DEGENERATION OF KÄHLER–RICCI SOLITONS ON FANO MANIFOLDS

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Abstract. We consider the space $KR(n, F)$ of Kähler–Ricci solitons on $n$-dimensional Fano manifolds with Futaki invariant bounded by $F$. We prove a partial $C^0$ estimate for $KR(n, F)$ as a generalization of the recent work of Donaldson-Sun for Fano Kähler–Einstein manifolds. In particular, any sequence in $KR(n, F)$ has a convergent subsequence in the Gromov-Hausdorff topology to a Kähler–Ricci soliton on a Fano variety with log terminal singularities.

1. Introduction.

Let $X$ be a Fano manifold admitting a smooth Kähler–Ricci soliton, that is a metric $g_{ij}$ satisfying the equation

$$Ric(g) = g + L_V g,$$

where $V$ is a holomorphic vector field, and $L_V$ is the Lie derivative along $V$. The holomorphic vector field can be expressed in terms of the Ricci potential $u$, with

$$R_{ij} = g_{ij} - u_{ij}, \quad u_{ij} = u_{ji} = 0, \quad V^i = -g^{i\bar{j}} u_{\bar{j}}.$$

The Futaki invariant associated to $(X, g, V)$ is given by

$$F_X(V) = \int_X |\nabla u|^2 dV_g = \int_X |V|^2 dV_g \geq 0.$$
Definition 1.1. Let $\mathcal{KR}(n, F)$ be the set of Kähler–Ricci solitons $(X, g)$ with
dim X = n, \( \text{Ric}(g) = g + L \gamma g, \mathcal{F}_X(\mathcal{V}) \leq F. \)

The main result of this paper is a partial $C^0$ estimate for Kähler–Ricci solitons. Let $(X, g) \in \mathcal{KR}(n, F)$ and $\omega_g$ be the Kähler form for $g$. Let $h$ be a hermitian metric on $K_X^{-1}$ with $\text{Ric}(h) = \omega_g$, which is unique up to a multiplicative normalization. We define the $L^2$-inner product on $H^0(X, K_X^{-k})$ by

$$\langle s, s' \rangle = k^n \int_X s \overline{s'} h^k \omega^n_g$$

for any $s, s' \in H^0(X, K_X^{-k})$. Let $\{s_j\}_{j=1}^{N_k}$ be an orthonormal basis in $H^0(X, K_X^{-k})$ with respect to $\langle \cdot, \cdot \rangle$. Then the Bergman kernel $\rho_{X,k}$ is defined to be

$$\rho_{X,k} = \sum_j |s_j|^2 h^k.$$  

The Bergman kernel $\rho_{X,k}$ is independent of the normalization of $h$. The partial $C^0$-estimate introduced and proved for smooth Fano surfaces with Kähler–Einstein metrics in [17], involves a uniform lower bound for the Bergman kernel $\rho_{X,k}$.

**Theorem 1.1.** There exist $k(n, F) \in \mathbb{Z}^+$ and $\epsilon(n, F) > 0$ such that for any $(X, g) \in \mathcal{KR}(n, F)$, the Bergman kernel $\rho_{X,k}$ of $H^0(X, K_X^{-k})$ is uniformly bounded below by $\epsilon$, i.e.,

$$\inf_{z \in X} \rho_{X,k}(z) \geq \epsilon.$$  

The proof of Theorem 1.1 relies on the arguments in [7] and [18, 22]. A consequence of Theorem 1.1 is the following compactness result, which is obtained by a suitable modification of the argument in [7].

**Theorem 1.2.** Any sequence $(X_i, g_i) \in \mathcal{KR}(n, F)$, after passing to a subsequence, converges in the Gromov–Hausdorff topology to a compact metric length space $(X_\infty, d_\infty)$ satisfying:

1. The singular set $\Sigma_{X_\infty}$ of $(X_\infty, d_\infty)$ is closed and has Hausdorff dimension no greater than $2n - 4$;
2. The complex structures $J_i$ and the Kähler metrics $g_i$ converge to a smooth complex structure $J_\infty$ and a smooth Kähler metric $g_\infty$ in $C^\infty$ on $X_\infty \setminus \Sigma_{X_\infty}$ satisfying the Kähler–Ricci soliton equation

$$\text{Ric}(g_\infty) = g_\infty + L \mathcal{V}_\infty g_\infty,$$

where $\mathcal{V}_\infty$ is a holomorphic vector field on $X_\infty \setminus \Sigma_{X_\infty}$. The upper bound of $\|\mathcal{V}_\infty\|_{L^\infty(X_\infty \setminus \Sigma_{X_\infty}, g_\infty)}$ only depends on $n$ and $F$;
3. The metric completion of \((X_\infty \setminus \Sigma_{X_\infty}, g_\infty)\) is homeomorphic to \((X_\infty, d_\infty)\) and \(J_\infty\) extends to a unique global complex structure on \(X_\infty\) such that \((X_\infty, J_\infty)\) is a projective \(\mathbb{Q}\)-Fano variety with log terminal singularities.

