SYSTEMS OF HOLOMORPHIC MULTIVALUED PROJECTIONS ON COMPLEX MANIFOLDS

by Kamil Drzyzga

Abstract. Let $M$ be a submanifold of a connected Stein manifold $X$. We construct a global system of holomorphic multivalued projections $X \to M$. In particular, for every locally bounded family $\mathcal{F} \subset \mathcal{O}(M)$ we get a continuous extension operator $\mathcal{F} \to \mathcal{O}(X)$.

1. Introduction. Let $M$ be a complex submanifold of a Stein manifold $X$. Using Bishop’s ideas of multivalued projections we proved in [4] that for every domain $U \subset X$ there exists a linear continuous extension operator $\mathcal{O}(M) \to \mathcal{O}(U)$. Now, we will study the problem of existence of global holomorphic multivalued projections $X \to M$ (see Definition 5.1 and Theorem 5.5). Note that in the paper [2] the author suggested that a holomorphic multivalued projections could exist. In particular, we prove that there is a continuous extension operator $\mathcal{F} \to \mathcal{O}(X)$ for each locally bounded family $\mathcal{F} \subset \mathcal{O}(M)$ and moreover as an application we get a linear continuous extension operator $L^2(M) \to \mathcal{O}(X)$.

2. Auxiliary Results. Let $M$ be a $d$-dimensional analytic subset of a connected Stein manifold $X$. In the sequel we denote by $\text{Reg} M$ the set of regular points of $M$. For a compact $K \subset X$, its holomorphic hull (with respect to the space $\mathcal{O}(X)$ of all holomorphic functions on $X$) will be denoted by $\hat{K}_{\mathcal{O}(X)}$. Put $D(r) := \{z \in \mathbb{C} : |z| < r\}$, $D := D(1)$.

2010 Mathematics Subject Classification. 32D15.

Key words and phrases. Multivalued projections, holomorphic continuation.

$^{1}L^2_h(M) := \{f \in \mathcal{O}(M) : \int_M |f|^2 < \infty\}.$
**Definition 2.1.** Let $f \in \mathcal{O}(X, \mathbb{C}^k)$. We say that a set $P \subset P_0 := M \cap f^{-1}(\mathbb{D}^k)$ is an analytic polyhedron in $M$ ($P \in \mathcal{P}(M, k, f)$) if $P \subset M$ and $P$ is the union of a family of connected components of $P_0$.

We say that an analytic polyhedron $P \in \mathcal{P}(M, k, f)$ is special if $d = k$.

**Theorem 2.2 (cf. [2]).** Assume that $P \in \mathcal{P}(M, k, f)$ and $S \subset P$, $T \subset f^{-1}(\mathbb{D}^k)$ are compact. Then there exists a special analytic polyhedron $Q \in \mathcal{P}(M, d, g)$ such that $S \subset Q \subset P$ and $g(T) \subset \mathbb{D}^d$.

**Theorem 2.3 (cf. [2]).** Assume that $X$ is Stein, $T \subset X$ is compact, and $U$ is an open neighborhood of $T$ such that $(U \setminus T) \cap \overline{\mathcal{O}(X)} = \emptyset$. Let $A$ stand for the closure of $\mathcal{O}(U)|_T$ in the space $\mathcal{C}(T)$ of all complex valued continuous functions on $T$. Then for every non-zero homomorphism $\xi : A \rightarrow \mathbb{C}$ there exists an $x_0 \in T$ such that $\xi(f) = f(x_0)$ for every $f \in A$. Consequently (cf. [1], Chapter I, Section II, Corollary 10), if $w_1, \ldots, w_m \in A$ have no common zeros on $T$, then there exist $c_1, \ldots, c_m \in A$ such that $c_1w_1 + \cdots + c_mw_m = 1$.

