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ALGEBRA OF OPERATORS AFFILIATED WITH A FINITE TYPE I VON NEUMANN ALGEBRA

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Abstract. The aim of the paper is to prove that the $*$-algebra of all (closed densely defined linear) operators affiliated with a finite type I von Neumann algebra admits a unique center-valued trace, which turns out to be, in a sense, normal. It is also demonstrated that for no other von Neumann algebras similar constructions can be performed.

1. Introduction. With every von Neumann algebra $\mathfrak{A}$ one can associate the set $\mathrm{Aff}(\mathfrak{A})$ of operators (unbounded, in general) which are *affiliated* with $\mathfrak{A}$. In [**11**] Murray and von Neumann discovered that, surprisingly, $\mathrm{Aff}(\mathfrak{A})$ turns out to be a unital $*$-algebra when $\mathfrak{A}$ is finite. This was in fact the first example of a rich set of *unbounded* operators in which one can define algebraic binary operations in a natural manner. This concept was later adapted by Segal [**17,18**], who distinguished a certain class of unbounded operators (namely, measurable with respect to a fixed normal faithful semi-finite trace) affiliated with an *arbitrary* semi-finite von Neumann algebra and equipped it with a structure of a $*$-algebra (for an alternative proof see e.g. [**12**] or §2 in Chapter IX of [**21**]). A more detailed investigations in algebras of the form $\mathrm{Aff}(\mathfrak{A})$ were initiated by a work of Stone [**19**], who described their models for commutative $\mathfrak{A}$ in terms of unbounded continuous functions densely defined on the Gelfand spectrum $\mathfrak{X}$ of $\mathfrak{A}$. Much later Kadison [**6**] studied this one-to-one correspondence between operators in $\mathrm{Aff}(\mathfrak{A})$ and functions on $\mathfrak{X}$. Recently Liu [**9**] established an interesting property of $\mathrm{Aff}(\mathfrak{A})$ concerning the Heisenberg uncertainty principle.

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Namely, she showed that the canonical commutation relation, which has the form $AB - BA = I$, fails to hold for any $A, B \in \mathrm{Aff}(\mathfrak{A})$, provided $\mathfrak{A}$ is finite. For finite *type I* algebras this result is a simple corollary of a fact, more or less known to the experts, that $\mathrm{Aff}(\mathfrak{A})$ has a uniquely determined center-valued trace, provided $\mathfrak{A}$ is a finite type I von Neumann algebra. In this paper we give a proof of this fact. Our result reads as follows:

THEOREM 1.1. *Let $\mathfrak{A}$ be a finite type I von Neumann algebra and let $\mathrm{Aff}(\mathfrak{A})$ be the $*$-algebra of all operators affiliated with $\mathfrak{A}$. Then there is a unique linear map $\mathrm{tr}_{\mathrm{Aff}}\colon \mathrm{Aff}(\mathfrak{A}) \to \mathfrak{Z}(\mathrm{Aff}(\mathfrak{A}))$ such that*

(tr1) $\mathrm{tr}_{\mathrm{Aff}}(A)$ *is non-negative, provided $A \in \mathrm{Aff}(\mathfrak{A})$ is so;*
(tr2) $\mathrm{tr}_{\mathrm{Aff}}(X \cdot Y) = \mathrm{tr}_{\mathrm{Aff}}(Y \cdot X)$ *for any $X, Y \in \mathrm{Aff}(\mathfrak{A})$;*
(tr3) $\mathrm{tr}_{\mathrm{Aff}}(Z) = Z$ *for each $Z \in \mathfrak{Z}(\mathrm{Aff}(\mathfrak{A}))$.*

Moreover,

$$\mathfrak{Z}(\mathrm{Aff}(\mathfrak{A})) = \mathrm{Aff}(\mathfrak{Z}(\mathfrak{A})) \tag{1-1}$$

and

(tr4) $\mathrm{tr}_{\mathrm{Aff}}(A) \neq 0$ *provided $A \in \mathrm{Aff}(\mathfrak{A})$ is non-zero and non-negative;*
(tr5) $\mathrm{tr}_{\mathrm{Aff}}(X \cdot Z) = \mathrm{tr}_{\mathrm{Aff}}(X) \cdot Z$ *for any $X \in \mathrm{Aff}(\mathfrak{A})$ and $Z \in \mathfrak{Z}(\mathrm{Aff}(\mathfrak{A}))$;*
(tr6) *every increasing net $(A_\sigma)_{\sigma\in\Sigma}$ of self-adjoint members of $\mathrm{Aff}(\mathfrak{A})$ which is majorized by a self-adjoint operator in $\mathrm{Aff}(\mathfrak{A})$ has its least upper bound in $\mathrm{Aff}(\mathfrak{A})$, and*

$$\sup_{\sigma\in\Sigma} \mathrm{tr}_{\mathrm{Aff}}(A_\sigma) = \mathrm{tr}_{\mathrm{Aff}}\Bigl(\sup_{\sigma\in\Sigma} A_\sigma\Bigr). \tag{1-2}$$

The above result, in a little bit different settings, was earlier established by Berberian [**1**], who used totally different methods. Besides, we establish more properties of the trace than he did. All they (as well as (1-1), which holds for all finite von Neumann algebras $\mathfrak{A}$ and most of the content of Sections 2 and 3) are, however, more or less known to the experts and can be deduced from the possibility of modelling $\mathrm{Aff}(\mathfrak{A})$ as a direct sum of finite-dimensional matrix algebras over corresponding algebras of bounded measurable functions (cf. [**15**] for a similar result in a more general setting). Nevertheless, it is likely that this result nowhere appears explicitly. For the reader's convenience, we prove most of auxiliary results here.

It is worth noting that (1-2) is a natural counterpart of normality (in the terminology of Takesaki – see Definition 2.1 in Chapter V of [**20**]) of center-valued traces in finite von Neumann algebras. The existence of the l.u.b. in item (tr6) was established by Yeadon [**22**] for more general algebras than $\mathrm{Aff}(\mathfrak{A})$.

It is natural to ask whether the above result may be generalised to a wider class of von Neumann algebras (e.g. for all finite ones). Our second goal is to show that the answer is negative, which is somewhat surprising. A precise formulation of the result is stated below. We recall that, in general, the set $\mathrm{Aff}(\mathfrak{A})$ admits no structure of a vector space, nevertheless, it is *always* homogeneous and for any $T \in \mathrm{Aff}(\mathfrak{A})$ and $S \in \mathfrak{A}$ the operator $T + S$ is a (well defined) member of $\mathrm{Aff}(\mathfrak{A})$. Based on this observation, we may formulate our result, which appears to be new, as follows.

PROPOSITION 1.2. *Let $\mathfrak{A}$ be a von Neumann algebra and let $\mathrm{Aff}(\mathfrak{A})$ be the set of all operators affiliated with $\mathfrak{A}$. Assume there exists a function $\varphi\colon \mathrm{Aff}(\mathfrak{A}) \to \mathrm{Aff}(\mathfrak{A})$ with the following properties:*

(a) *if $A, B \in \mathfrak{A}$ are such that $\varphi(A) \in \mathfrak{A}$, then $\varphi(\alpha A + \beta B) = \alpha\varphi(A) + \beta\varphi(B)$ for any scalars $\alpha, \beta \in \mathbb{C}$;*
(b) *$\varphi(A)$ is non-negative, provided $A \in \mathrm{Aff}(\mathfrak{A})$ is so;*
(c) *if $A \in \mathrm{Aff}(\mathfrak{A})$ and $B \in \mathfrak{A}$ are non-negative, and $\varphi(B) \in \mathfrak{A}$, then $\varphi(A+B) = \varphi(A) + \varphi(B)$;*
(d) *$\varphi(AB) = \varphi(BA)$ for all $A, B \in \mathfrak{A}$;*
(e) *$\varphi(Z) = Z$ for each $Z \in \mathfrak{Z}(\mathfrak{A})$;*
(f) *if A and U are members of $\mathfrak{A}$ and U is unitary, then $U^*\varphi(A)U = \varphi(A)$.*

Then $\mathfrak{A}$ is finite and type I.

The paper is organized as follows. In the next section we establish an interesting property of finite type I von Neumann algebras, which is crucial in this paper, since all other results, apart from Proposition 1.2, are its consequences. Its proof involves measure-theoretic techniques, which is in contrast to all other parts of the paper, where all arguments are, roughly speaking, intrinsic and algebraic. In Section 3 we establish most relevant properties of the set $\mathrm{Aff}(\mathfrak{A})$ (for a finite type I algebra $\mathfrak{A}$), including a new proof of the fact that $\mathrm{Aff}(\mathfrak{A})$ admits a structure of a $*$-algebra. In Section 4 we introduce the center-valued trace on $\mathrm{Aff}(\mathfrak{A})$ and prove all items of Theorem 1.1, except for (tr6), which is shown in Section 5, where we also establish other order properties of $\mathrm{Aff}(\mathfrak{A})$. Finally, Section 6 contains a proof of Proposition 1.2.

