UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA
doi: 10.4467/20843828AM.15.004.3730 FASCICULUS LII, 2015

DEGENERATION OF KÄHLER–RICCI SOLITONS ON FANO MANIFOLDS

by D. H. Phong, Jian Song and Jacob Sturm

Abstract. We consider the space $\mathcal{KR}(n,F)$ of Kähler–Ricci solitons on n-dimensional Fano manifolds with Futaki invariant bounded by F. We prove a partial C^0 estimate for $\mathcal{KR}(n,F)$ as a generalization of the recent work of Donaldson-Sun for Fano Kähler–Einstein manifolds. In particular, any sequence in $\mathcal{KR}(n,F)$ has a convergent subsequence in the Gromov-Hausdorff topology to a Kähler–Ricci soliton on a Fano variety with log terminal singularities.

1. Introduction.

Let X be a Fano manifold admitting a smooth Kähler–Ricci soliton, that is a metric $g_{i\bar{j}}$ satisfying the equation

$$Ric(g) = g + L_{\mathcal{V}} g.$$

where $\mathcal{V}$ is a holomorphic vector field, and $L_{\mathcal{V}}$ is the Lie derivative along V. The holomorphic vector field can be expressed in terms of the Ricci potential u, with

$$R_{i\bar{j}} = g_{i\bar{j}} - u_{i\bar{j}},\ u_{ij} = u_{\bar{i}\bar{j}} = 0,\ \mathcal{V}^i = -g^{i\bar{j}} u_{\bar{j}}. \tag{1.1}$$

The Futaki invariant associated to (X, g, V) is given by

$$\mathcal{F}_X(\mathcal{V}) = \int_X |\nabla u|^2 dV_g = \int_X |\mathcal{V}|^2 dV_g \geq 0.$$

2010 *Mathematics Subject Classification.* 53C55, 53C21, 53C44.
Key words and phrases. Bergman kernel, partial C^0 estimate, conformal transformations, Cheeger–Colding theory, Perelman pseudo-locality, Kodaira embeddings.
Work supported in part by National Science Foundation grants DMS-0757372, DMS-12-66033, DMS-0905873, and DMS-0847524.

DEFINITION 1.1. Let $\mathcal{KR}(n,F)$ be the set of Kähler–Ricci solitons (X,g) with

$$\dim X = n,\ Ric(g) = g + L_{\mathcal{V}}g,\ \mathcal{F}_X(\mathcal{V}) \le F.$$

The main result of this paper is a partial C^0 estimate for Kähler–Ricci solitons. Let $(X,g) \in \mathcal{KR}(n,F)$ and ω_g be the Kähler form for g. Let h be a hermitian metric on K_X^{-1} with $Ric(h) = \omega_g$, which is unique up to a multiplicative normalization. We define the L^2-inner product on $H^0(X, K_X^{-k})$ by

$$\langle s, s'\rangle = k^n \int_X s\overline{s'}h^k\omega_g^n$$

for any $s, s' \in H^0(X, K_X^{-k})$. Let $\{s_j\}_{j=1}^{N_k}$ be an orthonormal basis in $H^0(X, K_X^{-k})$ with respect to $\langle\,,\rangle$. Then the Bergman kernel $\rho_{X,k}$ is defined to be

$$\rho_{X,k} = \sum_j |s_j|^2_{h^k}. \tag{1.2}$$

The Bergman kernel $\rho_{X,k}$ is independent of the normalization of h. The partial C^0-estimate introduced and proved for smooth Fano surfaces with Kähler–Einstein metrics in [**17**], involves a uniform lower bound for the Bergman kernel $\rho_{X,k}$.

THEOREM 1.1. *There exist $k(n,F) \in \mathbb{Z}^+$ and $\epsilon(n,F) > 0$ such that for any $(X,g) \in \mathcal{KR}(n,F)$, the Bergman kernel $\rho_{X,k}$ of $H^0(X, K_X^{-k})$ is uniformly bounded below by ϵ, i.e.,*

$$\inf_{z\in X} \rho_{X,k}(z) \ge \epsilon. \tag{1.3}$$

The proof of Theorem 1.1 relies on the arguments in [**7**] and [**18, 22**]. A consequence of Theorem 1.1 is the following compactness result, which is obtained by a suitable modification of the argument in [**7**].

THEOREM 1.2. *Any sequence $(X_i, g_i) \in \mathcal{KR}(n,F)$, after passing to a subsequence, converges in the Gromov–Hausdorff topology to a compact metric length space (X_∞, d_∞) satisfying:*

1. *The singular set Σ_{X_∞} of (X_∞, d_∞) is closed and has Hausdorff dimension no greater than $2n-4$;*
2. *The complex structures J_i and the Kähler metrics g_i converge to a smooth complex structure J_∞ and a smooth Kähler metric g_∞ in C^∞ on $X_\infty \setminus \Sigma_{X_\infty}$ satisfying the Kähler–Ricci soliton equation*

$$Ric(g_\infty) = g_\infty + L_{\mathcal{V}_\infty}g_\infty, \tag{1.4}$$

where $\mathcal{V}_\infty$ is a holomorphic vector field on $X_\infty \setminus \Sigma_{X_\infty}$. The upper bound of $\|\mathcal{V}_\infty\|_{L^\infty(X_\infty\setminus\Sigma_{X_\infty}, g_\infty)}$ only depends on n and F;

3. *The metric completion of* $(X_\infty \setminus \Sigma_{X_\infty}, g_\infty)$ *is homeomorphic to* (X_∞, d_∞) *and* J_∞ *extends to a unique global complex structure on* X_∞ *such that* (X_∞, J_∞) *is a projective* $\mathbb{Q}$*-Fano variety with log terminal singularities;*
4. *The smooth Kähler metric* g_∞ *on* $X_\infty \setminus \Sigma_{X_\infty}$ *extends to a global Kähler current on* (X_∞, J_∞) *in* $c_1(X_\infty)$ *with bounded local potentials and the algebraic singular set of* (X_∞, J_∞) *coincides with the analytic singular set* Σ_{X_∞} *of* (X_∞, d_∞).