**Definition 2.4 (cf. [2]).** A continuous map $f : X \rightarrow Y$, where $X, Y$ are topological spaces, is called almost proper if each connected component of $f^{-1}(S)$ is compact for every compact subset $S$ of $Y$.

**Theorem 2.5 (cf. [2]).** Let $Y$ be a 0-dimensional analytic subset of $\text{Reg}(M)$. Then there exists an $f \in \mathcal{O}(X, \mathbb{C}^d)$ such that $f|_{\overline{M}}$ is almost proper and the mapping $f$ gives local coordinates on $M$ at $x$ for each $x \in Y$.

**Theorem 2.6 (cf. [2]).** Assume that $M$ is pure $d$-dimensional and let $f \in \mathcal{O}(X, \mathbb{C}^d)$ be such that $f|_{\overline{M}}$ is almost proper. Let $\{S_j\}_{j=1}^\infty$ be an increasing sequence of compact subsets of $M$, each of which has finitely many connected components and $\bigcup_{j=1}^\infty S_j = M$. Let $\alpha : \mathbb{N} \rightarrow \mathbb{R}_0^+$ such that

$$S_j \subset F_j := M \cap f^{-1}(\mathbb{D}^d(\alpha(j)))$$

for all $j \in \mathbb{N}$. Let $H_j$ be the union of all those connected components of $F_j$ which intersect $S_j$. Then $H_j$ is compact. For each $j \in \mathbb{N}$ put

$$G_j := (H_{j+1} \cap F_j) \setminus H_j.$$

Let $\{g_j\}_{j=1}^\infty \subset \mathcal{O}(M)$ and $\{\varepsilon_j\}_{j=1}^\infty \subset \mathbb{R}_0^+$. Then there exists an $s \in \mathcal{O}(M)$ such that

$$|s(x) - g_j(x)| < \varepsilon_j, \quad x \in G_j, \quad j \in \mathbb{N}.$$ 

Moreover, given a countable set $A \subset M$, the function $s$ can be chosen to have different values modulo $2\pi i$, i.e. $e^{s(x)} \neq e^{s(y)}$ for $x, y \in M$ and $x \neq y$.

**Remark 2.7.** Observe that:

(a) $H_j \subset H_{j+1}$ for $j \in \mathbb{N}$;
(b) $\bigcup_{j=1}^\infty H_j = M$.  

3. Symmetric products. The aim of this section is to present some properties of the symmetric products. Details can be found in [7], Appendix V.

Let $X$ be a Hausdorff topological space. We define an equivalence relation on $X^k$ by $(x_1, \ldots, x_k) \sim (y_1, \ldots, y_k)$ is a reordering of $(x_1, \ldots, x_k)$. $\overrightarrow{X^k} := X^k / \sim$ is called the $k$-symmetric product of $X$. In the case $k = 1$, we get $\overrightarrow{X^1} = X$. Now, we define the projection $\pi : X^k \rightarrow \overrightarrow{X^k}$, $\pi(x) := [x]$. We put $[x_1, \ldots, x_k] := [(x_1, \ldots, x_k)]$, $\{[x_1, \ldots, x_k]\} := \{x_1, \ldots, x_k\}$.

Moreover, we put $[x_1: \mu_1, \ldots, x_\ell: \mu_\ell] := [x_1, \ldots, x_1, \underbrace{x_\ell, \ldots, x_\ell}_{\mu_\ell \text{-times}}]$, provided that $x_j \neq x_t$ for $j \neq t$, $\mu_1, \ldots, \mu_\ell \in \mathbb{N}$, $\mu_1 + \cdots + \mu_\ell = k$. We define $[A_1, \ldots, A_k] := \{[x_1, \ldots, x_k] : x_i \in A_i, \ i = 1, \ldots, k\}$.

The topology on $\overrightarrow{X^k}$ is defined by the basis $[U_1, \ldots, U_m], \ U_i \text{ is open in } X, \ i = 1, \ldots, k$.