Notation and terminology. In this paper $\mathfrak{A}$ is used to denote an arbitrary von Neumann algebra acting on a (complex) Hilbert space $\mathcal{H}$. All *operators* are linear, closed and densely defined in a Hilbert space, *projections* are orthogonal while *non-negative* operators are, by definition, self-adjoint. The algebra of all bounded operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. An operator T in $\mathcal{H}$ is *affiliated* with $\mathfrak{A}$ if $UTU^{-1} = T$ for any unitary operator U belonging to the commutant $\mathfrak{A}'$ of $\mathfrak{A}$. $\mathrm{Aff}(\mathfrak{A})$ stands for the set of all operators affiliated with $\mathfrak{A}$. If $\mathfrak{A}$ is finite, $\mathrm{Aff}(\mathfrak{A})$ may be naturally equipped with the structure of a $*$-algebra (see e.g. [**11**]). In that case we denote binary algebraic operations in $\mathrm{Aff}(\mathfrak{A})$ by '$\boldsymbol{+}$' (for addition), '$\boldsymbol{-}$' (for subtraction) and '$\boldsymbol{\cdot}$' (for multiplication). For any ring $\mathfrak{R}$, $\mathfrak{Z}(\mathfrak{R})$ stands for the center of $\mathfrak{R}$ (that is, $\mathfrak{Z}(\mathfrak{R})$ consists of all elements of $\mathfrak{R}$

which commute with any element of $\mathfrak{R}$). This mainly applies to $\mathfrak{R} = \mathfrak{A}$, and $\mathfrak{R} = \mathrm{Aff}(\mathfrak{A})$, provided $\mathfrak{A}$ is finite. For any operator S, we use $\mathcal{D}(S)$, $\mathcal{N}(S)$ and $\mathcal{R}(S)$ to denote the domain, kernel and range of S, respectively. By $|S|$ we denote the operator $(S^*S)^{\frac{1}{2}}$. For any collection $\{T_s\}_{s\in S}$ of operators, $\bigoplus_{s\in S} T_s$ is understood as an operator with a maximal domain defined naturally; that is, $\oplus_{s\in S} x_s$ belongs to the domain of $\bigoplus_{s\in S} T_s$ if $x_s \in \mathcal{D}(T_s)$ for each $s \in S$, and $\sum_{s\in S} \|T_s x_s\|^2 < \infty$ (and, of course, $(\bigoplus_{s\in S} T_s)(\oplus_{s\in S} x_s) = \oplus_{s\in S}(T_s x_s)$). The center-valued trace in a finite von Neumann algebra $\mathfrak{W}$ is denoted by $\mathrm{tr}_{\mathfrak{W}}$. The center-valued trace on the algebra of operators affiliated with a finite type I von Neumann algebra will be denoted by $\mathrm{tr}_{\mathrm{Aff}}$. All vector spaces are assumed to be over the field $\mathbb{C}$ of complex numbers. For two C^*-algebras $\mathfrak{C}_1$ and $\mathfrak{C}_2$, we write $\mathfrak{C}_1 \cong \mathfrak{C}_2$ when $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are $*$-isomorphic. The direct product of a collection $\{\mathfrak{C}_s\}_{s\in S}$ of C^*-algebras is denoted by $\prod_{s\in S} \mathfrak{C}_s$ and it consists of all systems $(a_s)_{s\in S}$ with $a_s \in \mathfrak{C}_s$ and $\sup_{s\in S} \|a_s\| < \infty$ (cf. Definition II.8.1.2 in [**2**]). By I we denote the identity operator on $\mathcal{H}$.

2. Key result. As we will see in the sequel, all our main results depend on the following theorem, whose proof is the purpose of this section.

THEOREM 2.1. *Assume $\mathfrak{A}$ is finite and type I. Then for any $T \in \mathfrak{A}$ the following conditions are equivalent:*

(a) *$\|T\xi\| < \|\xi\|$ for each non-zero vector $\xi \in \mathcal{H}$;*
(b) *there is a sequence $Z_1, Z_2, \ldots \in \mathfrak{Z}(\mathfrak{A})$ of mutually orthogonal projections such that $\sum_{n=1}^{\infty} Z_n = I$ and $\|TZ_n\| < 1$ for any $n \geqslant 1$.*

We will derive the above theorem as a combination of a classical result on classification of type I von Neumann algebras and a measure-theoretic result due to Maharam [**10**].

For any positive integer n, let M_n be the C^*-algebra of all $n \times n$ complex matrices. Whenever $(X, \mathfrak{M}, \mu)$ is a finite measure space, we use $L^\infty(X, \mu, M_n)$ to denote the C^*-algebra of all M_n-valued essentially bounded measurable functions on X (a function $f = [f_{jk}]\colon X \to M_n$ is *measurable* if each of the functions $f_{jk}\colon X \to \mathbb{C}$ is measurable; in other words, $L^\infty(X, \mu, M_n) \cong L^\infty(X, \mu)\bar{\otimes} M_n$). The following result is well known and may easily be derived from Theorems 1.22.13 and 2.3.3 in [**16**] (cf. also Theorem 6.6.5 in [**7**]).

THEOREM 2.2. *For every finite type I von Neumann algebra $\mathfrak{A}$ there are a collection $\{(X_j, \mathfrak{M}_j, \mu_j)\}_{j\in J}$ of probabilistic measure spaces and a corresponding collection $\{\nu_j\}_{j\in J}$ of positive integers such that*

$$\mathfrak{A} \cong \prod_{j\in J} L^\infty(X_j, \mu_j, M_{\nu_j}). \tag{2-1}$$

We need a slight modification of (2-1) (see Theorem 2.4 below). To this end, let us introduce certain classical measure spaces, which we call *canonical.* Let α be an infinite cardinal and S_α be a fixed set of cardinality α. We consider the set $D_\alpha = \{0,1\}^{S_\alpha}$ (of all functions from S_α to $\{0,1\}$) equipped with the *product* σ-algebra $\mathfrak{M}_\alpha$ and the *product* probabilistic measure m_α; that is, $\mathfrak{M}_\alpha$ coincides with the σ-algebra on D_α generated by all sets of the form

$$\mathrm{Cyl}(G) \stackrel{\mathrm{def}}{=} \{u \in D_\alpha\colon\ u\big|_F \in G\} \tag{2-2}$$

where F is a finite subset of S_α and G is any subset of $\{0,1\}^F$, while m_α is the unique probabilistic measure on $\mathfrak{M}_\alpha$ such that $m_\alpha(\mathrm{Cyl}(G)) = \mathrm{card}(G)/2^{\mathrm{card}(F)}$ for any such sets G and F. It is worth noting that when α is uncountable and D_α is considered with the product topology, not every open set in D_α belongs to $\mathfrak{M}_\alpha$. Additionally, we denote by $(D_0, \mathfrak{M}_0, m_0)$ the unique probabilistic measure space with $D_0 = \{0\}$. For simplicity, let Card_∞ stand for the class of all infinite cardinal numbers. We call the measure spaces $(D_\alpha, \mathfrak{M}_\alpha, m_\alpha)$ with $\alpha \in \mathrm{Card}_\infty \cup \{0\}$ *canonical.* In the sequel we will apply the following consequence of a deep result due to Maharam [**10**]:

Theorem 2.3. *For any probabilistic measure space* $(X, \mathfrak{M}, \mu)$ *there is a sequence* (*finite or not*) $\alpha_1, \alpha_2, \ldots \in \mathrm{Card}_\infty \cup \{0\}$ *such that the* C^**-algebras* $L^\infty(X,\mu)$ *and* $\prod_{n\geqslant 1} L^\infty(D_{\alpha_n}, m_{\alpha_n})$ *are* $*$*-isomorphic.*

The above result is not explicitly stated in [**10**], but may simply be deduced from Theorems 1 and 2 included there.

As a consequence of Theorems 2.2 and 2.3 (and the fact that $L^\infty(X,\mu,M_n) \cong L^\infty(X,\mu)\bar{\otimes} M_n$), we obtain

Theorem 2.4. *For every finite type I von Neumann algebra* $\mathfrak{A}$ *there are collections* $\{\alpha_j\}_{j\in J} \subset \mathrm{Card}_\infty \cup \{0\}$ *and* $\{\nu_j\}_{j \in J} \subset \{1,2,\ldots\}$ *such that*

$$\mathfrak{A} \cong \prod_{j\in J} L^\infty(D_{\alpha_j}, m_{\alpha_j}, M_{\nu_j}).$$

The following simple lemma will also prove useful for us.

Lemma 2.5. *If* T *is an arbitrary member of* $\mathfrak{A}$, *then* $\mathcal{N}(T) = \{0\}$ *iff the mapping*

$$\mathfrak{A} \ni X \mapsto TX \in \mathfrak{A} \tag{2-3}$$

is one-to-one.

Proof. If $\mathcal{N}(T) = \{0\}$ and $TX = 0$, then $\mathcal{R}(X) \subset \mathcal{N}(T) = \{0\}$ and hence $X = 0$. For the converse, let $E\colon \mathfrak{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$ be the spectral measure of $|T|$, defined on the σ-algebra $\mathfrak{B}(\mathbb{R})$ of all Borel subsets of $\mathbb{R}$. Then $E(\sigma) \in \mathfrak{A}$ for any Borel set $\sigma \subset \mathbb{R}$. Since $TE(\{0\}) = 0$, we conclude from the injectivity of (2-3) that $E(\{0\}) = 0$ and thus $\mathcal{N}(T) = \mathcal{N}(|T|) = \{0\}$. $\square$

Proposition 2.6. *Let $\mathfrak{A}$ be a finite type I von Neumann algebra. Let $\{\alpha_j\}_{j\in J}$ and $\{\nu_j\}_{j\in J}$ be two collections as in the assertion of Theorem 2.4. Further, let $\Phi\colon \mathfrak{A} \to \prod_{j\in J} L^\infty(D_{\alpha_j}, m_{\alpha_j}, M_{\nu_j})$ be any $*$-isomorphism. For an arbitrary operator T in $\mathfrak{A}$ and $(f_j)_{j\in J} \stackrel{\text{def}}{=} \Phi(T)$, the following conditions are equivalent:*

(a) *$\|T\xi\| < \|\xi\|$ for each non-zero vector $\xi \in \mathcal{H}$;*
(b) *for each $j \in J$, the set $\{x \in D_{\alpha_j}\colon\ \|f_j(x)\| < 1\}$ is of full measure m_{α_j}.*

Proof. Assume first that (b) holds. Observe that then $\|T\| = \|\Phi(T)\| \leqslant 1$ and $\Phi((I - T^*T)S) = (1 - \Phi(T)^*\Phi(T))\Phi(S) \neq 0$ for any non-zero operator $S \in \mathfrak{A}$. So, Lemma 2.5 ensures that $I - T^*T$ is one-to-one. Consequently, (a) is fulfilled.