4. The smooth Kähler metric \(g_\infty\) on \(X_\infty \setminus \Sigma_{X_\infty}\) extends to a global Kähler current on \((X_\infty, J_\infty)\) in \(c_1(X_\infty)\) with bounded local potentials and the algebraic singular set of \((X_\infty, J_\infty)\) coincides with the analytic singular set \(\Sigma_{X_\infty}\) of \((X_\infty, d_\infty)\).

We remark that the limiting holomorphic vector field \(V_\infty\) extends globally to \(X_\infty\) since \(X_\infty\) is normal. The limiting metric \(g_\infty\) is bounded below by a multiple of the Fubini–Study metric by applying estimates similar to Schwarz lemma. We can now obtain a compactification of \(\overline{KR}(n, F)\) in the Gromov–Hausdorff topology.

**Definition 1.2.** Let \(\overline{KR}(n, F)\) be the closure of \(KR(n, F)\) defined by the set of all Kähler–Ricci solitons \((X_\infty, g_\infty)\) such that there exists a convergent sequence \((X_i, g_i) \in KR(n, F)\) with \((X_\infty, g_\infty)\) being the limit in Theorem 1.2.

Theorem 1.2 also implies certain algebraic boundedness for \(\overline{KR}(n, F)\).

**Corollary 1.1.** There exist \(m = m(n, F) \in \mathbb{Z}^+, C = C(n, F) > 0\) and \(\delta = \delta(n, F) > 0\), such that for any \(X \in \overline{KR}(n, F)\),

\[
- mK_X \text{ is Cartier, } \left[ -K_X \right]^n \leq C, \, \text{discr}(X) > -1 + \delta.
\]

Here \(\text{discr}(X)\) is the discrepancy of \(X\), defined by the equation \((6.1)\) below.

Finally, we raise two natural questions closely related to the main results.

- Does there exist \(F = F(n) > 0\) such that for any Kähler–Ricci soliton \(g_{ij}\) on an \(n\)-dimensional Fano manifold \(X\) and \(V\) the corresponding holomorphic vector field, the Futaki invariant is uniformly bounded by

\[
\mathcal{F}_X(V) \leq F? \tag{1.6}
\]

If this holds, the compactness result will hold for all Kähler–Ricci solitons on \(n\)-dimensional Fano manifolds.

In general, \((1.6)\) does not hold for the space of Kähler–Ricci solitons on Fano varieties with log terminal singularities. For example, we can consider a weighted projective surface \(X_m\) defined by the polytope \(P_m\) as the convex hull of three points \((-1, -1), (2/m, -1)\) and \((-1, m + 1)\). The discrepancy of \(X_m\) is given by \(-1 + 2/m\). Hence \(\text{discr}(X_m)\) tends to \(-1\) and \(c_1(X_m)^2\) tends to \(\infty\) as \(m \to \infty\). There always exists a smooth orbifold Kähler–Ricci soliton \((g_m, V_m)\) on \(X_m\) by \([19]\) and the Futaki invariant of \(X_m\) tends to \(\infty\) as \(m \to \infty\).
singularities. This seems to suggest that the Futaki invariant for Kähler–Ricci solitons are related to the boundedness problem for Fano varieties in birational geometry.

• For any \((X, g) \in \overline{KR}(n, F)\), is the Ricci curvature of \(g\) uniformly bounded on the regular part of \(X\)? This is equivalent to saying that the potential of the holomorphic vector field \(V\) is a quasi-plurisubharmonic function with respect to a multiple of \(g\).

2. Geometric estimates.
   Since smooth Fano manifolds with fixed dimension can only have finitely many deformation types \([11, 13]\), the intersection number \([-K_X]^n\) is uniformly bounded.

   Lemma 2.1. For any \(n > 0\), there exists \(c = c(n) > 0\) such that for any Fano manifold \(X\),

   \[
   c^{-1} \leq c_1^n(X) \leq c.
   \]  

   We consider Perelman’s entropy functional for a Fano manifold \((X, g)\) with the associated Kähler form \(\omega_g \in c_1(X)\), which is defined by

   \[
   W(g, f) = \frac{1}{V} \int_X (R + |\nabla f|^2 + f - n)e^{-f}dV_g,
   \]

   where \(R\) is the scalar curvature of \(g\) and \(V = c_1^n(X)\). The \(\mu\)-functional is defined by

   \[
   \mu(g) = \inf_f \left\{ W(g, f) \left| \frac{1}{V} \int_X e^{-f}dV_g = 1 \right. \right\}.
   \]

   For compact gradient shrinking solitons, we have the following well-known identities (cf. [6])

   \[
   R + \Delta u = n, \tag{2.4}
   \]

   \[
   R + |\nabla u|^2 = u + constant. \tag{2.5}
   \]

   In the case of Kähler–Ricci solitons, we have

   \[
   \text{Ric}(g) = g - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u, \quad u_{ij} = u_{\bar{i}\bar{j}} = 0.
   \]

   From now on, we always assume the following normalizing condition for \(u\)

   \[
   \frac{1}{V} \int_X e^{-u}dV_g = 1, \quad V = c_1^n(X). \tag{2.6}
   \]
Integrating (2.4) against $e^{-u}$, one can determine the constant in (2.5) after an integration by parts,

$$R + |\nabla u|^2 = u - \frac{1}{V} \int_X ue^{-u}dV_g + n.$$  \hfill (2.7)

The following lemma is due to Tian–Zhang \textsuperscript{18}. Since the proof is short, we include it here for the convenience of the reader.