Observe that $\pi$ is continuous, open, and $\overrightarrow{X^k}$ is Hausdorff.

DEFINITION 3.1. Let $Y$ be Hausdorff topological space and let $F : X \rightarrow \overrightarrow{Y^k}$ be continuous. Then we put $X_F^{(k)} := \{x \in X : \#\{F(x)\} = k\}$, $\chi_F := \max\{k : X_F^{(k)} \neq \emptyset\}$, $X_F := X_F^{(\chi_F)}$.

Note that $X_F$ is open.

PROPOSITION 3.2. Let $F$ be as above. Suppose that $a \in X_F$, $F(a) = [b_1: \mu_1, \ldots, b_\ell: \mu_\ell], \ \mu_1 + \cdots + \mu_\ell = k \ \ell := \chi_F$.

Then there is a neighborhood $U \subset X_F$ of $a$ and there are uniquely defined continuous functions $f_i : U \rightarrow Y$, $i = 1, \ldots, \ell$, such that $F(x) = [f_1(x): \mu_1, \ldots, f_\ell(x): \mu_\ell], \ x \in U$.

In the above situation, we will write $F = \mu_1 f_1 \oplus \cdots \oplus \mu_\ell f_\ell$ on $U$.

PROPOSITION 3.3. Let $F : X^k \rightarrow Y$ be continuous. Then $F$ is symmetric if and only if there exists a continuous function $\overrightarrow{F} : \overrightarrow{X^k} \rightarrow Y$ such that $F = \overrightarrow{F} \circ \pi$. 
4. Holomorphic multivalued functions and system of multivalued projections. All propositions below and their proofs are taken from [4]. We recall only those facts which will be used in this paper.

**Definition 4.1.** Let $M, N$ be complex manifolds with $M$ connected. We say a continuous mapping $F: M \rightarrow \overline{N^d}$ is holomorphic on $M$ ($F \in \mathcal{O}(M, \overline{N^d})$) if:

- $M \setminus M_F$ is thin, i.e. every point $x_0 \in M \setminus M_F$ has open connected neighborhood $V \subset M$ and a function $\varphi \in \mathcal{O}(V)$, $\varphi \not\equiv 0$, such that $(M \setminus M_F) \cap V \subset \varphi^{-1}(0)$,
- for every $a \in M_F$, if $F = \mu_1f_1 \oplus \cdots \oplus \mu_{\ell}f_{\ell}$ on $V$ as in Proposition 3.2, then $f_1, \ldots, f_{\ell} \in \mathcal{O}(V)$.

If $M$ is disconnected, then we say that $F$ is holomorphic on $M$ if $F|_C \in \mathcal{O}(C, \overline{N^d})$ for any connected component $C \subset M$.

**Proposition 4.2.** Let $M, N, K$ be complex manifolds and let $f \in \mathcal{O}(M, N)$, $g \in \mathcal{O}(N, \overline{K^d})$. Assume that $f(M) \cap N_g \neq \emptyset$ and $M$ is connected. Then $g \circ f \in \mathcal{O}(M_g \circ f, \overline{K^d})$.

**Proposition 4.3.** Let $f \in \mathcal{O}(M, \overline{N^d})$ and $g \in \mathcal{O}(N^d, K)$ be symmetric. Then $\overline{g^\circ f} \in \mathcal{O}(M, K)$.

**Theorem 4.4 (cf. [2]; see also [6], Chapter 7).** Assume that $P \in \mathcal{P}(M, d, f)$ is special. Then there exist a $k \in \mathbb{N}$ and a holomorphic mapping $\omega: \mathbb{D}^d \rightarrow \mathbb{P}^k$ such that:

- $f^{-1}(z) \cap P = \{\omega(z)\}$, $z \in \mathbb{D}^d$,
- $\#\{\omega(z)\} = k$ for $z \in \mathbb{D}^d \setminus \Sigma'$, where $\Sigma'$ is a proper analytic set.