Now assume that (a) is satisfied, or, equivalently, that $I - T^*T \geqslant 0$ and the mapping $\mathfrak{A} \ni X \mapsto (I - T^*T)X \in \mathfrak{A}$ is one-to-one. This means that

$$1 - \Phi(T)^*\Phi(T) \geqslant 0 \tag{2-4}$$

and $(1-\Phi(T)^*\Phi(T))g \neq 0$ for each non-zero $g \in \mathfrak{L} \stackrel{\text{def}}{=} \prod_{j\in J} L^\infty(D_{\alpha_j}, m_{\alpha_j}, M_{\nu_j})$. Suppose, on the contrary, that

$$m_{\alpha_k}(\{x \in D_{\alpha_k}\colon\ \|f_k(x)\| < 1\}) < 1 \tag{2-5}$$

for some $k \in J$. To get a contradiction, it is enough to find a bounded measurable function $u\colon D_{\alpha_k} \to M_{\nu_k}$ such that u is a non-zero vector in $L^\infty(D_{\alpha_k}, m_{\alpha_k}, M_{\nu_k})$ and $(1 - f_k^* f_k)u = 0$ (because then it suffices to put $g_k = u$ and $g_j = 0$ for $j \neq k$ in order to obtain a non-zero vector $g \stackrel{\text{def}}{=} (g_j)_{j\in J} \in \mathfrak{L}$ for which $(1 - \Phi(T)^*\Phi(T))g = 0$). It is now that we will make use of the form of the measurable space $(D_{\alpha_k}, \mathfrak{M}_{\alpha_k})$. If $\alpha_k = 0$, the existence of u is trivial. We therefore assume that α_k is infinite. Since $f_k\colon \{0,1\}^{S_{\alpha_k}} \to M_{\nu_k}$ is measurable and $\mathfrak{M}_{\alpha_k}$ is the product σ-algebra, we conclude that there exist a **countable** infinite set $F \subset S_{\alpha_k}$ and a measurable function $f\colon \{0,1\}^F \to M_{\nu_k}$ such that

$$f_k(\eta) = f(\eta\big|_F) \tag{2-6}$$

for any $\eta \in \{0,1\}^{S_{\alpha_k}}$. For simplicity, we put $\Omega = \{0,1\}^F$, $\mathfrak{M} = \{G \subset \Omega\colon\ \mathrm{Cyl}(G) \in \mathfrak{M}_{\alpha_k}\}$ (cf. (2-2)) and define a measure $\lambda\colon \mathfrak{M} \to [0,1]$ by $\lambda(G) = m_{\alpha_k}(\mathrm{Cyl}(G))$ for any $G \in \mathfrak{M}$. Note that the probabilistic measure space $(\Omega, \mathfrak{M}, \lambda)$ is naturally *isomorphic* to $(D_{\aleph_0}, \mathfrak{M}_{\aleph_0}, m_{\aleph_0})$ and hence it is a standard measure space (which is relevant for us). It suffices to find a measurable bounded function $v\colon \Omega \to M_n$ such that v is a non-zero vector in $L^\infty(\Omega, \lambda, M_n)$ and

$$(1 - f^*f)v \equiv 0 \tag{2-7}$$

(because then u may be defined by $u(\eta) \stackrel{\text{def}}{=} v(\eta\big|_F)$). Let $G \stackrel{\text{def}}{=} \{\omega \in \Omega\colon\ \|f(\omega)\| = 1\}(\in \mathfrak{M})$. It follows from (2-4), (2-5) and (2-6) that

$$\lambda(G) > 0. \tag{2-8}$$

Since we deal with (finite-dimensional) matrices, we see that

$$\forall \omega \in G\colon\ \mathcal{N}(I - f(\omega)^* f(\omega)) \neq \{0\}. \tag{2-9}$$

Now we consider a multifunction Ψ on Ω which to each $\omega \in \Omega$ assigns the kernel of $I - f(\omega)^* f(\omega)$. Equipping the set of all linear subspaces of $\mathbb{C}^n$ with the Effros-Borel structure (see [**3, 4**] or §6 in Chapter V in [**20**] and Appendix there), we conclude that Ψ is measurable (this is a kind of folklore; it may also be simply deduced from, e.g., a combination of Proposition 2.4 in [**5**] and Corollary A.18 in [**20**]). So, from Effros' theory it follows that there exist measurable functions $h_1, h_2, \ldots\colon \Omega \to \mathbb{C}^n$ such that the set $\{h_k(\omega)\colon\ k \geqslant 1\}$ is a dense subset of $\Psi(\omega)$ for each $\omega \in \Omega$ (to convince oneself of that, consult e.g. subsection A.16 of Appendix in [**21**]). We infer from (2-8) that there is $k \geqslant 1$ such that the set $D \stackrel{\text{def}}{=} \{\omega \in G\colon\ h_k(\omega) \neq 0\}$ has positive measure λ. Finally, we define $v\colon \Omega \to M_n$ as follows: for $\omega \in D$, $v(\omega)$ is the matrix which (in the canonical basis of $\mathbb{C}^n$) corresponds to a linear operator

$$\mathbb{C}^n \ni \xi \mapsto \frac{\langle \xi, h_k(\omega)\rangle}{\langle h_k(\omega), h_k(\omega)\rangle} h_k(\omega) \in \mathbb{C}^n$$

(where $\langle \cdot, - \rangle$ denotes the standard inner product in $\mathbb{C}^n$), and $v(\omega) = 0$ otherwise. It is readily seen that v is measurable and bounded. Moreover, since $\lambda(D) > 0$, we see that v is a non-zero element of $L^\infty(\Omega, \lambda, M_n)$. Finally, (2-7) holds, because $h_k(\omega) \in \Psi(\omega) = \mathcal{N}(I - f^*(\omega) f(\omega))$ for each ω. This completes the proof. □

Now we are ready to give

Proof of Theorem 2.1. It is clear that (b) is followed by (a). Now assume (a) holds. Let collections $\{\alpha_j\}_{j\in J}$ and $\{\nu_j\}_{j\in J}$ and a $*$-isomorphism

$$\Phi\colon \mathfrak{A} \to \prod_{j\in J} L^\infty(D_{\alpha_j}, m_{\alpha_j}, M_{\nu_j})$$

be as in Proposition 2.6. Define $(f_j)_{j\in J} \in \mathfrak{L} \stackrel{\text{def}}{=} \prod_{j\in J} L^\infty(D_{\alpha_j}, m_{\alpha_j}, M_{\nu_j})$ as $\Phi(T)$. We infer from Proposition 2.6 that for any $j \in J$,

$$m_{\alpha_j}(\{x \in D_{\alpha_j}\colon\ \|f_j(x)\| < 1\}) = 1. \tag{2-10}$$

We put $W_{j,n} = \{x \in D_{\alpha_j}\colon\ 1 - 2^{1-n} \leqslant \|f_j(x)\| < 1 - 2^{-n}\}$ and let $w_{j,n} \in L^\infty(D_{\alpha_j}, m_{\alpha_j}, M_{\nu_j})$ be (constantly) equal to the unit $\nu_j \times \nu_j$ matrix on $W_{j,n}$ and 0 off $W_{j,n}$. Observe that $w_n \stackrel{\text{def}}{=} (w_{j,n})_{j\in J}$ is a central projection in $\mathfrak{L}$ and

$\sum_{n=1}^{\infty} w_n = 1$ (thanks to (2-10)). Further, it follows from the definition of the sets $W_{j,n}$ that $\|\Phi(T)w_n\| \leqslant 1 - 2^{-n}$ for any $n \geqslant 1$. Thus, it remains to define Z_n as $\Phi^{-1}(w_n)$ to finish the proof. □

For simplicity, let us introduce

DEFINITION 2.7. A *partition* (in $\mathfrak{A}$) is an arbitrary collection $\{Z_s\}_{s\in S}$ of mutually orthogonal projections such that $\sum_{s\in S} Z_s = I$ and $Z_s \in \mathfrak{Z}(\mathfrak{A})$ for any $s \in S$.

We will need the following corollary, which is a strengthening of Theorem 2.1. Since its proof is a slight modification of the argument used in the proof of Theorem 2.1, we skip it and leave it to the reader.

COROLLARY 2.8. *Let Λ be a countable infinite set of indices and $\{a_\lambda\colon\ \lambda \in \Lambda\}$ be a set of positive real numbers such that*

$$\sup\{a_\lambda\colon\ \lambda \in \Lambda\} = 1. \tag{2-11}$$

For $T \in \mathfrak{A}$, the following conditions are equivalent:

(a) *$\|T\xi\| < \|\xi\|$ for every non-zero vector $\xi \in \mathcal{H}$;*
(b) *there exists a partition $\{Z_\lambda\}_{\lambda\in\Lambda}$ such that $\|TZ_\lambda\| \leqslant a_\lambda$ for every $\lambda \in \Lambda$.*

3. Algebra of affiliated operators. The aim of this part is to show the following result.

THEOREM 3.1. *Let $\mathfrak{A}$ be finite and type I, and $\{c_\lambda\}_{\lambda\in\Lambda}$ be a countable and **unbounded** set of positive real numbers. For any operator T in $\mathcal{H}$ the following conditions are equivalent:*

(a) *$T \in \operatorname{Aff}(\mathfrak{A})$;*
(b) *there are $S \in \mathfrak{A}$ and a partition $\{Z_\lambda\}_{\lambda\in\Lambda}$ for which $T = \sum_{\lambda\in\Lambda} c_\lambda S Z_\lambda$.*

The above result can be deduced from Saitô's [**15**]. Here we present an alternative proof.