We remark that the limiting holomorhpic vector field $\mathcal{V}_\infty$ extends globally to X_∞ since X_∞ is normal. The limiting metric g_∞ is bounded below by a multiple of the Fubini–Study metric by applying estimates similar to Schwarz lemma. We can now obtain a compactification of $\mathcal{KR}(n, F)$ in the Gromov–Hausdorff topology.

Definition 1.2. Let $\overline{\mathcal{KR}(n, F)}$ be the closure of $\mathcal{KR}(n, F)$ defined by the set of all Kähler–Ricci solitons (X_∞, g_∞) such that there exists a convergent sequence $(X_i, g_i) \in \mathcal{KR}(n, F)$ with (X_∞, g_∞) being the limit in Theorem 1.2.

Theorem 1.2 also implies certain algebraic boundedness for $\overline{\mathcal{KR}(n, F)}$.

Corollary 1.1. *There exist* $m = m(n, F) \in \mathbb{Z}^+$, $C = C(n, F) > 0$ *and* $\delta = \delta(n, F) > 0$, *such that for any* $X \in \overline{\mathcal{KR}(n, F)}$,

$$-mK_X \text{ is Cartier}, \ [-K_X]^n \le C, \ discr(X) > -1 + \delta. \tag{1.5}$$

Here $discr(X)$ *is the discrepancy of* X, *defined by the equation (6.1) below.*

Finally, we raise two natural questions closely related to the main results.

• Does there exist $F = F(n) > 0$ such that for any Kähler–Ricci soliton $g_{i\bar{j}}$ on an n-dimensional Fano manifold X and V the corresponding holomorphic vector field, the Futaki invariant is uniformly bounded by

$$\mathcal{F}_X(\mathcal{V}) \le F? \tag{1.6}$$

If this holds, the compactness result will hold for all Kähler–Ricci solitons on n-dimensional Fano manifolds.

In general, (1.6) does not hold for the space of Kähler–Ricci solitons on Fano varieties with log terminal singularities. For example, we can consider a weighted projective surface X_m defined by the polytope P_m as the convex hull of three points $(-1, -1)$, $(2/m, -1)$ and $(-1, m+1)$. The discrepancy of X_m is given by $-1 + 2/m$. Hence $discr(X_m)$ tends to -1 and $c_1(X_m)^2$ tends to ∞ as $m \to \infty$. There always exists a smooth orbifold Kähler–Ricci soliton (g_m, V_m) on X_m by [**19**] and the Futaki invariant of X_m tends to ∞ as $m \to \infty$. The compactness for singular Kähler–Ricci solitons on $\mathbb{Q}$-Fano varieties might still hold with bounds such as the Futaki invariant, c_1^n and the discrepancy of the singularities. This seems to suggest that the Futaki invariant for Kähler–Ricci solitons are related to the boundedness problem for Fano varieties in birational geometry.

• For any $(X, g) \in \overline{\mathcal{KR}(n, F)}$, is the Ricci curvature of g uniformly bounded on the regular part of X? This is equivalent to saying that the potential of the holomorphic vector field $\mathcal{V}$ is a quasi-plurisubharmonic function with respect to a multiple of g.

2. Geometric estimates.

Since smooth Fano manifolds with fixed dimension can only have finitely many deformation types [**11,13**], the intersection number $[-K_X]^n$ is uniformly bounded.

LEMMA 2.1. *For any $n > 0$, there exists $c = c(n) > 0$ such that for any Fano manifold X,*

$$c^{-1} \leq c_1^n(X) \leq c. \tag{2.1}$$

We consider Perelman's entropy functional for a Fano manifold (X, g) with the associated Kähler form $\omega_g \in c_1(X)$, which is defined by

$$\mathcal{W}(g, f) = \frac{1}{V} \int_X (R + |\nabla f|^2 + f - n) e^{-f} dV_g, \tag{2.2}$$

where R is the scalar curvature of g and $V = c_1^n(X)$. The μ-functional is defined by

$$\mu(g) = \inf_f \left\{ \mathcal{W}(g, f) \;\middle|\; \frac{1}{V} \int_X e^{-f} dV_g = 1 \right\}. \tag{2.3}$$

For compact gradient shrinking solitons, we have the following well-known identities (cf. [**6**])

$$R + \Delta u = n, \tag{2.4}$$

$$R + |\nabla u|^2 = u + constant. \tag{2.5}$$

In the case of Kähler–Ricci solitons, we have

$$Ric(g) = g - \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} u, \;\; u_{ij} = u_{\overline{i}\overline{j}} = 0.$$

From now on, we always assume the following normalizing condition for u

$$\frac{1}{V} \int_X e^{-u} dV_g = 1, \;\; V = c_1^n(X). \tag{2.6}$$

Integrating (2.4) against e^{-u}, one can determine the constant in (2.5) after an integration by parts,

$$R + |\nabla u|^2 = u - \frac{1}{V}\int_X ue^{-u} dV_g + n. \tag{2.7}$$

The following lemma is due to Tian–Zhang [**18**]. Since the proof is short, we include it here for the convenience of the reader.