The number $k$ in the above theorem is called the multiplicity of $f$ on $P$.

**Definition 4.5.** Let $M$ be an analytic submanifold of a manifold $X$. Let $U \subset X$ be a domain such that $U \cap M \neq \emptyset$. We say a holomorphic function $\Delta: U \rightarrow \overline{(M \times \mathbb{C})^d}$ is a holomorphic multivalued projection $U \rightarrow M$ if for any $x \in U \cap M$ such that $\Delta(x) = [(x_1, z_1), \ldots, (x_n, z_n)]$ we have $x_{j_0} = x$ for some $j_0 \in \{1, \ldots, n\}$ and $z_j = 0$ for any $j \in \{1, 2, \ldots, n\} \setminus \{j_0\}$.

Let $\mathfrak{P}$ denote the set of all holomorphic multivalued projections $U \rightarrow M$. Then we define the map

$$\Xi: (U \cap M) \times \mathfrak{P} \rightarrow \mathbb{C}, \quad \Xi(x, \Delta) := z_{j_0}.$$ 

Observe that $\Xi$ is well defined.
Definition 4.6. We say \( \Pi = (\Delta_s)_{s=1}^k \) is a system of holomorphic multivalued projections \( U \rightarrow M \) if \( \Delta_s : U \rightarrow (M \times \mathbb{C})^k_s, \ s = 1, \ldots , k \), are holomorphic multivalued projections and \( \sum_{s=1}^k \Xi(x, \Delta_s) = 1 \) for any \( x \in U \cap M \).

Theorem 4.7. Assume that there exists a system \( \Pi \) of holomorphic multivalued projections on \( U \). Then there exists a linear continuous operator
\[
L_\Pi : \mathcal{O}(M) \rightarrow \mathcal{O}(U)
\]
such that \( L_\Pi(u)(x) = u(x) \) for \( x \in U \cap M \).

Theorem 4.8. Let \( M \) be an analytic submanifold of a Stein manifold \( X \). Let \( U \) be a relatively compact domain of \( X \) such that \( U \cap M \neq \emptyset \). Then there exists a system of multivalued holomorphic projections \( U \rightarrow M \). Theorems 4.7 and 4.8 immediately imply the following result.

Theorem 4.9. Let \( M \) be an analytic submanifold of a Stein manifold \( X \). Let \( U \) be a relatively compact domain of \( X \) such that \( U \cap M \neq \emptyset \). Then there exists a linear continuous extension operator \( L : \mathcal{O}(M) \rightarrow \mathcal{O}(U) \).

Proposition 4.10. Let \( \omega, f, X, P \) be as above. Additionally assume that \( f(U) \subset \mathbb{D}^d \), where \( U \subset X \) is a domain and \( U \cap P \neq \emptyset \). Then \( \omega \circ f|_U \in \mathcal{O}(U, P^k) \).

Proposition 4.11. Let \( \omega, f, X, P \) be as above. Then \( \omega \circ f|_P \in \mathcal{O}(P, P^k) \).

5. Global system of holomorphic multivalued projections. Let \( X \) be a connected complex manifold and \( M \) be a complex submanifold.

Definition 5.1. A sequence \( \Pi = (\Delta_{s,j})_{(s,j) \in \{1, \ldots , r\} \times \mathbb{N}} \) is called a global system of holomorphic multivalued projections \( X \rightarrow M \) if for each \( j \in \mathbb{N} \) the mapping \( \Delta_{s,j} : U_j \rightarrow (M \times \mathbb{C})^{k_{s,j}}_j (k_{s,j} \in \mathbb{N}) \) is a holomorphic multivalued projection (in the sense of Definition 4.5), \( s = 1, \ldots , r \), having the following properties
(a) \( U_j \subset X \) is a domain with \( U_j \cap M \neq \emptyset, U_j \subset U_{j+1}, \bigcup_{j \in \mathbb{N}} U_j = X \);
(b) \( \lim_{n \rightarrow \infty} \sum_{s=1}^r \Xi(x, \Delta_{s,n}) = 1, \ x \in M \).