To make the above result more precise (and understandable), let us introduce the following

DEFINITION 3.2. Let $\{Z_\lambda\}_{\lambda\in\Lambda}$ be a partition and $\{S_\lambda\}_{\lambda\in\Lambda}$ be any collection of operators in $\mathfrak{A}$. An operator $\sum_{\lambda\in\Lambda} S_\lambda Z_\lambda$ is defined as follows:

$$\mathcal{D}\Big(\sum_{\lambda\in\Lambda} S_\lambda Z_\lambda\Big) \stackrel{\text{def}}{=} \Big\{\xi \in \mathcal{H}\colon\ \sum_{\lambda\in\Lambda} \|S_\lambda Z_\lambda \xi\|^2 < \infty\Big\}$$

and $(\sum_{\lambda\in\Lambda} S_\lambda Z_\lambda)\xi = \sum_{\lambda\in\Lambda}(S_\lambda Z_\lambda \xi)$ (notice that $S_\lambda Z_\lambda \xi = Z_\lambda S_\lambda \xi$ and thus the vectors $S_\lambda Z_\lambda \xi$, $\lambda \in \Lambda$, are mutually orthogonal).

The following simple result will find many applications in the sequel.

Lemma 3.3. *Let $(Z_\lambda)_{\lambda\in\Lambda}$ be a partition. Denote the range of Z_λ by $\mathcal{H}_\lambda$. Then there exists a unitary operator $U\colon \mathcal{H} \to \bigoplus_{\lambda\in\Lambda} \mathcal{H}_\lambda$ such that for any collection $\{S_\lambda\}_{\lambda\in\Lambda} \subset \mathfrak{A}$,*

$$U\Big(\sum_{\lambda\in\Lambda} S_\lambda Z_\lambda\Big)U^{-1} = \bigoplus_{\lambda\in\Lambda} S_\lambda\big|_{\mathcal{H}_\lambda}.$$

Proof. For each $\xi \in \mathcal{H}$, it is enough to define $U\xi$ as $\oplus_{\lambda\in\Lambda} Z_\lambda \xi$. □

Now we list only most basic consequences of Lemma 3.3. Below $\{Z_\lambda\}_{\lambda\in\Lambda}$ is a partition in $\mathfrak{A}$, $\{S_\lambda\}_{\lambda\in\Lambda}$ is an arbitrary collection of operators in $\mathfrak{A}$ and $\mathcal{E}$ stands for the linear span of $\bigcup_{\lambda\in\Lambda} \mathcal{R}(Z_\lambda)$. Notice that $\mathcal{E}$ is dense in $\mathcal{H}$.

(Σ1) $\sum_{\lambda\in\Lambda} S_\lambda Z_\lambda$ is closed;
(Σ2) $\mathcal{E} \subset \mathcal{D}(\sum_{\lambda\in\Lambda} S_\lambda Z_\lambda)$ and $\mathcal{E}$ is a core of $\sum_{\lambda\in\Lambda} S_\lambda Z_\lambda$;
(Σ3) $(\sum_{\lambda\in\Lambda} S_\lambda Z_\lambda)^* = \sum_{\lambda\in\Lambda} S_\lambda^* Z_\lambda$;
(Σ4) $\sum_{\lambda\in\Lambda} S_\lambda Z_\lambda \in \mathrm{Aff}(\mathfrak{A})$;
(Σ5) if $S_\lambda = c_\lambda S$ with $c_\lambda > 0$ for each $\lambda \in \Lambda$, then $\sum_{\lambda\in\Lambda} S_\lambda Z_\lambda$ is self-adjoint (resp. non-negative; normal) iff S is so.

In the proof of Theorem 3.1 we will make use of a certain transformation which to any closed densely defined operator assigns a contraction. In the existing literature there are at least two such transformations. The first was studied e.g. by Kaufman [**8**] and with every closed densely defined operator T it associates the operator $T(I+T^*T)^{-\frac{1}{2}}$. The second, quite similar to the first, is the so-called $\mathfrak{b}$-transform introduced in [**13**] and given by $\mathfrak{b}(T) = T(I+|T|)^{-1}$. We will use the following properties of the latter transform.

Lemma 3.4. *Let T and T_s, $s \in S$, be closed densely defined operators in $\mathcal{H}$ and $\mathcal{H}_s$, respectively. Then:*

($\mathfrak{b}$1) *the $\mathfrak{b}$-transform establishes a one-to-one correspondence between the set of all closed densely defined operators in $\mathcal{H}$ and the set of all bounded operators S on $\mathcal{H}$ such that $\|S\xi\| < \|\xi\|$ for each non-zero vector $\xi \in \mathcal{H}$; the inverse transform is given by $S \mapsto \mathfrak{ub}(S) \stackrel{\text{def}}{=} S(I-|S|)^{-1}$;*
($\mathfrak{b}$2) *T is bounded iff $\|\mathfrak{b}(T)\| < 1$; conversely, if $S \in \mathcal{B}(\mathcal{H})$ and $\|S\| < 1$, then $\mathfrak{ub}(S) \in \mathcal{B}(\mathcal{H})$;*
($\mathfrak{b}$3) *$T \in \mathrm{Aff}(\mathfrak{A}) \iff \mathfrak{b}(T) \in \mathfrak{A}$;*
($\mathfrak{b}$4) *$\mathfrak{b}(UTU^{-1}) = U\mathfrak{b}(T)U^{-1}$ for any unitary operator $U\colon \mathcal{H} \to \mathcal{K}$;*
($\mathfrak{b}$5) *$\mathfrak{b}(\bigoplus_{s\in S} T_s) = \bigoplus_{s\in S} \mathfrak{b}(T_s)$.*

Below we use the $\mathfrak{b}$- and $\mathfrak{ub}$-transforms also as complex-valued functions defined on $\mathbb{C}$, given by appropriate analogous formulas.

Proof of Theorem 3.1. Property (Σ4) shows that (a) is implied by (b). Now assume that $T \in \mathrm{Aff}(\mathfrak{A})$. Then $\mathfrak{b}(T) \in \mathfrak{A}$ and $\|\mathfrak{b}(T)\xi\| < \|\xi\|$ for each

$\xi \neq 0$ (see ($\mathfrak{b}1$) and ($\mathfrak{b}3$)). Using Corollary 2.8 with $a_\lambda \stackrel{\text{def}}{=} \mathfrak{b}(c_\lambda) = \frac{c_\lambda}{1+c_\lambda}$, we obtain a partition $\{Z_\lambda\}_{\lambda\in\Lambda}$ such that $\|\mathfrak{b}(T)Z_\lambda\| \leqslant a_\lambda < 1$. We now infer from ($\mathfrak{b}2$) that there exist operators $S_\lambda \in \mathcal{B}(\mathcal{H})$, $\lambda \in \Lambda$, such that

$$\mathfrak{b}(S_\lambda) = \mathfrak{b}(T)Z_\lambda. \tag{3-1}$$

We can express S_λ directly as $S_\lambda = \mathfrak{b}(T)Z_\lambda\big(I - |\mathfrak{b}(T)Z_\lambda|\big)^{-1}$ and this formula clearly implies that $S_\lambda \in \mathfrak{A}$. It is a well-known property of the functional calculus for self-adjoint (bounded) operators that $\|\mathfrak{ub}(A)\| = \mathfrak{ub}(\|A\|) = \frac{\|A\|}{1-\|A\|}$ for any non-negative operator A of norm less than 1. We will apply this property to $A = |\mathfrak{b}(T)S_\lambda|$. We have:

$$S_\lambda Z_\lambda = \mathfrak{b}(T)Z_\lambda^2(I - |\mathfrak{b}(T)Z_\lambda|)^{-1} = \mathfrak{b}(T)Z_\lambda(I - |\mathfrak{b}(T)Z_\lambda|)^{-1} = S_\lambda \tag{3-2}$$

and

$$\begin{aligned} \|S_\lambda\| &= \|\mathfrak{b}(T)Z_\lambda(I - |\mathfrak{b}(T)Z_\lambda|)^{-1}\| \\ &= \Big\||\mathfrak{b}(T)Z_\lambda|(I - |\mathfrak{b}(T)Z_\lambda|)^{-1}\Big\| \\ &= \|\mathfrak{ub}(|\mathfrak{b}(T)Z_\lambda|)\| = \mathfrak{ub}\big(\big\||\mathfrak{b}(T)Z_\lambda|\big\|\big) \leqslant \mathfrak{ub}(a_\lambda) = c_\lambda. \end{aligned} \tag{3-3}$$