LEMMA 2.2. *There exists $A = A(n, F) > 0$ such that for any $(X, g) \in \mathcal{KR}(n, F)$,*

$$\mu(g) \geq -A. \tag{2.8}$$

PROOF. It is well-known that, for solitons, the minimum of the functional $W(g, f)$ is achieved at u. Straightforward calculations using (2.7) show that

$$\mu(g) = W(g, u) = \frac{1}{V}\int_X (R + |\nabla u|^2 + u - n)e^{-u} dV_g = \frac{1}{V}\int_X ue^{-u} dV_g. \tag{2.9}$$

It then suffices to show that $\frac{1}{V}\int_X ue^{-u} dV_g$ is uniformly bounded below. By (2.7),

$$\int_X ue^{-u} dV_g \geq \int_X u dV_g - F$$

or equivalently,

$$\begin{aligned}
\int_X ue^{-u} dV_g &\geq \int_X 2u dV_g - \int_X ue^{-u} dV_g - 2F \\
&= \int_{u\leq -1} u(2 - e^{-u}) dV_g + \int_{u\geq -1} u(2 - e^{-u}) dV_g - 2F \\
&\geq -(2 + \max_{x\geq -1} xe^{-x})V - 2F.
\end{aligned}$$

□

The following lemma is well-known and due to Ivey [**10**].

LEMMA 2.3. *The scalar curvature R is positive for all compact shrinking gradient solitons.*

Then following Perelman's argument (see [**16**]) combined with the above two lemmas, one obtains the following lemma.

PROPOSITION 2.1. *There exists $C = C(n, F) > 0$ such that for all $(X, g) \in \mathcal{KR}(n, F)$,*

$$|u| + |\nabla u|_g^2 + |R(g)| + Diam_g(X) \leq C. \tag{2.10}$$

PROOF. We give a sketch of the proof. The Kähler–Ricci soliton can be considered as a solution of the Kähler–Ricci flow

$$\frac{\partial g(t)}{\partial t} = -Ric(g(t)) + g(t), \; g(0) = g$$

after applying the holomorphic vector field $\mathcal{V}$ (cf. [**15**]). In particular, if we let Φ_t be the automorphisms induced by the real part of $\mathcal{V}$, then

$$g(t) = (\Phi_t)^* g.$$

Let $u(t)$ be the Ricci potential of $g(t)$ defined by

$$Ric(g(t)) = g(t) - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u(t), \; \int_X e^{-u(t)} dV_{g(t)} = V.$$

Then

$$u(t) = (\Phi_t)^* u(0).$$

We note that Perelman's estimates for the Fano Kähler–Ricci flow [**16**] only depend on the dimension, the lower bound of the μ-functional and the upper bound of the volume at the initial time. Since the volume of g is uniformly bounded and $\mu(g(t)) = \mu(g)$ is uniformly bounded below for all $(X, g) \in \mathcal{KR}(n, F)$, following [**16**], $\int_X u(t)e^{-u(t)} dV_{g(t)}$ is uniformly bounded and $u(t)$ is uniformly bounded below. Notice that for any continuous function $h(z, t) = F(u(t), |\nabla u(t)|_{g(t)}, \Delta_{g(t)} u(t))$,

$$\max_{z\in X} h(z, t) = \max_{z\in X} h(z, 0)$$

as $h(\cdot, t) = (\Phi_t)^* h(\cdot, 0)$. Hence using Perelman's argument of the maximum principle, one has uniform bounds for

$$\frac{|\nabla u(t)|_{g(t)}}{u(t) + 1 - \min_z u(t)}, \; \frac{-\Delta_{g(t)} u(t)}{u(t) + 1 - \min_z u(t)}.$$

This will lead to the uniform bound of the diameter of $g(t)$ using the uniform lower bound of $\mu(g(t))$. The proposition then easily follows. □

3. Conformal transformation and analytic compactness.

The following is the idea of Z. Zhang [**22**]. Let $(X, g) \in \mathcal{KR}(n, F)$, we apply a conformal transformation using the Ricci potential u

$$\tilde{g} = e^{-\frac{1}{n-1}u} g. \tag{3.1}$$

Then the uniform bounds on u and on $|\nabla u|_g$ imply that $\tilde{g}$ and g are C^1 equivalent.

The Ricci curvatures of the metrics $\tilde{g}$ and g are related by the well-known equation (see e.g. [**1**], section 6.1)

$$\tilde{R}_{ij} = R_{ij} + \nabla_i \nabla_j u + \frac{1}{2(n-1)} \nabla_i u \nabla_j u - \frac{1}{2(n-1)} (|\nabla u|_g^2 - \Delta u) g_{ij}. \tag{3.2}$$

It follows that from the soliton equation (1.1) and Proposition 2.1 that the Ricci curvature of $\tilde{g}$ is bounded:

LEMMA 3.1. *There exists $C = C(n, F)$ such that for any $(M, g) \in \mathcal{KR}(n, F)$,*

$$-C\tilde{g} \leq Ric(\tilde{g}) \leq C\tilde{g}.$$

With Lemma 3.1, one can apply the general compactness results as in [**2–5**]. The uniform bound of u implies that the diameter of $(X, \tilde{g})$ is uniformly bounded above and the volume of $(X, \tilde{g})$ is uniformly bounded on both sides. In addition, one has the uniform nonlocal collapsing property for $\tilde{g}$. All the constants only depend on n and F. We also have the following volume comparison:

COROLLARY 3.1. *There exist $\kappa = \kappa(n, F) > 0$ such that for any $(X, g) \in \mathcal{KR}(n, F)$,*

$$\kappa^{-1} r^{2n} \leq Vol(B_g(z, r)) \leq \kappa r^{2n}, \tag{3.3}$$

for any $z \in X$ and $r \leq 1$.