Remark 5.2. Let \( \Pi = (\Delta_{s,j})_{(s,j) \in \{1, \ldots , r\} \times \mathbb{N}} \) be as above.
(a) For each \( j \in \mathbb{N} \) we get a linear continuous operator (cf. the proof of Theorem 4.7) in [4.]
\[
L_{\Pi,j} : \mathcal{O}(M) \rightarrow \mathcal{O}(U_j), \quad L_{\Pi,j} := \sum_{s=1}^r \hat{u}_{s,j} \circ \Delta_{s,j}, \text{ where}
\]
\[
\hat{u}_{s,j} : (M \times \mathbb{C})^{k_{s,j}} \rightarrow \mathbb{C}, \quad \hat{u}_{s,j}((\xi_1, \lambda_1), \ldots , (\xi_{k_{s,j}}, \lambda_{k_{s,j}})) = \sum_{m=1}^{k_{s,j}} u(\xi_m)\lambda_m.
\]
(b) Using Definition 5.1,(b), for \( u \in \mathcal{O}(M) \) and \( x \in M \) we get
\[
\lim_{j \to \infty} L_{\Pi,j}(u)(x) = \lim_{j \to \infty} \sum_{s=1}^{r} \frac{1}{L_{s,j}} \circ \Delta_{s,j}(x) = \lim_{j \to \infty} \sum_{s=1}^{r} u(x) \Xi(x, \Delta_{s,j}) = u(x).
\]

Let \( \emptyset \neq \mathcal{F} \subset \mathcal{O}(M) \).

**Definition 5.3.** We say a global system of holomorphic multivalued projections \( \Pi = (\Delta_{s,j})_{(s,j) \in \{1, \ldots, N\} \times \mathbb{N}} \) is an \( \mathcal{F} \)-extension if for each \( u \in \mathcal{F} \) the sequence \( (L_{\Pi,j}(u))_{j=1}^{\infty} \) converges locally uniformly in \( X \).

Set \( L_{\Pi}(u) := \lim_{j \to \infty} L_{\Pi,j}(u), \ u \in \mathcal{F} \).

**Remark 5.4.** Let \( \Pi = (\Delta_{s,j})_{(s,j) \in \{1, \ldots, N\} \times \mathbb{N}} \) be an \( \mathcal{F} \)-extension.

(a) By Remark 5.2(b), \( L_{\Pi} : \mathcal{F} \to \mathcal{O}(X) \) is an extension operator.
(b) If \( u, v \in \mathcal{F} \) and \( u + v \in \mathcal{F} \), then \( L_{\Pi}(u + v) = L_{\Pi}(u) + L_{\Pi}(v) \).
(c) If \( u \in \mathcal{F} \), \( \alpha \in \mathbb{C} \) and \( \alpha u \in \mathcal{F} \), then \( L_{\Pi}(\alpha u) = \alpha L_{\Pi}(u) \).
(d) If \( \mathcal{F} \) is a vector space, then \( L_{\Pi} \) is linear.
(e) If \( u_1, \ldots, u_m \in \mathcal{F} \) are linearly independent (in \( \mathcal{O}(M) \)), then the formula
\[
L_{\Pi}(\alpha_1 u_1 + \cdots + \alpha_m u_m) := \alpha_1 L_{\Pi}(u_1) + \cdots + \alpha_m L_{\Pi}(u_m), \quad \alpha_1, \ldots, \alpha_m \in \mathbb{C},
\]
extends the operator \( L_{\Pi} \) to the vector space \( \text{span}\{u_1, \ldots, u_m\} \).

The main result of the paper is the following theorem.