Define $S \stackrel{\text{def}}{=} \sum_{\lambda\in\Lambda} \frac{1}{c_\lambda} S_\lambda$. From (3-2) and (3-3) we infer that the series converges in the strong operator topology. Consequently, $S \in \mathfrak{A}$. Moreover, $c_\lambda S Z_\lambda = S_\lambda Z_\lambda$. In order to prove that $T = \sum_{\lambda\in\Lambda} c_\lambda S Z_\lambda$, it is enough to show that $\mathfrak{b}(T) = \mathfrak{b}(\sum_{\lambda\in\Lambda} c_\lambda S Z_\lambda)$. Using Lemma 3.3, a unitary operator U and subspaces $\mathcal{H}_\lambda$ appearing there, properties ($\mathfrak{b}4$) and ($\mathfrak{b}5$) formulated in Lemma 3.4, and (3-1), we get

$$\begin{aligned} \mathfrak{b}\Big(\sum_{\lambda\in\Lambda} c_\lambda S Z_\lambda\Big) &= U^{-1}\Big(\bigoplus_{\lambda\in\Lambda} \mathfrak{b}\big(c_\lambda S\big|_{\mathcal{H}_\lambda}\big)\Big)U = U^{-1}\Big(\bigoplus_{\lambda\in\Lambda} \mathfrak{b}\big(S_\lambda\big|_{\mathcal{H}_\lambda}\big)\Big)U \\ &= U^{-1}\Big(\bigoplus_{\lambda\in\Lambda} \mathfrak{b}(S_\lambda)\big|_{\mathcal{H}_\lambda}\Big)U = \bigoplus_{\lambda\in\Lambda} \mathfrak{b}(T)\big|_{\mathcal{H}_\lambda} = \mathfrak{b}(T) \end{aligned}$$

and we are done. □

As the first application of Theorem 3.1 we obtain

Corollary 3.5. *Let $\mathfrak{A}$ be finite and type I, and $\Lambda \stackrel{\text{def}}{=} \{\nu = (\nu_1, \dots, \nu_k)\colon \nu_1, \dots, \nu_k \geqslant 1\}$. For any collection $T_1, \dots, T_k \in \mathrm{Aff}(\mathfrak{A})$ there exist a partition $\{Z_\nu\}_{\nu\in\Lambda}$ in $\mathfrak{A}$ and operators $S_1, \dots, S_k \in \mathfrak{A}$ such that for each $j \in \{1, \dots, k\}$,*

$$T_j = \sum_{\nu\in\Lambda} \nu_j S_j Z_\nu. \tag{3-4}$$

Proof. Using Theorem 3.1, write each T_j as $\sum_{n=1}^\infty n S_j Z_n^{(j)}$ and put $Z_\nu = Z_{\nu_1}^{(1)} \cdot \ldots \cdot Z_{\nu_k}^{(k)}$. □

Remark 3.6. Corollary 3.5 gives an alternative proof of the Murray–von Neumann theorem [**11**] that $\mathrm{Aff}(\mathfrak{A})$ can be naturally equipped with the structure of a $*$-algebra provided $\mathfrak{A}$ is finite and type I (the assumption that $\mathfrak{A}$ is type I is superfluous; however, our proof works only in that case). Indeed, if $T_1, \ldots, T_k$ are arbitrary members of $\mathrm{Aff}(\mathfrak{A})$, and $\{Z_\nu\}_{\nu\in\Lambda}$ and $S_1, \ldots, S_k \in \mathfrak{A}$ are as in (3-4), then for any polynomial $p(x_1, \ldots, x_n)$ in n non-commuting variables we may define $p(T_1, \ldots, T_n)$ as $\sum_{\nu\in\Lambda} p(\nu_1 S_1, \ldots, \nu_k S_k) Z_\nu$. With such a definition, the linear span of $\bigcup_{\nu\in\Lambda} \mathcal{R}(Z_\nu)$ is a core for each operator of the form $p(T_1, \ldots, T_k)$. Furthermore, the representation (3-4) enables us to prove briefly that $T_1 = T_2$, provided $T_1 \subset T_2$. It is now easy to conclude from all these observations that $\mathrm{Aff}(\mathfrak{A})$ admits the structure of a $*$-algebra (in particular, all algebraic laws for an algebra, such as associativity, are satisfied). We leave the details to interested readers.

4. Trace. We now turn to the concept of a center-valued trace on $\mathrm{Aff}(\mathfrak{A})$. **In this section $\mathfrak{A}$ is assumed to be finite and type I.** We recall that '$+$', '$-$' and '$\cdot$' denote the binary operations in $\mathrm{Aff}(\mathfrak{A})$. Our main goal is to prove all items of Theorem 1.1, except for (tr6), which will be shown in the next section.

We begin with the following result, which is well known for arbitrary finite von Neumann algebras.

Proposition 4.1. $\mathrm{Aff}(\mathfrak{Z}(\mathfrak{A})) = \mathfrak{Z}(\mathrm{Aff}(\mathfrak{A}))$.

Proof. Take $T \in \mathrm{Aff}(\mathfrak{Z}(\mathfrak{A})) \subset \mathrm{Aff}(\mathfrak{A})$. Since $\mathfrak{Z}(\mathfrak{A})$ is also finite and type I, it follows from Theorem 3.1 that T has the form $T = \sum_{n=1}^{\infty} nSZ_n$ with $S, Z_n \in \mathfrak{Z}(\mathfrak{A})$. Similarly, any $X \in \mathrm{Aff}(\mathfrak{A})$ has the form $X = \sum_{n=1}^{\infty} nYW_n$ with $Y \in \mathfrak{A}$ and $W_n \in \mathfrak{Z}(\mathfrak{A})$. Then $SY = YS$ and it follows from Remark 3.6 that for $\Lambda = \{\nu = (\nu_1, \nu_2)\colon\ \nu_1, \nu_2 \geqslant 1\}$ and $Z_\nu = Z_{\nu_1} W_{\nu_2}$, $T \cdot X = \sum_{\nu\in\Lambda} \nu_1\nu_2 SYZ_\nu = \sum_{\nu\in\Lambda} \nu_2\nu_1 YSZ_\nu = X \cdot T$, which shows that $T \in \mathfrak{Z}(\mathrm{Aff}(\mathfrak{A}))$. In particular, $\mathfrak{Z}(\mathfrak{A}) \subset \mathfrak{Z}(\mathrm{Aff}(\mathfrak{A}))$.

Conversely, take $T \in \mathfrak{Z}(\mathrm{Aff}(\mathfrak{A}))$ of the form $T = \sum_{n=1}^{\infty} nSZ_n$ (with $S \in \mathfrak{A}$ and $Z_n \in \mathfrak{Z}(\mathfrak{A})$). Then $SZ_n = \frac{1}{n} T \cdot Z_n$ belongs to $\mathfrak{Z}(\mathrm{Aff}(\mathfrak{A})) \cap \mathfrak{A} \subset \mathfrak{Z}(\mathfrak{A})$ and thus $S = \sum_{n=1}^{\infty} SZ_n \in \mathfrak{Z}(\mathfrak{A})$. Applying Theorem 3.1 again (this time to the von Neumann algebra $\mathfrak{Z}(\mathfrak{A})$) we obtain $T \in \mathrm{Aff}(\mathfrak{Z}(\mathfrak{A}))$. $\square$

Lemma 4.2. *Let $\{Z_\lambda\}_{\lambda\in\Lambda}$ and $\{W_\gamma\}_{\gamma\in\Gamma}$ be two partitions in $\mathfrak{A}$ and let $\{T_\lambda\}_{\lambda\in\Lambda}$ and $\{S_\gamma\}_{\gamma\in\Gamma}$ be two collections of operators in $\mathfrak{A}$ such that*

$$\sum_{\lambda\in\Lambda} T_\lambda Z_\lambda = \sum_{\gamma\in\Gamma} S_\gamma W_\gamma. \tag{4-1}$$

Then

$$\sum_{\lambda\in\Lambda} \mathrm{tr}_{\mathfrak{A}}(T_\lambda) Z_\lambda = \sum_{\gamma\in\Gamma} \mathrm{tr}_{\mathfrak{A}}(S_\gamma) W_\gamma.$$

Proof. For $P_{\lambda,\gamma} \stackrel{\text{def}}{=} Z_\lambda W_\gamma$, by (4-1), we have, $T_\lambda P_{\lambda,\gamma} = S_\gamma P_{\lambda,\gamma}$ for any $\lambda \in \Lambda$ and $\gamma \in \Gamma$. Consequently, $\operatorname{tr}_{\mathfrak{A}}(T_\lambda)P_{\lambda,\gamma} = \operatorname{tr}_{\mathfrak{A}}(T_\lambda P_{\lambda,\gamma}) = \operatorname{tr}_{\mathfrak{A}}(S_\gamma P_{\lambda,\gamma}) = \operatorname{tr}_{\mathfrak{A}}(S_\gamma)P_{\lambda,\gamma}$. So,

$$\sum_{\lambda\in\Lambda} \operatorname{tr}_{\mathfrak{A}}(T_\lambda)Z_\lambda = \sum_{\lambda\in\Lambda}\sum_{\gamma\in\Gamma} \operatorname{tr}_{\mathfrak{A}}(T_\lambda)P_{\lambda,\gamma} = \sum_{\gamma\in\Gamma}\sum_{\lambda\in\Lambda} \operatorname{tr}_{\mathfrak{A}}(S_\gamma)P_{\lambda,\gamma} = \sum_{\gamma\in\Gamma} \operatorname{tr}_{\mathfrak{A}}(S_\gamma)W_\gamma$$

and we are done. $\square$

Now we are ready to introduce

Definition 4.3. The *center-valued trace* in $\operatorname{Aff}(\mathfrak{A})$ is a mapping

$$\operatorname{tr}_{\operatorname{Aff}}\colon \operatorname{Aff}(\mathfrak{A}) \to \mathfrak{Z}(\operatorname{Aff}(\mathfrak{A}))$$

defined as follows. For any partition $\{Z_\lambda\}_{\lambda\in\Lambda}$ in $\mathfrak{A}$ and a collection $\{S_\lambda\}_{\lambda\in\Lambda} \subset \mathfrak{A}$,

$$\operatorname{tr}_{\operatorname{Aff}}\Big(\sum_{\lambda\in\Lambda} S_\lambda Z_\lambda\Big) = \sum_{\lambda\in\Lambda} \operatorname{tr}_{\mathfrak{A}}(S_\lambda)Z_\lambda.$$

Theorem 3.1 and Lemma 4.2 ensure that the definition is full and correct, while Proposition 4.1 (and its proof) shows that indeed $\operatorname{tr}_{\operatorname{Aff}}(T)$ belongs to $\mathfrak{Z}(\operatorname{Aff}(\mathfrak{A}))$ for any $T \in \operatorname{Aff}(\mathfrak{A})$.