Corollary 3.1 also holds for $\tilde{g}$ as g and $\tilde{g}$ differ by a uniformly bounded conformal factor. One can now apply the results of Cheeger–Colding to $\tilde{g}$. With a careful treatment for the tangent cones, one derives the following theorem [**18, 22**], making use of the uniform C^1 equivalence between g and $\tilde{g}$.

THEOREM 3.1. *Let $(X_i, g_i) \in \mathcal{KR}(n, F)$ be a sequence in $\mathcal{KR}(n, F)$ with uniformly bounded volumes. Then after passing to a subsequence if necessary, the sequence (X_i, g_i) converges in the Gromov–Hausdorff sense to a compact metric length space (X_∞, g_∞) satisfying the following:*

1. *The singular set Σ_{X_∞} of X_∞ is of codimension no less than 4;*
2. *On $X_\infty \backslash \Sigma_{X_\infty}$; g_∞ is a smooth Kähler metric satisfying the Kähler–Ricci soliton equation. The metric completion of $(X_\infty \setminus \Sigma_{X_\infty}, g_\infty)$ coincides with (X_∞, g_∞);*
3. *g_i converges to g_∞ in C^∞ topology on $X_\infty \setminus \Sigma_{X_\infty}$.*

The C^∞ convergence on the regular part of X_∞ is achieved by making use of a variant of Perelman's pseudolocality theorem due to [**9**] since the soliton metric is a solution of the Ricci flow. The goal of the rest of the paper is to show that X_∞ is isomorphic to a projective variety equipped with a canonical Kähler–Ricci soliton metric.

4. L^2-estimates.

In this section, we will obtain some uniform L^2-estimates for $H^0(X, K_X^{-k})$ when $X \in \mathcal{KR}(n, F)$. Using the same notations in [**7**], we denote

$$K_X^\sharp = K_X^{-k},\ h^\sharp = h^k,\ \omega^\sharp = k\omega,\ L^{p,\sharp}(X) = L^p(X, \omega^\sharp),$$

where h is the hermitian metric on K_X^{-1} with its curvature $Ric(h) = \omega$. The hermitian metric on K_X^{-1} is equivalent to a volume form on X and since g satisfies the soliton equation, we can normalize h such that

$$h = e^{-u}\omega^n,\ \int_X e^{-u}\omega^n = \int_X \omega^n = c_1(X)^n.$$

We also note that the Bergman kernel $\rho_{X,k}$ is invariant under any scaling for h.

Since the Sobolev constant is uniform for $\tilde{g}$, so it is for g as g and $\tilde{g}$ are uniformly equivalent, when $(X, g) \in \mathcal{KR}(n, F)$. The following proposition, which shows that Proposition 2.1 in [**7**] can be extended to the case of Kähler–Ricci solitions, is one of the key components in the proof of Theorem 1.1:

Proposition 4.1. *There exist $a = a(n, F)$, $K_1 = K_1(n, F)$, $K_2 = K_2(n, F) > 0$ such that if $(X, g) \in \mathcal{KR}(n, F)$ and $s \in H^0(X, K_X^{-k})$ for $k \geq 1$, then*

1. $\|s\|_{L^{\infty,\sharp}} \leq K_1 \|s\|_{L^{2,\sharp}}$*;*
2. $\|\nabla s\|_{L^{\infty,\sharp}} \leq K_2 \|s\|_{L^{2,\sharp}}$*;*
3. *We consider the L^2 inner product for any K_X^{-k}-valued $(0,1)$-form σ defined by*

$$\int_X |\sigma|^2_{h^\sharp, g^\sharp} e^{-u} dV_{g^\sharp}$$

and its induced adjoint operator $\overline{\partial}_u^*$ of $\overline{\partial}$. Then the Beltrami–Laplace operator $\Delta^\sharp_{\overline{\partial},u} = \overline{\partial}\overline{\partial}_u^* + \overline{\partial}_u^*\overline{\partial}$ is invertible with

$$\Delta^\sharp_{\overline{\partial},u} \geq a. \tag{4.1}$$

Proof. The proof proceeds in a similar way as in [**7**].

1. Let (X, g) be any element in $\mathcal{KR}(n, F)$. The bound on the Sobolev constant of (X, g) only depends on n and F, and so does the Sobolev constant for the rescaled metric (X, kg, h^k). For simplicity, we write $|s|$ for $|s|_{h^k}$ and $s \in H^0(X, K_X^{-k})$. The case of $\|s\|_{L^\infty}$ is straightforward, and follows from pointwise estimates of the form

$$\Delta|s|^2 = -n|s|^2 + |\nabla s|^2 \geq -n|s|^2. \tag{4.2}$$

Bounds for $\|s\|_{L^\infty}$ follow by Moser iteration,

$$\|s\|_{L^\infty} \leq C\,\|s\|_{L^4} \leq C\,\|s\|^{\frac{1}{2}}_{L^\infty}\|s\|^{\frac{1}{2}}_{L^2} \tag{4.3}$$

and hence $\|s\|_{L^\infty} \leq C\|s\|_{L^2}$, as desired.