**Theorem 5.5.** Let \( X \) be a Stein manifold and \( \mathcal{F} \subset \mathcal{O}(M) \) be locally bounded (i.e. \( \sup_{u \in \mathcal{F}} \|u\|_K < +\infty \) for every compact set \( K \subset M \), e.g. \( \mathcal{F} \) is finite). Then there exists an \( \mathcal{F} \)-extension \( \Pi = (\Delta_{s,j})_{(s,j) \in \{1, \ldots, d\} \times \mathbb{N}} \) with \( d := \dim M \). Consequently, there exists a continuous extension operator \( L_{\Pi} : \mathcal{F} \to \mathcal{O}(X) \).

**Corollary 5.6.** Let \( X \) be a Stein manifold and \( \mathcal{V} \) be a finitely dimensional vector subspace of \( \mathcal{O}(M) \). Then there exists a linear continuous extension operator \( L : \mathcal{V} \to \mathcal{O}(X) \).

**Proposition 5.7.** Assume that \( \mathcal{H} \subset \mathcal{O}(M) \) is a Hilbert space such that the unit ball \( B := \{ f \in \mathcal{H} : \|f\|_H \leq 1 \} \) is locally uniformly bounded and the convergence in the sense of \( \mathcal{H} \) implies the locally uniform convergence in \( M \).

Then there exists a linear continuous extension operator \( L : \mathcal{H} \to \mathcal{O}(X) \). In particular, there exists a linear continuous extension operator \( L : L^2_h(M) \to \mathcal{O}(X) \).

**Proof.** We put \( \mathcal{F} := B \). By Theorem 5.5 there exists a continuous extension operator \( \tilde{L} : \mathcal{F} \to \mathcal{O}(X) \). Moreover, since \( \text{span}(\mathcal{F}) = \mathcal{H} \), we conclude that there exists a linear continuous extension operator \( L : \mathcal{H} \to \mathcal{O}(X) \).

\[ 2\|f\|_K := \sup_K |f| .\]
Indeed, suppose that $(f_j)_{j=1}^\infty \subset \mathcal{F}$ is an orthonormal basis of $\mathcal{H}$. Set $\tilde{f}_j := \tilde{L}(f_j)$. Let $f \in \mathcal{H}$ be such that $f = \sum_{j=1}^\infty c_j f_j$. Put

- $L(f) := \sum_{j=1}^\infty c_j \tilde{f}_j = C_f \tilde{L}(f/C_f)$,
- $s_N := \sum_{j=1}^N c_j \tilde{f}_j$,

where $C_f := \|f\|_H$. Since $(f_j)_{j=1}^\infty$ is orthonormal, hence

- $f_j, \frac{c_j}{\|f_j\|} \tilde{f}_j \in \mathcal{F}$,
- $\frac{c_j}{\|f_j\|} \tilde{f}_j + \frac{c_k}{\|f_k\|} \tilde{f}_k \in \mathcal{F}$ for $j, k \in \mathbb{N}, j \neq k$.

Therefore, $C_f \tilde{L}(s_N/C_f) = \sum_{j=1}^N c_j \tilde{f}_j$. As $s_N/C_f \to f/C_f$ locally uniformly and $s_N/C_f \in \mathcal{F}$, we get $L(f) = C_f \tilde{L}(f/C_f)$. By assumption on topologies, $L$ is continuous.

Now, assume that $\mathcal{H} = L^2_k(M)$. It is known that for any compact set $K \subset M$ there are $C_K > 0$ and open neighborhood $K \subset \Omega \subset M$ such that $\|f\|_K \leq C_K \|f\|_{L^2(\Omega,dV)}$. It follows that $B$ is locally uniformly bounded. □

**Corollary 5.8.** Let $X \in \{\mathbb{D}^n, \mathbb{B}_n\}$. There exists a linear continuous extension operator $L : L^2_k(\mathbb{D}^3) \to \mathcal{O}(X)$.