For transparency, let us isolate the uniqueness part of Theorem 1.1 in the following

Lemma 4.4. *If* $\operatorname{tr}'\colon \operatorname{Aff}(\mathfrak{A}) \to \mathfrak{Z}(\operatorname{Aff}(\mathfrak{A}))$ *is a linear mapping which satisfies axioms* (tr1)–(tr3) (*with* $\operatorname{tr}_{\operatorname{Aff}}$ *replaced by* tr'), *then* $\operatorname{tr}' = \operatorname{tr}_{\operatorname{Aff}}$.

Proof. Fix a partition $\{Z_n\}_{n=1}^\infty$ in $\mathfrak{A}$ and consider the map

$$f\colon \mathfrak{A} \ni S \mapsto \operatorname{tr}'\Big(\sum_{n=1}^\infty nSZ_n\Big) \cdot \Big(\sum_{n=1}^\infty \frac{1}{n} Z_n\Big) \in \mathfrak{Z}(\operatorname{Aff}(\mathfrak{A})).$$

It is clear that f is linear. Moreover, for any $S_1, S_2 \in \mathfrak{A}$, using (tr2), we get

$$\begin{aligned}
f(S_1S_2) &= \operatorname{tr}'\Big(\sum_{n=1}^\infty nS_1S_2Z_n\Big) \cdot \Big(\sum_{n=1}^\infty \frac{1}{n}Z_n\Big) \\
&= \operatorname{tr}'\Big(\sum_{n=1}^\infty \sqrt{n}S_1Z_n \cdot \sum_{n=1}^\infty \sqrt{n}S_2Z_n\Big) \cdot \Big(\sum_{n=1}^\infty \frac{1}{n}Z_n\Big) \\
&= \operatorname{tr}'\Big(\sum_{n=1}^\infty \sqrt{n}S_2Z_n \cdot \sum_{n=1}^\infty \sqrt{n}S_1Z_n\Big) \cdot \Big(\sum_{n=1}^\infty \frac{1}{n}Z_n\Big) \\
&= \operatorname{tr}'\Big(\sum_{n=1}^\infty nS_2S_1Z_n\Big) \cdot \Big(\sum_{n=1}^\infty \frac{1}{n}Z_n\Big) = f(S_2S_1).
\end{aligned}$$

Further, if $S \in \mathfrak{A}$ in non-negative, then $T \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} nSZ_n$ is non-negative as well (by (Σ5)). Consequently, $\operatorname{tr}'(T)$ is non-negative and, therefore, so is $f(S)$. Also for $C \in \mathfrak{Z}(\mathfrak{A})$ we have $\sum_{n=1}^{\infty} nCZ_n \in \operatorname{Aff}(\mathfrak{Z}(\mathfrak{A})) = \mathfrak{Z}(\operatorname{Aff}(\mathfrak{A}))$ (cf. the proof of Proposition 4.1), thus, thanks to (tr3),

$$f(C) = \sum_{n=1}^{\infty} nCZ_n \cdot \sum_{n=1}^{\infty} \frac{1}{n} Z_n = \sum_{n=1}^{\infty} CZ_n = C.$$

Finally, we claim that $f(S)$ is bounded for any $S \in \mathfrak{A}$. (This will imply that $f(\mathfrak{A}) \subset \mathfrak{Z}(\operatorname{Aff}(\mathfrak{A})) \cap \mathfrak{A} = \mathfrak{Z}(\mathfrak{A})$.) To see this, it is enough to assume that $S \in \mathfrak{A}$ is non-negative. Then the operator $\|S\|I - S$ is non-negative as well and hence both $f(S)$ and $f(\|S\|I - S)$ are non-negative. But $f(S) + f(\|S\|I - S) = f(\|S\|I) = \|S\|I$. Consequenty, $f(S)$ is bounded, as we claimed.

As $f\colon \mathfrak{A} \to \mathfrak{Z}(\mathfrak{A})$ satisfies all axioms of the center-valued trace in $\mathfrak{A}$ (cf. e.g. Theorem 8.2.8 in [**7**]), we have $f = \operatorname{tr}_{\mathfrak{A}}$ and consequently for each $S \in \mathfrak{A}$:

$$\begin{aligned}\operatorname{tr}'\Big(\sum_{n=1}^{\infty} nSZ_n\Big) &= f(S) \cdot \sum_{n=1}^{\infty} nZ_n = \operatorname{tr}_{\mathfrak{A}}(S) \cdot \sum_{n=1}^{\infty} nZ_n \\ &= \sum_{n=1}^{\infty} n \operatorname{tr}_{\mathfrak{A}}(S) Z_n = \operatorname{tr}_{\mathrm{Aff}}\Big(\sum_{n=1}^{\infty} nSZ_n\Big).\end{aligned}$$

Since the partition was arbitrary, an application of Theorem 3.1 completes the proof. $\square$

Proof of Theorem 1.1. As we announced, property (tr6) will be established in the next section. The linearity of $\operatorname{tr}_{\mathrm{Aff}}$ follows from Corollary 3.5 and the very definition of $\operatorname{tr}_{\mathrm{Aff}}$ (see also Remark 3.6 and Lemma 4.2). Conditions (tr1) and (tr4) are immediate consequences of (Σ5). Property (tr3) follows from the fact that each $C \in \mathfrak{Z}(\operatorname{Aff}(\mathfrak{A}))$ may be written in the form $C = \sum_{n=1}^{\infty} nWZ_n$ where $W \in \mathfrak{Z}(\mathfrak{A})$ and $\{Z_n\}_{n=1}^{\infty}$ is a partition (see the proof of Proposition 4.1). Further, (tr2) and (tr5) are implied by suitable properties of $\operatorname{tr}_{\mathfrak{A}}$ and the way the multiplication in $\operatorname{Aff}(\mathfrak{A})$ is defined (below we use Corollary 3.5 with $\Lambda = \{\nu = (\nu_1, \nu_2)\colon\ \nu_1, \nu_2 \geqslant 1\}$): if $T = \sum_{\nu \in \Lambda} \nu_1 SZ_\nu$ and $X = \sum_{\nu \in \Lambda} \nu_2 YZ_\nu$ (with $Y \in \mathfrak{Z}(\mathfrak{A})$, provided $X \in \mathfrak{Z}(\operatorname{Aff}(\mathfrak{A}))$), then $T \cdot X = \sum_{\nu \in \Lambda} \nu_1\nu_2 SYZ_\nu$ and hence

$$\operatorname{tr}_{\mathrm{Aff}}(T \cdot X) = \sum_{\nu \in \Lambda} \nu_1\nu_2 \operatorname{tr}_{\mathfrak{A}}(SY) Z_\nu = \sum_{\nu \in \Lambda} \nu_1\nu_2 \operatorname{tr}_{\mathfrak{A}}(YS) Z_\nu = \operatorname{tr}_{\mathrm{Aff}}(X \cdot T);$$

and if $X \in \mathfrak{Z}(\operatorname{Aff}(\mathfrak{A}))$, we get

$$\operatorname{tr}_{\mathrm{Aff}}(T \cdot X) = \sum_{\nu \in \Lambda} \nu_1\nu_2 \operatorname{tr}_{\mathfrak{A}}(S) YZ_\nu = \sum_{\nu \in \Lambda} \nu_1 \operatorname{tr}_{\mathfrak{A}}(S) Z_\nu \cdot \sum_{\nu \in \Lambda} \nu_2 YZ_\nu = \operatorname{tr}_{\mathrm{Aff}}(T) \cdot X.$$

Finally, uniqueness of $\mathrm{tr}_{\mathrm{Aff}}$ has already been established in Lemma 4.4 and (1-1) is just the assertion of Proposition 4.1. $\square$

It is worth noting that $\mathrm{tr}_{\mathrm{Aff}}(S) = \mathrm{tr}_{\mathfrak{A}}(S)$ for each $S \in \mathfrak{A}$, the proof of which is left as a simple exercise.

As an immediate consequence of Theorem 1.1, we get the following

COROLLARY 4.5. *Suppose that* $\mathfrak{A}$ *is finite and type I. There are no* $X, Y \in \mathrm{Aff}(\mathfrak{A})$ *such that* $X \cdot Y - Y \cdot X = I$.

PROOF. Apply the trace to both sides. $\square$

The above result for arbitrary finite von Neumann algebras was proved in [**9**].