**Lemma 2.2.** There exists $A = A(n,F) > 0$ such that for any $(X,g) \in K\mathcal{R}(n,F)$,

$$\mu(g) \geq -A.$$  \hfill (2.8)

**Proof.** It is well-known that, for solitons, the minimum of the functional $W(g,f)$ is achieved at $u$. Straightforward calculations using (2.7) show that

$$\mu(g) = W(g,u) = \frac{1}{V} \int_X (R + |\nabla u|^2 + u - n)e^{-u}dV_g = \frac{1}{V} \int_X ue^{-u}dV_g.$$  \hfill (2.9)

It then suffices to show that $\frac{1}{V} \int_X ue^{-u}dV_g$ is uniformly bounded below. By (2.7),

$$\int_X ue^{-u}dV_g \geq \int_X udV_g - F$$

or equivalently,

$$\int_X ue^{-u}dV_g \geq \int_X 2udV_g - \int_X ue^{-u}dV_g - 2F$$

$$= \int_{u \leq -1} u(2 - e^{-u})dV_g + \int_{u \geq -1} u(2 - e^{-u})dV_g - 2F$$

$$\geq -(2 + \max_{x \geq -1} xe^{-x})V - 2F.$$  \hfill \Box

The following lemma is well-known and due to Ivey \textsuperscript{10}.

**Lemma 2.3.** The scalar curvature $R$ is positive for all compact shrinking gradient solitons.

Then following Perelman’s argument (see \textsuperscript{16}) combined with the above two lemmas, one obtains the following lemma.

**Proposition 2.1.** There exists $C = C(n,F) > 0$ such that for all $(X,g) \in K\mathcal{R}(n,F)$,

$$|u| + |\nabla u|^2_g + |R(g)| + Diam_g(X) \leq C.$$  \hfill (2.10)
Proof. We give a sketch of the proof. The Kähler–Ricci soliton can be considered as a solution of the Kähler–Ricci flow
\[
\frac{\partial g(t)}{\partial t} = -\text{Ric}(g(t)) + g(t), \quad g(0) = g
\]
after applying the holomorphic vector field $\mathcal{V}$ (cf. [15]). In particular, if we let $\Phi_t$ be the automorphisms induced by the real part of $\mathcal{V}$, then
\[
g(t) = (\Phi_t)^*g.
\]
Let $u(t)$ be the Ricci potential of $g(t)$ defined by
\[
\text{Ric}(g(t)) = g(t) - \frac{1}{2\pi} \partial \bar{\partial} u(t), \quad \int_X e^{-u(t)} dV_{g(t)} = V.
\]
Then
\[
u(t) = (\Phi_t)^*u(0).
\]
We note that Perelman’s estimates for the Fano Kähler–Ricci flow [16] only depend on the dimension, the lower bound of the $\mu$-functional and the upper bound of the volume at the initial time. Since the volume of $g$ is uniformly bounded and $\mu(g(t)) = \mu(g)$ is uniformly bounded below for all $(X, g) \in \mathcal{K}\mathcal{R}(n, F)$, following [16], $\int_X u(t)e^{-u(t)}dV_{g(t)}$ is uniformly bounded and $u(t)$ is uniformly bounded below. Notice that for any continuous function $h(z, t) = F(u(t), |\nabla u(t)|_{g(t)}, \Delta_{g(t)}u(t))$,
\[
\max_{z \in X} h(z, t) = \max_{z \in X} h(z, 0)
\]
as $h(\cdot, t) = (\Phi_t)^*h(\cdot, 0)$. Hence using Perelman’s argument of the maximum principle, one has uniform bounds for
\[
\frac{|\nabla u(t)|_{g(t)}}{u(t) + 1 - \min_z u(t)}, \quad \frac{-\Delta_{g(t)}u(t)}{u(t) + 1 - \min_z u(t)}.
\]
This will lead to the uniform bound of the diameter of $g(t)$ using the uniform lower bound of $\mu(g(t))$. The proposition then easily follows. \qed

3. Conformal transformation and analytic compactness.

The following is the idea of Z. Zhang [22]. Let $(X, g) \in \mathcal{K}\mathcal{R}(n, F)$, we apply a conformal transformation using the Ricci potential $u$
\[
\tilde{g} = e^{-\frac{1}{n-1}u}g.
\]
Then the uniform bounds on $u$ and on $|\nabla u|_g$ imply that $\tilde{g}$ and $g$ are $C^1$ equivalent.
The Ricci curvatures of the metrics $\tilde{g}$ and $g$ are related by the well-known equation (see e.g. [1], section 6.1)

$$R_{ij} = R_{ij} + \nabla_i \nabla_j u + \frac{1}{2(n-1)} \nabla_i u \nabla_j u - \frac{1}{2(n-1)} (|\nabla u|^2 - \Delta u) g_{ij}.$$  