2. We drop the index $\sharp$ for simplicity. The case of $\|\nabla s\|^2_{L^\infty}$ is more delicate, and the soliton equation together with the fact that the potential u is bounded have to be taken into account. This time we find

$$\Delta|\nabla s|^2 = -2|\nabla s|^2 + n|s|^2 + |\nabla\nabla s|^2 + R_{\bar{j}k}\langle\nabla_j s, \nabla_k s\rangle. \tag{4.4}$$

The new term is $R_{\bar{j}k}\langle\nabla_j s, \nabla_k s\rangle$, and it leads to

$$\int_X R_{\bar{j}k}\langle\nabla_j s, \nabla_k s\rangle|\nabla s|^p = \int_X |\nabla s|^{p+2} - \int_X u_{\bar{j}k}\langle\nabla_j s, \nabla_k s\rangle|\nabla s|^p. \tag{4.5}$$

The non-trivial term is the second term on the right-hand side, and we integrate by parts

$$\begin{aligned}-\int_X u_{\bar{j}k}\langle\nabla_j s, \nabla_k s\rangle|\nabla s|^p \quad &= \quad \int_X u_{\bar{j}}\Big(\langle\nabla_k\nabla_j s, \nabla_k s\rangle|\nabla s|^p - n\langle\nabla_j s, s\rangle|\nabla s|^p \\ &\quad -\frac{p}{2}\langle\nabla_j s, \nabla_k s\rangle\nabla_k|\nabla s|^2\,|\nabla s|^{p-2}\Big).\end{aligned}$$

Since $|\nabla u|$ is bounded, we can estimate each of these terms by

$$\int |\nabla\nabla s|\,|\nabla s|^{p+1} \leq \Big(\int |\nabla\nabla s|^2|\nabla s|^p\Big)^{\frac{1}{2}}\Big(\int |\nabla s|^{p+2}\Big)^{\frac{1}{2}},$$

$$\int |s|\,|\nabla s|^{p+1} \leq \Big(\int |s|^2|\nabla s|^p\Big)^{\frac{1}{2}}\Big(\int |\nabla s|^{p+2}\Big)^{\frac{1}{2}}.$$

The terms $|s|^2|\nabla s|^p$ and $|\nabla\nabla s|^2|\nabla s|^p$ on the right hand side of (4.4) can absorb these terms, up to $-C\int|\nabla s|^{p+2}$. Thus we obtain

$$\int |\nabla(|\nabla s|^{\frac{p}{2}+1})|^2 \leq (Cp^2)\int|\nabla s|^{p+2}. \tag{4.6}$$

Moser iteration can now take place as before, giving a bound for $\|\nabla s\|_{L^\infty}$ in terms of $\|\nabla s\|_{L^4}$, and hence in terms of $\|\nabla s\|_{L^2}$, by the same argument as above. Since $\|\nabla s\|^2_{L^2} = n\|s\|^2_{L^2}$, the desired estimate follows.

3. The last inequality follows from the Bochner–Kodaira–Nakano identity, where the weight e^{-u} eliminates the $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u$ in the soliton equation for $Ric(g)$. More precisely, let

$$\langle\sigma,\sigma\rangle = \int_X |\sigma|^2 e^{-u}(\omega^\sharp)^n$$

be the L^2-product for $\sigma \in \Omega^{0,1}\otimes K_X^{-k}$,

$$\Delta_{\bar{\partial},u} = \bar{\partial}\bar{\partial}^*_u + \bar{\partial}^*_u\bar{\partial},\ \ \Delta_{D_u} = -(D_u)^{\bar{j}}(D_u)_{\bar{j}},$$

where (D_u) is the covariant derivative on $\Omega^{(0,1)} \otimes K_X^{-k}$ with respect to the Kähler metric g and the hermitian metric he^{-u}. We have the following Bochner–Kodaira–Nakano identity (cf. [**12**]).

$$\begin{aligned}(\Delta_{\overline{\partial},u}\sigma)_{\bar{j}} &= (\Delta_{D_u}\sigma)_{\bar{j}} + (g)^{i\bar{q}}\left(g_{i\bar{j}} + u_{i\bar{j}} + (Ric(g))_{i\bar{j}}\right)\sigma_{\bar{q}} \\ &= (\Delta_{D_u}\sigma)_{\bar{j}} + \frac{k+1}{k}\sigma_{\bar{j}}.\end{aligned}$$

This immediately implies that

$$\langle \Delta_{\overline{\partial},u}\sigma, \sigma\rangle \geq e^{-\sup u}||\sigma||^2_{L^2}. \tag{4.7}$$

The proof of the proposition is complete. □

5. Partial C^0 estimate.

We now consider a slight modification of the H-property introduced by Donaldson–Sun [**7**].

Definition 5.1. We consider the following data $(p_*, D, U, \Lambda, J, g, h, A)$ satisfying

1. (p_*, U, J, g) is an open bounded Kähler manifold with a complex structure J, a Kähler metric g and a base point $p_* \in U$;
2. $\Lambda \to U$ is a hermitian line bundle equipped with a hermitian metric h. A is the connection induced by the hermitian metric h on Λ, with curvature $\Omega(A) = g$. D is an open disc with $p_* \in D \subset\subset U$.

