb{B}(a, r)$ , the function  $\lambda \mapsto f(\lambda z)$  is holomorphic in a neighborhood of  $\Delta$ . In particular,

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\lambda z) d\lambda}{\lambda - 1}, \quad z \in \mathbb{B}(a, r),$$

where  $\Gamma$  is the positively oriented boundary of  $\Delta$ . On the other hand, for  $z \in \mathbb{B}(a, r)$  we get

$$\sigma_n(z) = \sum_{j=0}^{k_n} c_{n,j} \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\lambda z) d\lambda}{\lambda^{j+1}} = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda z) W_n\left(\frac{1}{\lambda}\right) \frac{d\lambda}{\lambda}.$$

Consequently,

$$f(z) - \sigma_n(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\lambda z)}{\lambda} \left( \frac{1}{1 - \frac{1}{\lambda}} - W_n\left(\frac{1}{\lambda}\right) \right) d\lambda, \quad z \in \mathbb{B}(a, r).$$

Since  $W_n(\frac{1}{\lambda}) \rightarrow \frac{1}{1 - \frac{1}{\lambda}}$  uniformly for  $\lambda \in \partial\Delta$ , we conclude that  $\sigma_n \rightarrow f$  uniformly on  $\mathbb{B}(a, r)$ .  $\square$

**4. Proof of Theorem 1.3.** The case of the Bergman boundary follows from Remark 1.2(e).

(a) It suffices to show that  $\mathcal{A}(D) \subset \mathcal{A}(\tilde{D})|_{\overline{D}}$  (cf. Remark 1.2(b)). Fix an  $f \in \mathcal{A}(D)$  and let  $\{\sigma_n\}_{n=1}^{\infty}$  be as in § 2. Using Theorem 2.1 and the equation (\*) of Remark 1.1 we conclude that the sequence  $\{\sigma_n\}_{n=1}^{\infty}$  is uniformly convergent on  $\tilde{G}$  to a function  $\tilde{f} \in \mathcal{A}(\tilde{D})$ , which completes the proof of (a).

(b) Fix a sequence  $\varepsilon_n \searrow 0$  and let  $D_n := (1 + \varepsilon_n)D \supset \overline{D}$ ,  $n \in \mathbb{N}$ . Take an  $f \in \mathcal{A}(D)$  and let  $f_n(z) := f(\frac{z}{1+\varepsilon_n})$ ,  $z \in D_n$ . Then  $f_n|_{\overline{D}} \in \mathcal{O}(\overline{D})$  and  $f_n|_{\overline{D}} \rightarrow f$  uniformly on  $\overline{D}$ . Thus  $f \in \mathcal{B}(D)$ . Now, the result follows from Remark 1.2(e)(f).  $\square$

**REMARK 4.1.** Observe that Theorem 1.3(b) may be also proved via Theorem 3.1. In fact, applying Theorem 3.1 to  $(D_n, f_n)$ , we conclude that for each  $n \in \mathbb{N}$ , there exists a polynomial  $P_n$  such that  $|f_n(z) - P_n(z)| \leq \frac{1}{n}$ ,  $z \in \overline{D}$ . Thus the sequence  $\{P_n\}_{n=1}^{\infty}$  converges to  $f$  uniformly on  $\overline{D}$ . Now, we can finish the proof as in (a).

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