It follows that from the soliton equation (1.1) and Proposition 2.1 that the Ricci curvature of $\tilde{g}$ is bounded:

**Lemma 3.1.** There exists $C = C(n,F)$ such that for any $(M,g) \in \mathcal{KR}(n,F)$,

$$-C \tilde{g} \leq \text{Ric}(\tilde{g}) \leq C \tilde{g}.$$  

With Lemma 3.1 one can apply the general compactness results as in [2–5]. The uniform bound of $u$ implies that the diameter of $(X,\tilde{g})$ is uniformly bounded above and the volume of $(X,\tilde{g})$ is uniformly bounded on both sides. In addition, one has the uniform nonlocal collapsing property for $\tilde{g}$. All the constants only depend on $n$ and $F$. We also have the following volume comparison:

**Corollary 3.1.** There exist $\kappa = \kappa(n,F) > 0$ such that for any $(X,g) \in \mathcal{KR}(n,F)$,

$$\kappa^{-1} r^{2n} \leq \text{Vol}(B_g(z,r)) \leq \kappa r^{2n},$$

for any $z \in X$ and $r \leq 1$.

Corollary 3.1 also holds for $\tilde{g}$ as $g$ and $\tilde{g}$ differ by a uniformly bounded conformal factor. One can now apply the results of Cheeger–Colding to $\tilde{g}$. With a careful treatment for the tangent cones, one derives the following theorem [18,22], making use of the uniform $C^1$ equivalence between $g$ and $\tilde{g}$.

**Theorem 3.1.** Let $(X_i, g_i) \in \mathcal{KR}(n,F)$ be a sequence in $\mathcal{KR}(n,F)$ with uniformly bounded volumes. Then after passing to a subsequence if necessary, the sequence $(X_i, g_i)$ converges in the Gromov–Hausdorff sense to a compact metric length space $(X_\infty, g_\infty)$ satisfying the following:

1. The singular set $\Sigma_{X_\infty}$ of $X_\infty$ is of codimension no less than 4;
2. On $X_\infty \setminus \Sigma_{X_\infty}$; $g_\infty$ is a smooth Kähler metric satisfying the Kähler–Ricci soliton equation. The metric completion of $(X_\infty \setminus \Sigma_{X_\infty}, g_\infty)$ coincides with $(X_\infty, g_\infty)$;
3. $g_i$ converges to $g_\infty$ in $C^\infty$ topology on $X_\infty \setminus \Sigma_{X_\infty}$.

The $C^\infty$ convergence on the regular part of $X_\infty$ is achieved by making use of a variant of Perelman’s pseudolocality theorem due to [9] since the soliton metric is a solution of the Ricci flow. The goal of the rest of the paper is to show that $X_\infty$ is isomorphic to a projective variety equipped with a canonical Kähler–Ricci soliton metric.
4. $L^2$-estimates.

In this section, we will obtain some uniform $L^2$-estimates for $H^0(X, K_X^{-k})$ when $X \in \mathcal{KR}(n, F)$. Using the same notations in [7], we denote

$$K_X, h^k = h^k, \omega^k = k\omega, L^p(X) = L^p(X, \omega^k),$$

where $h$ is the hermitian metric on $K_X^1$ with its curvature $Ric(h) = \omega$. The hermitian metric on $K_X^1$ is equivalent to a volume form on $X$ and since $g$ satisfies the soliton equation, we can normalize $h$ such that

$$h = e^{-u}\omega^n, \int_X e^{-u}\omega^n = \int_X \omega^n = c_1(X)^n.$$ We also note that the Bergman kernel $\rho_{X,k}$ is invariant under any scaling for $h$.

Since the Sobolev constant is uniform for $\tilde{g}$, so it is for $g$ as $g$ and $\tilde{g}$ are uniformly equivalent, when $(X, g) \in \mathcal{KR}(n, F)$. The following proposition, which shows that Proposition 2.1 in [7] can be extended to the case of Kähler–Ricci solitons, is one of the key components in the proof of Theorem 1.1:

**Proposition 4.1.** There exist $a = a(n, F), K_1 = K_1(n, F), K_2 = K_2(n, F) > 0$ such that if $(X, g) \in \mathcal{KR}(n, F)$ and $s \in H^0(X, K_X^{-k})$ for $k \geq 1$, then

1. $\|s\|_{L^\infty} \leq K_1\|s\|_{L^2}$;
2. $\|\nabla s\|_{L^\infty} \leq K_2\|s\|_{L^2}$;
3. We consider the $L^2$ inner product for any $K_X^{-k}$-valued $(0,1)$-form $\sigma$ defined by

$$\int_X |\sigma|_{h^k, g}\omega^k e^{-u}dV_g$$

and its induced adjoint operator $\overline{\partial}^* u$ of $\overline{\partial}$. Then the Beltrami–Laplace operator $\Delta^2_{\overline{\partial}, u} = (\overline{\partial}^* u + \overline{\partial} u) \overline{\partial}$ is invertible with

$$\|s\|_{L^\infty} \leq C\|s\|_{L^4} \leq C\|s\|_{L^2}^{1/2} \|s\|_{L^4}^{1/2}$$

and hence $\|s\|_{L^\infty} \leq C\|s\|_{L^2}$, as desired.