The data $(p_*, D, U, \Lambda, J, g, h, A)$ is said to have the H'-property if there exist $C > 0$ and a compactly supported smooth section $\sigma : U \to \Lambda$ satisfying

H'_1: $\|\sigma\|_{L^2} < (2\pi)^{n/2}$;
H'_2: $|\sigma(p_*)| > 3/4$;
H'_3: for any holomorphic section τ of Λ over a neighborhood of $\overline{D}$,

$$|\tau(p_*)| \leq C\|\tau\|_{L^2(D)};$$

H'_4: $\|\overline{\partial}\sigma\|_{L^2} < \min\left(\frac{a^{1/2}}{4C}, \frac{(2\pi)^{n/2}}{10\sqrt{2}}\right)$, where $a = a(n, F)$ is the constant in Proposition 4.1;
H'_5: σ is constant in D.

It is straightforward to check that the H'-property is open with respect to C^l variations in (g, J, A) for any $l \geq 0$ with (p_*, D, U, Λ) being fixed.

The standard application of L^2-estimate implies the following lemma (cf. [**7**]).

LEMMA 5.1. *Suppose* $(X,g) \in \mathcal{KR}(n,F)$. *There exists* $b = b(n,F) > 0$ *such that if* $p \in D \subset\subset U \subset X$ *satisfies property* H *with* $\Lambda = K_X^{-k}$ *for some* $k > 0$, *then*

$$\rho_{X,k}(p) > b. \tag{5.1}$$

PROOF. Let σ be a smooth section in the definition of H'-property. We define

$$\tau = \overline{\partial}_u^*(\Delta_{\overline{\partial},u})^{-1}\overline{\partial}\sigma, \quad s = \sigma - \tau.$$

$\overline{\partial}\tau = \overline{\partial}\sigma$ since $\overline{\partial}\Delta_{\overline{\partial},u} = \Delta_{\overline{\partial},u}\overline{\partial}$. Therefore $\overline{\partial}s = 0$ and so $s \in H^0(X, K_X^{-k})$. The L^2 norm of s is bounded by

$$\begin{aligned} ||s||_{L^2} &\leq ||\sigma||_{L^2} + ||\tau||_{L^2} \\ &\leq (2\pi)^{n/2} + a^{-1/2}||\overline{\partial}\sigma||_{L^2} \\ &\leq (2\pi)^{n/2}(1 + (200a)^{-1/2}). \end{aligned}$$

On the other hand, by H_3' and the calculations above,

$$\begin{aligned} |s|(p) &\geq |\sigma|(p) - |\tau|(p) \\ &> 3/4 - C\|\tau|_{L^2(D)} \\ &\geq 3/4 - Ca^{-1/2}\|\overline{\partial}\sigma\|_{L^2} \\ &> 1/2. \end{aligned}$$

The lemma then immediately follows. □

PROOF OF THEOREM 1.1. For any sequence $(X_i, g_i) \in \mathcal{KR}(n,F)$, a subsequence will converge to some (X_∞, g_∞) as in Theorem 3.1. For any $p \in X_\infty$, any tangent cone $C(Y)$ at p is a metric cone whose link Y has a singular set Σ_Y of real codimension at least 4. The cone metric on $C(Y)$ is a smooth Ricci flat Kähler metric $g_{C(Y)}$ on the regular part with

$$g_{C(Y)} = dr^2 + r^2 g_Y = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}r^2/2, \tag{5.2}$$

where $r = |z|$ is the distance from the vertex p to z. With the L^2 estimates proved in Proposition 4.1, the argument of Donaldson–Sun in section 3 of [**7**] can be immediately applied. In particular, there exist $k > 0$ and (p_*, D, U) with $p_* \in D \subset U \subset\subset C(Y)_{reg}$, such that $(p_*, D, U, \Lambda^k, J_{C(Y)}, kg_{C(Y)}, A_{C(Y)}^{\otimes k})$ satisfies Property H' for sufficiently small perturbations of $g_{C(Y)}$ and $J_{C(Y)}$ in $C^0(U)$, where $\Lambda \to U$ is a trivial hermitian line bundle with hermitian metric $h_{C(Y)} = e^{-|z|^2/2}$. □

6. Limiting Kähler–Ricci solitons.

DEFINITION 6.1. Let X be a normal variety with K_X being a $\mathbb{Q}$-Cartier divisor. Let $\pi : \tilde{X} \to X$ be a log resolution of singularities with

$$K_{\tilde{X}} = \pi^* K_X + \sum a_i E_i, \tag{6.1}$$

where $a_i \in \mathbb{Q}$ and E_i are the exceptional divisors of π. X is said to have log terminal singularities if $a_i > -1$, for all i. The discrepancy of X is defined by

$$discr(X) = \inf_{\pi,i}(1, a_i), \tag{6.2}$$

for any log resolution $\pi : \tilde{X} \to X$.