5. Ordering. Throughout this section, $\mathfrak{A}$ continues to be finite and type I; and $\langle \cdot, - \rangle$ stands for the inner product of $\mathcal{H}$. Similarly as in C^*-algebras, we may distinguish *real* part of $\mathrm{Aff}(\mathfrak{A})$ and introduce a natural ordering in it. To this end, we introduce

DEFINITION 5.1. The *real* part $\mathrm{Aff}_s(\mathfrak{A})$ of $\mathrm{Aff}(\mathfrak{A})$ is the set of all self-adjoint operators in $\mathrm{Aff}(\mathfrak{A})$. Additionally, we put $\mathfrak{A}_s = \mathfrak{A} \cap \mathrm{Aff}_s(\mathfrak{A})$. For $A \in \mathrm{Aff}_s(\mathfrak{A})$ we write $A \geqslant 0$ if A is non-negative (that is, if $\langle A\xi, \xi \rangle \geqslant 0$ for each $\xi \in \mathcal{D}(A)$; or, equivalently, if the spectrum of A is contained in $[0, \infty)$). For two operators $A_1, A_2 \in \mathrm{Aff}_s(\mathfrak{A})$ we write $A_1 \leqslant A_2$ or $A_2 \geqslant A_1$ if $A_2 - A_1 \geqslant 0$.

The least upper bound (in $\mathrm{Aff}_s(\mathfrak{A})$) of a collection $\{B_s\}_{s \in S} \subset \mathrm{Aff}_s(\mathfrak{A})$ is denoted by $\sup_{s \in S} B_s$, provided it exists.

The following simple result gives another description of the ordering defined above.

LEMMA 5.2. *Let* A *and* B *be arbitrary members of* $\mathrm{Aff}_s(\mathfrak{A})$.

(a) *If both* A *and* B *are non-negative, so is* $A + B$.
(b) *The following conditions are equivalent:*
   (i) $A \leqslant B$;
   (ii) $\langle A\xi, \xi \rangle \leqslant \langle B\xi, \xi \rangle$ *for any* $\xi \in \mathcal{D}(A) \cap \mathcal{D}(B)$.

PROOF. All properties follow from the fact that $\mathcal{D}(A) \cap \mathcal{D}(B)$ is a core of the self-adjoint operators $B - A$ and $A + B$. $\square$

It is now readily seen that the ordering '$\leqslant$' in $\mathrm{Aff}_s(\mathfrak{A})$ is reflexive, transitive and antisymmetric (which means that $A = B$, provided $A \leqslant B$ and $B \leqslant A$), and that it is compatible with the linear structure of $\mathrm{Aff}_s(\mathfrak{A})$. Another property, well known for arbitrary von Neumann algebras, is for finite type I algebras established below.

Proposition 5.3. *If* $A, B \in \mathrm{Aff}_s(\mathfrak{A})$ *are non-negative and* $A \cdot B = B \cdot A$, *then* $A \cdot B$ *is non-negative as well.*

Proof. By Corollary 3.5 and (Σ5), we may express A and B in the forms $A = \sum_{\nu \in \Lambda} \nu_1 S Z_\nu$ and $B = \sum_{\nu \in \Lambda} \nu_2 T Z_\nu$ where $S, T \in \mathfrak{A}$ are non-negative. Moreover, we know that then

$$A \cdot B = \sum_{\nu \in \Lambda} \nu_1 \nu_2 S T Z_\nu \tag{5-1}$$

and $B \cdot A = \sum_{\nu \in \Lambda} \nu_2 \nu_1 T S Z_\nu$. From these connections and the assumption we now deduce that $ST = TS$ and, consequently, $ST \geqslant 0$. Now the assertion follows from (5-1) and (Σ5). $\square$

For transparency, we isolate a part of (tr6) (in Theorem 1.1) below. It was proved in a more general setting by Yeadon [**22**].

Lemma 5.4. *Let* $\mathcal{A} = \{A_\sigma\}_{\sigma \in \Sigma} \in \mathrm{Aff}_s(\mathfrak{A})$ *be an increasing net* (*indexed by a directed set* Σ), *bounded above by* $A \in \mathrm{Aff}_s(\mathfrak{A})$. *Then* $\mathcal{A}$ *has a least upper bound in* $\mathrm{Aff}_s(\mathfrak{A})$.

Proof. First of all, we may and do assume that $A_\sigma \geqslant 0$ for any $\sigma \in \Sigma$. (Indeed, fixing $\sigma_0 \in \Sigma$ and putting $\Sigma' \stackrel{\text{def}}{=} \{\sigma \in \Sigma \colon\ \sigma \geqslant \sigma_0\}$, $\mathcal{A}' \stackrel{\text{def}}{=} \{A'_{\sigma'}\}_{\sigma' \in \Sigma'}$ with $A'_{\sigma'} \stackrel{\text{def}}{=} A_\sigma - A_{\sigma_0}$ and $A' \stackrel{\text{def}}{=} A - A_{\sigma_0}$, it is easy to verify that $\mathcal{A}'$ is an increasing net of non-negative operators upper bounded by A', and $\sup_{\sigma \in \Sigma} A_\sigma = A_{\sigma_0} + \sup_{\sigma' \in \Sigma'} A'_{\sigma'}$.) Using Theorem 3.1 and (Σ5), we may express A in the form $A = \sum_{n=1}^\infty n B Z_n$ where $B \in \mathfrak{A}$ is non-negative and $\{Z_n\}_{n=1}^\infty$ is a partition in $\mathfrak{A}$. Fix $k \geqslant 1$. It follows from Proposition 5.3 that the operators $(A - A_\sigma) \cdot Z_k$ and $A_\sigma \cdot Z_k$ are non-negative for any $\sigma \in \Sigma$. So, $0 \leqslant A_\sigma \cdot Z_k \leqslant A \cdot Z_k$. Since $A \cdot Z_k = k B Z_k$ is bounded, we now conclude (e.g. from Lemma 5.2) that $A_\sigma \cdot Z_k$ is bounded as well. Moreover, the same argument shows that the net $\{A_\sigma \cdot Z_k\}_{\sigma \in \Sigma} \subset \mathfrak{A}_s$ is increasing and upper bounded by $A \cdot Z_k \in \mathfrak{A}_s$. From a classical property of von Neumann algebras we infer that this last net has a least upper bound in $\mathfrak{A}_s$, say G_k. We now put $G \stackrel{\text{def}}{=} \sum_{k=1}^\infty G_k Z_k$. Note that $G \in \mathrm{Aff}_s(\mathfrak{A})$ (see (Σ4) and (Σ5)). Since $0 \leqslant A_\sigma \cdot Z_k \leqslant G_k \leqslant A \cdot Z_k = (A \cdot Z_k) Z_k$, we have $G_k = G_k Z_k (= G \cdot Z_k)$ and $A_\sigma \cdot Z_k = (A_\sigma \cdot Z_k) Z_k$, and thus $A_\sigma \cdot Z_k \leqslant G \cdot Z_k \leqslant A \cdot Z_k$ for any $\sigma \in \Sigma$ and $k \geqslant 1$. These inequalities imply that

$$A_\sigma \leqslant G \leqslant A \qquad (\sigma \in \Sigma) \tag{5-2}$$

(because for $X \in \{A_\sigma, G, A\}$, $X = \sum_{k=1}^\infty (X \cdot Z_k) Z_k$ in the sense of Definition 3.2; then apply Lemma 5.2). We will check that $G = \sup_{\sigma \in \Sigma} A_\sigma$. To this end, take an arbitrary upper bound $A' = \sum_{n=1}^\infty n B' Z'_n$ (where $B' \in \mathfrak{A}$ is self-adjoint) of $\mathcal{A}$. It remains to check that $G \leqslant A'$. In what follows, to avoid misunderstandings, '$\sup^{\mathfrak{A}}$' will stand for the least upper bound in $\mathfrak{A}_s$ of suitable families of bounded operators.

For an arbitrary positive n and m we have $0 \leqslant A_\sigma \cdot (Z_n Z_m') \leqslant A' \cdot (Z_n Z_m') = mB' Z_n Z_m' \in \mathfrak{A}_s$, which yields

$$G \cdot (Z_n Z_m') = G_n Z_m' = [\sup_{\sigma\in\Sigma}{}^{\mathfrak{A}} (A_\sigma \cdot Z_n)] Z_m' = \sup_{\sigma\in\Sigma}{}^{\mathfrak{A}} [(A_\sigma \cdot Z_n) Z_m'] \\ = \sup_{\sigma\in\Sigma}{}^{\mathfrak{A}} [A_\sigma \cdot (Z_n Z_m')] \leqslant A' \cdot (Z_n Z_m').$$

Now, as before, it suffices to note that $X = \sum_{n=1}^\infty \sum_{m=1}^\infty (X \cdot (Z_n Z_m')) Z_n Z_m'$ for $X \in \{G, A'\}$ and then apply Lemma 5.2. □

The argument presented above contains a proof of the following convenient property.

Corollary 5.5. *If $\mathcal{T}$ is an increasing net in $\mathfrak{A}_s$ which is upper bounded in $\mathfrak{A}_s$, then its least upper bounds in $\mathfrak{A}_s$ and* $\mathrm{Aff}_s(\mathfrak{A})$ *coincide.*

We need one more simple lemma.