**Proof.** The proof proceeds in a similar way as in [7].

1. Let $(X, g)$ be any element in $\mathcal{KR}(n, F)$. The bound on the Sobolev constant of $(X, g)$ only depends on $n$ and $F$, and so does the Sobolev constant for the rescaled metric $(X, kg, h^k)$. For simplicity, we write $|s|$ for $|s|_{h^k}$ and $s \in H^0(X, K_X^{-k})$. The case of $\|s\|_{L^\infty}$ is straightforward, and follows from pointwise estimates of the form

$$\Delta |s|^2 = -n|s|^2 + |\nabla s|^2 \geq -n|s|^2.$$ Bound for $\|s\|_{L^\infty}$ follow by Moser iteration,

$$\|s\|_{L^\infty} \leq C\|s\|_{L^4} \leq C\|s\|_{L^2}^{1/2} \|s\|_{L^4}^{1/2}$$

and hence $\|s\|_{L^\infty} \leq C\|s\|_{L^2}$, as desired.
2. We drop the index ♯ for simplicity. The case of $\|\nabla s\|_{L^\infty}^2$ is more delicate, and the soliton equation together with the fact that the potential $u$ is bounded have to be taken into account. This time we find

\begin{equation}
\Delta |\nabla s|^2 = -2|\nabla s|^2 + n|s|^2 + |\nabla \nabla s|^2 + R_{jk}(\nabla j s, \nabla k s).
\end{equation}

The new term is $R_{jk}(\nabla j s, \nabla k s)$, and it leads to

\begin{equation}
\int_X R_{jk}(\nabla j s, \nabla k s)|\nabla s|^p = \int_X |\nabla s|^{p+2} - \int_X u_{jk}(\nabla j s, \nabla k s)|\nabla s|^p.
\end{equation}

The non-trivial term is the second term on the right-hand side, and we integrate by parts

\begin{equation}
-\int_X u_{jk}(\nabla j s, \nabla k s)|\nabla s|^p = \int_X u_j \left( \langle \nabla k \nabla j s, \nabla k s \rangle |\nabla s|^p - n \langle \nabla j s, s \rangle |\nabla s|^p \right) - \frac{p}{2} \langle \nabla j s, \nabla k s \rangle \nabla k |\nabla s|^2 |\nabla s|^{p-2}.
\end{equation}

Since $|\nabla u|$ is bounded, we can estimate each of these terms by

\begin{align*}
\int |\nabla \nabla s| |\nabla s|^{p+1} &\leq \left( \int |\nabla \nabla s|^2 |\nabla s|^p \right)^{\frac{1}{2}} \left( \int |\nabla s|^{p+2} \right)^{\frac{1}{2}}, \\
\int |s| |\nabla s|^{p+1} &\leq \left( \int |s|^2 |\nabla s|^p \right)^{\frac{1}{2}} \left( \int |\nabla s|^{p+2} \right)^{\frac{1}{2}}.
\end{align*}

The terms $|s|^2 |\nabla s|^p$ and $|\nabla \nabla s|^2 |\nabla s|^p$ on the right hand side of (4.4) can absorb these terms, up to $-C \int |\nabla s|^{p+2}$. Thus we obtain

\begin{equation}
\int |\nabla (|\nabla s|^\frac{p}{2+1})|^{2} \leq (Cp^2) \int |\nabla s|^{p+2}.
\end{equation}

Moser iteration can now take place as before, giving a bound for $|\nabla s|_{L^\infty}$ in terms of $|\nabla s|_{L^4}$, and hence in terms of $|\nabla s|_{L^2}$, by the same argument as above. Since $\|\nabla s\|_{L^2}^2 = n \|s\|_{L^2}^2$, the desired estimate follows.

3. The last inequality follows from the Bochner–Kodaira–Nakano identity, where the weight $e^{-u}$ eliminates the $\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u$ in the soliton equation for $Ric(g)$. More precisely, let

\begin{equation}
\langle \sigma, \sigma \rangle = \int_X |\sigma|^2 e^{-u}(\omega^3)^n
\end{equation}

be the $L^2$-product for $\sigma \in \Omega^{0,1} \otimes K^{-k}$,

\begin{align*}
\Delta \sigma, u &= \partial \bar{\partial} \sigma, u + \bar{\partial} \sigma \bar{\partial} u, \\
\Delta D, u &= -(D, u)^j (D, u)_{\bar{j}},
\end{align*}
where \((D_u)\) is the covariant derivative on \(\Omega^{(0,1)} \otimes K^{-k}_X\) with respect to the Kähler metric \(g\) and the hermitian metric \(he^{-u}\). We have the following Bochner–Kodaira–Nakano identity (cf. [12]).

\[
(\Delta_{\bar{\partial},u}\sigma)_j = (\Delta_{D,u}\sigma)_j + (g^{i\bar{q}} (g_{i\bar{j}} + u_{ij}) \sigma_{\bar{q}} = (\Delta_{D,u}\sigma)_j + \frac{k+1}{k} \sigma_{\bar{j}}.
\]

This immediately implies that

\[
\langle \Delta_{\bar{\partial},u}\sigma, \sigma \rangle \geq e^{-\sup u} \|\sigma\|_{L^2}^2.
\]

The proof of the proposition is complete.