After establishing the partial C^0 estimate in Theorem 1.1, the arguments in sections 4.1, 4.2 and 4.3 of [**7**] can be faithfully applied to show that there exists $k = k(n, F) > 0$ such that any sequence $(X_i, g_i, K_{X_i}^{-k}) \in \mathcal{KR}(n, F)$, after passing to a subsequence, converges to a polarized limit $(X_\infty, g_\infty, K_{X_\infty}^{-k})$ with $X_\infty = Proj(R(X_\infty, K_{X_\infty}^{-k}))$ being a normal projective variety, where $R(X_\infty, K_{X_\infty}^{-k}) = \oplus_m H^0(X_\infty, K_{X_\infty}^{-mk})$. Without loss of generality, we can embed X_i and X_∞ in a fixed $\mathbb{P}^{N_k}$ using the L^2-orthonormal basis $\{s_j^{(i)}\}_{j=0}^{N_k}$ of $H^0(X_i, K_{X_i}^{-k})$ and $\{s_j^{(\infty)}\}_{j=0}^{N_k}$ of $H^0(X_\infty, K_{X_\infty}^{-k})$ respectively. Let $\rho_{X_i,k} = \sum |s_j^{(i)}|^2_{h_i^k}$ and $\rho_{X_\infty,k} = \sum_j |s_j^{(\infty)}|^2_{h_\infty^k}$ be the Bergman kernels.

PROPOSITION 6.1. *X_∞ is a projective $\mathbb{Q}$-Fano variety with log terminal singularities. In particular, the algebraic singular set coincides with the singular set of g_∞.*

PROOF. By Proposition 4.1 and Theorem 1.1, $\log \rho_{X_\infty,k}$ is uniformly bounded. For any point $p \in X_\infty$, there exists a holomorphic section $s \in H^0(X_\infty, K_{X_\infty}^{-k})$ such that s does not vanish on an open U neighborhood with $\inf_U |s| \geq \epsilon$. Then $\Theta_s = (s \wedge \bar{s})^{-1/k}$ is a volume measure and

$$\int_{X_\infty \cap U} (s \wedge \bar{s})^{-1/k} = \int_{X_\infty \cap U} |s|^{-2/k} dV_{g_\infty} \leq (\epsilon)^{-2/k} V. \tag{6.3}$$

Let $\pi : \tilde{X} \to X_\infty$ be a resolution of singularities. Then $\pi^*\Theta_s$ is L^1-integrable on $\tilde{X}$, and since Θ_s can have only algebraic singularities, $\pi^*\Theta_s$ is $L^{1+\epsilon}$-integrable on $\tilde{X}$ for some $\epsilon > 0$. This implies that X_∞ has at worst log terminal singularities. □

Proposition 6.2. *The limiting variety* (X_∞, g_∞) *arising from Proposition 6.1 solves the Kähler–Ricci soliton on* X_∞ *in the following sense.*

1. g_∞ *is a global Kähler current on* X_∞ *with bounded local Kähler potentials.*
2. g_∞ *solves the Kähler–Ricci soliton equation on* X_∞^{reg}

$$Ric(g_\infty) + \nabla^2 u_\infty = g_\infty, \tag{6.4}$$

for some smooth real valued potential function u_∞ *on* X_∞^{reg}.

3. $\|u_\infty\|_{C^1(X_\infty^{reg})} < \infty$, *and thus the holomorphic vector field* $\mathcal{V}_\infty = \uparrow \overline{\partial} u_\infty$ *(i.e.* $(\mathcal{V}_\infty)^i = (g_\infty)^{i\bar{j}}(u_\infty)_{\bar{j}}$*) extends to a global holomorphic vector field on* X_∞ *with* $\|\mathcal{V}_\infty\|_{L^\infty(X_\infty, g_\infty)} < \infty$. *In particular, the Futaki invariant of* (X_∞, g_∞) *can be bounded by* F,

$$\mathcal{F}_{X_\infty}(\mathcal{V}_\infty) = \int_{X_\infty} |\mathcal{V}_\infty|^2 dV_{g_\infty} \le F.$$

Proof. We first prove that the local Kähler potentials of g_∞ are uniformly bounded. Let

$$\omega_{FS,i} = k^{-1}\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\sum_j s_j^{(i)} \wedge \overline{s_j^{(i)}},\ \omega_{FS,\infty} = k^{-1}\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\sum_j s_j^{(\infty)} \wedge \overline{s_j^{(\infty)}}$$

be the Fubini–Study metrics from the embeddings by $\{s_j^{(i)}\}_j$ and $\{s_j^{(\infty)}\}_j$. Let ω_{g_i} be the Kähler form associated to the soliton metric g_i. We define $\rho_{X_i,k}$ to be the Bergman kernel for $K_{X_i}^k$ with respect to g_i defined by

$$\rho_{X_i,k} = \sum_j |s_j^{(i)}|^2_{h_i^k},$$

where $h_i = e^{-u_i}\omega_i^n$ is the hermitian metric on $K_{X_i}^{-1}$. Then

$$\omega_{g_i} = \omega_{FS,i} + k^{-1}\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\rho_{X_i,k},\ \omega_{g_\infty} = \omega_{FS,\infty} + k^{-1}\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\rho_{X_\infty,k}.$$

By Proposition 4.1, $\rho_{X_i,k}$ and $\rho_{X_\infty,k}$ are uniformly bounded in L^∞ for some fixed sufficiently large k. By the partial C^0 estimate, $\rho_{X_i,k}$ and $\rho_{X_\infty,k}$ are uniformly bounded below away from 0. Therefore $\varphi_i = k^{-1}\log\rho_{k,i}$ and $\varphi_\infty = k^{-1}\log\rho_{X_\infty,k}$ are uniformly bounded in L^∞.