Lemma 5.6. *Let T be any member of* $\mathrm{Aff}(\mathfrak{A})$ *and $\{Z_\lambda\}_{\lambda\in\Lambda}$ be a partition in $\mathfrak{A}$. If $T_\lambda \stackrel{\text{def}}{=} T \cdot Z_\lambda$ is a bounded operator for any $\lambda \in \Lambda$, then $T_\lambda Z_\lambda = T_\lambda$ for all $\lambda \in \Lambda$ and $T = \sum_{\lambda\in\Lambda} T_\lambda Z_\lambda$.*

Proof. Since T_λ is bounded, we get $T_\lambda Z_\lambda = T_\lambda \cdot Z_\lambda = T \cdot Z_\lambda = T_\lambda$. Express T in the form $T = \sum_{n=1}^\infty nSW_n$ with $S \in \mathfrak{A}$ and $W_n \in \mathfrak{Z}(\mathfrak{A})$. Then $T_\lambda = \sum_{n=1}^\infty T_\lambda W_n = \sum_{n=1}^\infty (T \cdot Z_\lambda) \cdot W_n = \sum_{n=1}^\infty (T \cdot W_n) \cdot Z_\lambda = \sum_{n=1}^\infty nB(W_n Z_\lambda)$ and hence

$$\sum_{\lambda\in\Lambda} T_\lambda Z_\lambda = \sum_{\lambda\in\Lambda} \sum_{n=1}^\infty nB(W_n Z_\lambda) = \sum_{n=1}^\infty \sum_{\lambda\in\Lambda} nB(W_n Z_\lambda) = \sum_{n=1}^\infty nBW_n = T$$

and we are done. □

Now we are ready to give

Proof of item (tr6) in Theorem 1.1. We already know from Lemma 5.4 and (tr1) that both $A \stackrel{\text{def}}{=} \sup_{\sigma\in\Sigma} A_\sigma$ and $A' \stackrel{\text{def}}{=} \sup_{\sigma\in\Sigma} \mathrm{tr}_{\mathrm{Aff}}(A_\sigma)$ are well defined. As in the proof of Lemma 5.4, we may and do assume that each operator A_σ is non-negative. As usual, we express A in the form $A = \sum_{n=1}^\infty nBZ_n$ where $B \in \mathfrak{A}$. Then, from the very definition of $\mathrm{tr}_{\mathrm{Aff}}$ we deduce that $\mathrm{tr}_{\mathrm{Aff}}(A) = \sum_{n=1}^\infty n\, \mathrm{tr}_{\mathfrak{A}}(B) Z_n$. Further, the proof of Lemma 5.4, combined with Corollary 5.5, yields

$$A = \sum_{n=1}^\infty [\sup_{\sigma\in\Sigma} (A_\sigma \cdot Z_n)] Z_n.$$

Consequently, $nBZ_n = A \cdot Z_n = \sup_{\sigma\in\Sigma}(A_\sigma \cdot Z_n)$. Now the normality of $\mathrm{tr}_{\mathfrak{A}}$ implies that $n\,\mathrm{tr}_{\mathfrak{A}}(B)Z_n = \sup_{\sigma\in\Sigma}\mathrm{tr}_{\mathfrak{A}}(A_\sigma\cdot Z_n)$. But $\mathrm{tr}_{\mathfrak{A}}(A_\sigma\cdot Z_n) = \mathrm{tr}_{\mathrm{Aff}}(A_\sigma)\cdot Z_n$ (see (tr5)). We claim that $\sup_{\sigma\in\Sigma}(\mathrm{tr}_{\mathrm{Aff}}(A_\sigma)\cdot Z_n) = A'\cdot Z_n$. (To convince oneself of that, first note that inequality '$\leqslant$' is immediate. To see the reverse inequality, denote by A_1' and A_2', respectively, $\sup_{\sigma\in\Sigma}(A_\sigma \cdot Z_n)$ and $\sup_{\sigma\in\Sigma}(A_\sigma\cdot(I-Z_n))$, and observe that $A_\sigma \leqslant A_1' + A_2'$ and consequently $A' \leqslant A_1' + A_2'$, from which one infers that $A'\cdot Z_n \leqslant A_1'\cdot Z_n + A_2'\cdot Z_n$, but $A_2' \leqslant A'\cdot(I-Z_n)$ and thus $A_2'\cdot Z_n = 0$.) These observations lead us to $A'\cdot Z_n = n\,\mathrm{tr}_{\mathfrak{A}}(B)Z_n \in \mathfrak{A}$. So, Lemma 5.6 yields $A' = \sum_{n=1}^{\infty} n\,\mathrm{tr}_{\mathfrak{A}}(B)Z_n = \mathrm{tr}_{\mathrm{Aff}}(A)$. □

As we have noted in the introductory part, condition (tr6) is a counterpart of normality (in the terminology of Takesaki; see Definition 2.1 in Chapter V of [**20**]) of center-valued traces in finite von Neumann algebras. Thus, the question of whether it is possible to equip $\mathrm{Aff}(\mathfrak{A})$ with a (*naturally* defined) topology with respect to which the center-valued trace $\mathrm{tr}_{\mathrm{Aff}}$ is continuous naturally arises. This will be a subject of further investigations.

6. Trace-like mappings in $\mathrm{Aff}(\mathfrak{A})$ and the type of $\mathfrak{A}$. As Proposition 1.2 shows, finite type I von Neumann algebras may be characterised (among all von Neumann algebras) as those whose (full) sets of affiliated operators admit mappings which resemble center-valued traces. The aim of the section is to prove Proposition 1.2, which we now turn to.

Proof of Proposition 1.2. First observe that if $A \in \mathfrak{A}$, then $\varphi(A)$ is bounded and consequently $\varphi(A) \in \mathfrak{A}$. Indeed, it is enough to show this for non-negative $A \in \mathfrak{A}$. Such A satisfies $\|A\|I - A \geqslant 0$; therefore, $\varphi(\|A\|I - A)$ and $\varphi(A)$ are non-negative (by (b)). But it follows from (e) and (a) that

$$\varphi(\|A\|I - A) = \varphi(\|A\|I) - \varphi(A) = \|A\|I - \varphi(A),$$

which means that $0 \leqslant \varphi(A) \leqslant \|A\|I$ and hence $\varphi(A)$ is bounded. As $\varphi(A)$ commutes with each unitary operator in $\mathfrak{A}$ (by (f)), we conclude that $\varphi(A) \in \mathfrak{Z}(\mathfrak{A})$ for each $A \in \mathfrak{A}$. So, $\psi = \varphi\big|_{\mathfrak{A}}\colon \mathfrak{A} \to \mathfrak{Z}(\mathfrak{A})$ is linear (thanks to (a)) and satisfies all axioms of a center-valued trace (see (b), (d) and (e)), hence $\mathfrak{A}$ is finite. Note also that $\varphi(X) = \mathrm{tr}_{\mathfrak{A}}(X)$ for each $X \in \mathfrak{A}$.

Suppose that $\mathfrak{A}$ is not type I. Then one can find a non-zero projection $Z \in \mathfrak{Z}(\mathfrak{A})$ such that $\mathfrak{A}_0 \stackrel{\mathrm{def}}{=} \mathfrak{A}Z$ is type II_1. Recall that $\mathrm{tr}_{\mathfrak{A}}\big|_{\mathfrak{A}_0}$ coincides with the center-valued trace $\mathrm{tr}_{\mathfrak{A}_0}$ of $\mathfrak{A}_0$.

Every type II_1 von Neumann algebra $\mathfrak{W}$ has the following property: for each projection $P \in \mathfrak{W}$ and an operator $C \in \mathfrak{Z}(\mathfrak{W})$ such that $0 \leqslant C \leqslant \mathrm{tr}_{\mathfrak{W}}(P)$ there exists a projection $Q \in \mathfrak{W}$ for which $Q \leqslant P$ and $\mathrm{tr}_{\mathfrak{W}}(Q) = C$ (to convince oneself of that, see Theorem 8.4.4 and item (vii) of Theorem 8.4.3, both in [**7**]). Involving this property, we by induction define a sequence $(P_n)_{n=1}^{\infty}$

of projections in $\mathfrak{A}_0$ as follows: $P_1 \in \mathfrak{A}_0$ is arbitrary such that $\operatorname{tr}_{\mathfrak{A}_0}(P_1) = \frac{1}{2}Z$; and for $n > 1$, $P_n \in \mathfrak{A}_0$ is such that $P_n \leqslant Z - \sum_{k=1}^{n-1} P_k$ and $\operatorname{tr}_{\mathfrak{A}_0}(P_n) = \frac{1}{2^n}Z$. Observe that the projections P_n, $n \geqslant 1$, are mutually orthogonal and for any $N \geqslant 1$,

$$\operatorname{tr}_{\mathfrak{A}}\Big(\sum_{k=1}^{N} 2^k P_k\Big) = \sum_{k=1}^{N} 2^k \operatorname{tr}_{\mathfrak{A}_0}(P_k) = NZ.$$

Now for $N \geqslant 0$, put $T_N \stackrel{\text{def}}{=} \sum_{k=N+1}^{\infty} 2^k P_k$ (the series understood pointwise, similarly as in Definition 3.2). As each P_k belongs to $\mathfrak{A}_0$, we see that $T_N \in \operatorname{Aff}(\mathfrak{A}_0)$. Moreover, T_N is non-negative and we infer from axiom (c) that

$$\varphi(T_0) = \varphi\Big(\sum_{k=1}^{N} 2^k P_k\Big) + \varphi(T_N) = \operatorname{tr}_{\mathfrak{A}}\Big(\sum_{k=1}^{N} 2^k P_k\Big) + \varphi(T_N) = NZ + \varphi(T_N).$$

Therefore, for $\xi \in \mathcal{D}(\varphi(T_0)) = \mathcal{D}(\varphi(T_N))$, we get:

$$\langle \varphi(T_0)\xi, \xi\rangle = N\|Z\xi\|^2 + \langle \varphi(T_N)\xi, \xi\rangle \geqslant N\|Z\xi\|^2$$

(here $\langle \cdot, - \rangle$ denotes the inner product in $\mathcal{H}$). Since N can be arbitrarily large, the above implies that $Z\xi = 0$ for every $\xi \in \mathcal{D}(\varphi(T_0))$. But this is impossible, because $Z \neq 0$ and $\mathcal{D}(\varphi(T_0))$ is dense in $\mathcal{H}$. The proof is complete. $\square$

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