5. Partial \(C^0\) estimate.

We now consider a slight modification of the \(H\)-property introduced by Donaldson–Sun [7].

**Definition 5.1.** We consider the following data \((p_*, D, U, \Lambda, J, g, h, A)\) satisfying

1. \((p_*, U, J, g)\) is an open bounded Kähler manifold with a complex structure \(J\), a Kähler metric \(g\) and a base point \(p_* \in U\);
2. \(\Lambda \to U\) is a hermitian line bundle equipped with a hermitian metric \(h\). \(A\) is the connection induced by the hermitian metric \(h\) on \(\Lambda\), with curvature \(\Omega(A) = g\). \(D\) is an open disc with \(p_* \in D \subset U\).

The data \((p_*, D, U, \Lambda, J, g, h, A)\) is said to have the \(H'\)-property if there exist \(C > 0\) and a compactly supported smooth section \(\sigma : U \to \Lambda\) satisfying

- \(H'_1\): \(\|\sigma\|_{L^2} < (2\pi)^{n/2}\);
- \(H'_2\): \(\|\sigma(p_*)\| > 3/4\);
- \(H'_3\): for any holomorphic section \(\tau\) of \(\Lambda\) over a neighborhood of \(\overline{D}\),

\[
|\tau(p_*)| \leq C\|\tau\|_{L^2(D)};
\]

- \(H'_4\): \(\|\bar{\partial}\sigma\|_{L^2} < \min \left( \frac{a^{1/2}}{4C}, \frac{(2\pi)^{n/2}}{10\sqrt{2}} \right)\), where \(a = a(n, F)\) is the constant in Proposition 4.1;
- \(H'_5\): \(\sigma\) is constant in \(D\).

It is straightforward to check that the \(H'\)-property is open with respect to \(C^l\) variations in \((g, J, A)\) for any \(l \geq 0\) with \((p_*, D, U, \Lambda)\) being fixed.

The standard application of \(L^2\)-estimate implies the following lemma (cf. [7]).
Lemma 5.1. Suppose \((X, g) \in \mathcal{KR}(n, F)\). There exists \(b = b(n, F) > 0\) such that if \(p \in D \subset \subset U \subset X\) satisfies property \(H\) with \(\Lambda = K_X^{-k}\) for some \(k > 0\), then

\[
\rho_{X,k}(p) > b.
\]

Proof. Let \(\sigma\) be a smooth section in the definition of \(H'\)-property. We define

\[
\tau = \overline{\partial}_u (\Delta_{\overline{\partial}_u})^{-1} \overline{\partial} \sigma, \quad s = \sigma - \tau.
\]

\(\overline{\partial}_u\) is smooth since \(\overline{\partial}_u \Delta_{\overline{\partial}_u} = \Delta_{\overline{\partial}_u} \overline{\partial}_u\). Therefore \(\overline{\partial}s = 0\) and so \(s \in H^0(X, K_X^{-k})\). The \(L^2\) norm of \(s\) is bounded by

\[
||s||_{L^2} \leq ||\sigma||_{L^2} + ||\tau||_{L^2} \\
\leq (2\pi)^{n/2} + a^{-1/2} ||\overline{\partial} \sigma||_{L^2} \\
\leq (2\pi)^{n/2} (1 + (200a)^{-1/2}).
\]

On the other hand, by \(H'\) and the calculations above,

\[
|s|(p) \geq |\sigma|(p) - |\tau|(p) \\
> 3/4 - C\|\tau\|_{L^2(D)} \\
\geq 3/4 - C a^{-1/2} ||\overline{\partial} \sigma||_{L^2} \\
> 1/2.
\]

The lemma then immediately follows.

Proof of Theorem 1. For any sequence \((X_i, g_i) \in \mathcal{KR}(n, F)\), a subsequence will converge to some \((X_\infty, g_\infty)\) as in Theorem 3.1. For any \(p \in X_\infty\), any tangent cone \(C(Y)\) at \(p\) is a metric cone whose link \(Y\) has a singular set \(\Sigma_Y\) of real codimension at least 4. The cone metric on \(C(Y)\) is a smooth Ricci flat Kähler metric \(g_{C(Y)}\) on the regular part with

\[
g_{C(Y)} = dr^2 + r^2 g_Y = \frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} r^2 / 2,
\]

where \(r = |z|\) is the distance from the vertex \(p\) to \(z\). With the \(L^2\) estimates proved in Proposition 4.1, the argument of Donaldson–Sun in section 3 of [7] can be immediately applied. In particular, there exist \(k > 0\) and \((p_*, D, U)\) with \(p_* \in D \subset \subset U \subset \subset C(Y)_{reg}\) such that \((p_*, D, U, K, J_{C(Y)}, k g_{C(Y)}, A_{C(Y)}^{\otimes k})\) satisfies Property \(H'\) for sufficiently small perturbations of \(g_{C(Y)}\) and \(J_{C(Y)}\) in \(C^0(U)\), where \(\Lambda \to U\) is a trivial hermitian line bundle with hermitian metric \(h_{C(Y)} = e^{-|z|^2/2}\). □