Note that the hermitian metrics on $K_{X_i}^{-1}$ and $K_{X_\infty}^{-1}$ are given by $h_i = e^{-u_i}\omega_{g_i}^n$ and $h_\infty = e^{-u_\infty}\omega_{g_\infty}^n$. Since u_i and $|\nabla u_i|_{g_i}$ are uniformly bounded, u_i converges in C^α on X_∞^{reg} to u_∞. From the smooth convergence of g_i to g_∞ on X_∞^{reg}, u_i converges in C^∞ to u_∞ on X_∞^{reg} with $|u_\infty|$ and $|\nabla u_\infty|$ uniformly bounded on X_∞^{reg}. Furthermore, g_∞ satisfies the soliton equation on X_∞^{reg}

$$Ric(g_\infty) = g_\infty - \nabla^2 u_\infty = g_\infty + L_{\mathcal{V}_\infty} g_\infty,$$

where $(\mathcal{V}_\infty)^i = -(g_\infty)^{i\bar{j}}(u_\infty)_{\bar{j}}$ is the holomorphic vector field on X_∞^{reg} induced by u_∞. Since X_∞ is normal, $\mathcal{V}_\infty$ extends to a bounded global holomorphic vector field on X_∞ with $||X_\infty||_{L^\infty(X_\infty, g_\infty)} < \infty$ and $\mathcal{F}_{X_\infty}(\mathcal{V}_\infty) \le F$. In fact, if we let $\Omega_{FS,\infty}$ be the smooth volume form on X_∞ with $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Omega_{FS,\infty} = -\omega_{FS,\infty}$, then φ_∞ satisfies a global Monge–Ampère equation on X_∞

$$(\omega_{FS,\infty} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_\infty)^n = e^{-\varphi_\infty + u_\infty}\Omega_{FS,\infty} \tag{6.5}$$

and on X_∞^{reg} (cf. [**8, 20**]). □

Acknowledgements. The authors would like to thank Xiaowei Wang, Ved Datar and Bin Guo for many valuable discussions.

References

1. Aubin T., *Non-linear analysis on manifolds. Monge–Ampère equations*, Springer-Verlag, New York, 1982.
2. Cheeger J., Colding T. H., *On the structure of spaces with Ricci curvature bounded below. I.*, J. Diff. Geom., **45** (1997), 406–480.
3. Cheeger J., Colding T. H., *On the structure of spaces with Ricci curvature bounded below. II.*, J. Diff. Geom., **54** (2000), 13–35.
4. Cheeger J., Colding T. H., *On the structure of spaces with Ricci curvature bounded below. III.*, J. Diff. Geom., **54** (2000), 37–74.
5. Cheeger J., Colding T. H., Tian G., *On the singularities of spaces with bounded Ricci curvature*, Geom. Funct. Anal., **12** (2002), 873–914.
6. Cao H. D., *Geometry of Ricci solitons*, Chinese Ann. Math. Ser. B, **27** (2006), no. **2**, 121–142.
7. Donaldson S., Sun S., *Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry*, Acta Math., **213** (2014), no. **1**, 63–106.
8. Eyssidieux P., Guedj V., Zeriahi A., *Singular Kähler-Einstein metrics*, J. Amer. Math. Soc., **22** (2009), no. **3**, 607–639.
9. Fang F., Zhang Y., Zhang Z., *Maximum solutions of normalized Ricci flow on 4-manifolds*, Comm. Math. Phys., **283** (2008), no. **1**, 1–24.
10. Ivey T., *Ricci solitons on compact three-manifolds*, Diff. Geom. Appl., **3** (1993), 301–307.
11. Kollar J., Miyaoka Y., Mori S., *Rational connectedness and boundedness of Fano manifolds*, J. Diff. Geom., **36** (1992), 765–779.
12. Morrow J., Kodaira K., *Complex manifolds*, Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, 1971.
13. Nadel A. M. *The boundedness of degree of Fano varieties with Picard number one*, J. Amer. Math. Soc., **4** (1991), no. **4**, 681–692.
14. Perelman G., *The entropy formula for the Ricci flow and its geometric applications*, `arXiv:math.DG/0211159`.
15. Phong D. H., Song J., Sturm J., Weinkove B., *On the convergence of the modified Kähler–Ricci flow and solitons*, Comment. Math. Helv., **86** (2011), no. **1**, 91–112.
16. Sesum N., Tian G., *Bounding scalar curvature and diameter along the Kähler Ricci flow (after Perelman) and some applications*, Jour. Inst. Math. Juss., **7** (2008), 575–587.

17. Tian G., *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math., **101** (1990), no. **1**, 101–172.
18. Tian G., Zhang Z., *Degeneration of Kähler–Ricci solitons*, Int. Math. Res. Not. IMRN, 2012, no. **5**, 957–985.
19. Wang X., Zhu X., *Kähler–Ricci solitons on toric manifolds with positive first Chern class*, Adv. Math., **188** (2004), no. **1**, 87–103.
20. Yau S.-T., *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I*, Comm. Pure Appl. Math., **31** (1978), 339–411.
21. Yau S.-T., *Open problems in geometry*, Proc. Symposia Pure Math., **54** (1993), 1–28.
22. Zhang Z., *Degeneration of shrinking Ricci solitons*, Int. Math. Res. Not. IMRN, 2010, no. **21**, 4137–4158.

Received June 16, 2015

Department of Mathematics
Columbia University
New York, NY 10027
e-mail: phong@math.columbia.edu

Department of Mathematics
Rutgers University
Piscataway, NJ 08854
e-mail: jiansong@math.rutgers.edu

Department of Mathematics
Rutgers University
Newark, NJ 07102
e-mail: sturm@andromeda.rutgers.edu