Definition 6.1. Let $X$ be a normal variety with $K_X$ being a $\mathbb{Q}$-Cartier divisor. Let $\pi : \tilde{X} \to X$ be a log resolution of singularities with

$$K_{\tilde{X}} = \pi^* K_X + \sum a_i E_i,$$

where $a_i \in \mathbb{Q}$ and $E_i$ are the exceptional divisors of $\pi$. $X$ is said to have log terminal singularities if $a_i > -1$, for all $i$. The discrepancy of $X$ is defined by

$$\text{discr}(X) = \inf_{\pi, i} (1, a_i),$$

for any log resolution $\pi : \tilde{X} \to X$.

After establishing the partial $C^0$ estimate in Theorem 1.1, the arguments in sections 4.1, 4.2 and 4.3 of [7] can be faithfully applied to show that there exists $k = k(n, F) > 0$ such that any sequence $(X_i, g_i, K_{X_i}^{-k}) \in \text{KR}(n, F)$, after passing to a subsequence, converges to a polarized limit $(X_\infty, g_\infty, K_{X_\infty}^{-k})$ with $X_\infty = \text{Proj}(R(X_\infty, K_{X_\infty}^{-k}))$ being a normal projective variety, where $R(X_\infty, K_{X_\infty}^{-k}) = \bigoplus_m H^0(X_\infty, K_{X_\infty}^{-mk})$. Without loss of generality, we can embed $X_i$ and $X_\infty$ in a fixed $\mathbb{P}^N_k$ using the $L^2$-orthonormal basis $\{s^{(i)}_j\}_{j=0}^{N_k}$ of $H^0(X_i, K_{X_i}^{-k})$ and $\{s^{(\infty)}_j\}_{j=0}^{N_k}$ of $H^0(X_\infty, K_{X_\infty}^{-k})$ respectively. Let $\rho_{X_i, k} = \sum |s^{(i)}_j|^2_{h^k_i}$ and $\rho_{X_\infty, k} = \sum |s^{(\infty)}_j|^2_{h^k_\infty}$ be the Bergman kernels.

Proposition 6.1. $X_\infty$ is a projective $\mathbb{Q}$-Fano variety with log terminal singularities. In particular, the algebraic singular set coincides with the singular set of $g_\infty$.

Proof. By Proposition 4.1 and Theorem 1.1 log $\rho_{X_\infty, k}$ is uniformly bounded. For any point $p \in X_\infty$, there exists a holomorphic section $s \in H^0(X_\infty, K_{X_\infty}^{-k})$ such that $s$ does not vanish on an open $U$ neighborhood with $\inf_U |s| \geq \epsilon$. Then $\Theta_s = (s \wedge \bar{s})^{-1/k}$ is a volume measure and

$$\int_{X_\infty \cap U} (s \wedge \bar{s})^{-1/k} = \int_{X_\infty \cap U} |s|^{-2/k} dV_{g_\infty} \leq (\epsilon)^{-2/k} V.$$

Let $\pi : \tilde{X} \to X_\infty$ be a resolution of singularities. Then $\pi^* \Theta_s$ is $L^1$-integrable on $\tilde{X}$, and since $\Theta_s$ can have only algebraic singularities, $\pi^* \Theta_s$ is $L^{1+\epsilon}$-integrable on $\tilde{X}$ for some $\epsilon > 0$. This implies that $X_\infty$ has at worst log terminal singularities. □
Proposition 6.2. The limiting variety \((X_{\infty}, g_{\infty})\) arising from Proposition 6.1 solves the Kähler–Ricci soliton on \(X_{\infty}\) in the following sense.

1. \(g_{\infty}\) is a global Kähler current on \(X_{\infty}\) with bounded local Kähler potentials.

2. \(g_{\infty}\) solves the Kähler–Ricci soliton equation on \(X_{\infty}^{reg}\)

\[
\text{Ric}(g_{\infty}) + \nabla^2 u_{\infty} = g_{\infty},
\]

for some smooth real valued potential function \(u_{\infty}\) on \(X_{\infty}^{reg}\).

3. \(\|u_{\infty}\|_{C^1(X_{\infty}^{reg})} < \infty\), and thus the holomorphic vector field \(\mathcal{V}_\infty = \bar{\partial} u_{\infty}\) (i.e. \((\mathcal{V}_\infty)^i = (g_{\infty})^{ij}(u_{\infty})\)) extends to a global holomorphic vector field on \(X_{\infty}\) with \(\|\mathcal{V}_\infty\|_{L^\infty(X_{\infty}, g_{\infty})} < \infty\). In particular, the Futaki invariant of \((X_{\infty}, g_{\infty})\) can be bounded by \(F\),

\[
\mathcal{F}_{X_{\infty}}(\mathcal{V}_\infty) = \int_{X_{\infty}} |\mathcal{V}_\infty|^2 dV_{g_{\infty}} \leq F.
\]

Proof. We first prove that the local Kähler potentials of \(g_{\infty}\) are uniformly bounded. Let

\[
\omega_{FS,i} = k^{-1}\frac{-1}{2\pi} \partial \bar{\partial} \log \sum j s^{(i)}_j \wedge \overline{s^{(i)}_j}, \quad \omega_{FS,\infty} = k^{-1}\frac{-1}{2\pi} \partial \bar{\partial} \log \sum j s^{(\infty)}_j \wedge \overline{s^{(\infty)}_j}
\]

be the Fubini–Study metrics from the embeddings by \(\{s^{(i)}_j\}_j\) and \(\{s^{(\infty)}_j\}_j\). Let \(\omega_{g_{\infty}}\) be the Kähler form associated to the soliton metric \(g_{\infty}\). We define \(\rho_{X_i,k}\) to be the Bergman kernel for \(K_{X_i}^{k}\) with respect to \(g_{i}\) defined by

\[
\rho_{X_i,k} = \sum_j |s^{(i)}_j|^2 h^{\frac{\gamma}{k}},
\]

where \(h_i = e^{-u_i} \omega^n_{i}\) is the hermitian metric on \(K_{X_i}^{-\gamma}\). Then

\[
\omega_{g_{\infty}} = \omega_{FS,i} + k^{-1}\frac{-1}{2\pi} \partial \bar{\partial} \log \rho_{X_i,k}, \quad \omega_{g_{\infty}} = \omega_{FS,\infty} + k^{-1}\frac{-1}{2\pi} \partial \bar{\partial} \log \rho_{X_{\infty,k}}.
\]

By Proposition 6.1, \(\rho_{X_i,k}\) and \(\rho_{X_{\infty,k}}\) are uniformly bounded in \(L^\infty\) for some fixed sufficiently large \(k\). By the partial \(C^0\) estimate, \(\rho_{X_i,k}\) and \(\rho_{X_{\infty,k}}\) are uniformly bounded below away from 0. Therefore \(\varphi_i = k^{-1} \log \rho_{k,i}\) and \(\varphi_{\infty} = k^{-1} \log \rho_{X_{\infty,k}}\) are uniformly bounded in \(L^\infty\).

Note that the hermitian metrics on \(K_{X_i}^{-1}\) and \(K_{X_{\infty}}^{-1}\) are given by \(h_i = e^{-u_i} \omega^n_{g_{\infty}}\) and \(h_{\infty} = e^{-u_{\infty}} \omega^n_{g_{\infty}}\). Since \(u_i\) and \(|\nabla u_i|_{g_{i}}\) are uniformly bounded, \(u_i\) converges in \(C^\alpha\) on \(X_{\infty}^{reg}\) to \(u_{\infty}\). From the smooth convergence of \(g_{i}\) to \(g_{\infty}\) on \(X_{\infty}^{reg}\), \(u_i\) converges in \(C^\infty\) to \(u_{\infty}\) on \(X_{\infty}^{reg}\) with \(|u_{\infty}|\) and \(|\nabla u_{\infty}|\) uniformly bounded on \(X_{\infty}^{reg}\). Furthermore, \(g_{\infty}\) satisfies the soliton equation on \(X_{\infty}^{reg}\)

\[
\text{Ric}(g_{\infty}) = g_{\infty} - \nabla^2 u_{\infty} = g_{\infty} + L_{\mathcal{V}_{\infty}} g_{\infty},
\]
where \((\mathcal{V}_\infty)\) is the holomorphic vector field on \(X^{reg}_\infty\) induced by \(u_\infty\). Since \(X_\infty\) is normal, \(\mathcal{V}_\infty\) extends to a bounded global holomorphic vector field on \(X_\infty\) with \(||X_\infty||_{L^\infty(X_\infty,g_\infty)} < \infty\) and \(\mathcal{F}_{X_\infty}(\mathcal{V}_\infty) \leq F\). In fact, if we let \(\Omega_{FS, \infty}\) be the smooth volume form on \(X_\infty\) with \(\sqrt{-1} \frac{1}{2\pi} \partial \bar{\partial} \log \Omega_{FS, \infty} = -\omega_{FS, \infty}\), then \(\varphi_\infty\) satisfies a global Monge–Ampère equation on \(X_\infty\)

\[
(\omega_{FS, \infty} + \sqrt{-1} \frac{1}{2\pi} \partial \bar{\partial} \varphi_\infty)^n = e^{-\varphi_\infty + u_\infty} \Omega_{FS, \infty}
\]

and on \(X^{reg}_\infty\) (cf. \([8, 20]\)).

\[\square\]

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