**Proof of Theorem 5.5.** Let $Y$ be an arbitrary 0-dimensional analytic subset of $M$. By Theorem 5.1 there exists a mapping $f \in \mathcal{O}(X, \mathbb{C}^d)$ such that $f|_M$ is almost proper and for each $x \in Y$ the mapping $f$ gives local coordinates on $M$ at $x$.

Let $S_k, \alpha(k), F_k, H_k$ and $G_k$ be as in Theorem 2.6. Observe that $Q_k := \text{int} H_k = H_k \cap f^{-1}(\mathbb{D}^d(\alpha(k)))$ is a special analytic polyhedron. Let $\lambda_k$ denote the multiplicity of $f$ in $Q_k$, defined via Theorem 4.1 with $\omega_k : \mathbb{D}^d(\alpha(k)) \to Q_k^\psi$. Set $\omega_k(f(x)) = [x_1^k, \ldots, x_k^k]$ (counted with multiplicities), $x \in f^{-1}(\mathbb{D}^d(\alpha(k)))$.

Observe that for arbitrary $x \in X$, the set $M \cap f^{-1}(f(x))$ is discrete. Let $(x_i)_{i=1}^\infty = M \cap f^{-1}(f(x))$ (points are counted with multiplicities). We assume that $x_1 = x$ for $x \in M$. Let

$$\Xi_k(x) := \{ j \in \mathbb{N} : x_j \in H_k \}.$$ 

Observe that for each $k \in \mathbb{N}$ and $x \in f^{-1}(Q_k)$ the set $\Xi_k(x)$ is finite and $\{x_j : j \in \Xi_k(x)\} = \{x_k^1, \ldots, x_k^k\}$.

Put $g_k := \lambda_{k+1} + k^2 + 1, k \in \mathbb{N}$. By Theorem 2.6 there exists an $f_{d+1} \in \mathcal{O}(X)$ such that $|f_{d+1} - g_k| < 1$ on $G_k$, $k \in \mathbb{N}$, and the function $w := e^{-f_{d+1}}$ separates points in $M \cap f^{-1}(f(x))$ for all $x \in Y$.

$\mathbb{D} \simeq \{(z,0) \in \mathbb{C}^n : z \in \mathbb{D}\}$.
Lemma 5.9. Let $\mathcal{F} \subset \mathcal{O}(M)$ be locally bounded. Then there exists a function $f_{d+1}^{\ast} \in \mathcal{O}(X)$ such that if $h := e^{f_{d+1}^{\ast}}$ and

$$\tilde{\varphi}_k(x) := \sum_{j \in \Xi_k(x)} \varphi(x_j) h(x_j) \prod_{\mu \in \Xi_k(x)} \left(1 - \frac{w(x_\mu)}{w(x)}\right), \quad \varphi \in \mathcal{F}, x \in X, k \in \mathbb{N},$$

then for every domain $U \subset X$ such that $U \cap M \neq \emptyset$,

- there exists a $k_0 \in \mathbb{N}$ such that $\tilde{\varphi}_k \in \mathcal{O}(U)$ for $k \geq k_0$ and
- the sequence $(\tilde{\varphi}_k)_{k=1}^\infty$ converges uniformly on $U$.

Suppose for a moment that the lemma is proved. Let $\tilde{\varphi}(x) := \lim_{k \to \infty} \tilde{\varphi}_k(x)$, $x \in U$. Then $\tilde{\varphi} \in \mathcal{O}(U)$. Since $x_1 = x$ for $x \in M \cap U$, we get

$$\tilde{\varphi}(x) = \varphi(x) h(x) \prod_{\mu=2}^\infty \left(1 - \frac{w(x_\mu)}{w(x)}\right) = \varphi(x) h(x) \tilde{w}_1(x), \quad x \in M \cap U,$$

where

$$\tilde{w}_1(x) := \prod_{\mu=2}^\infty \left(1 - \frac{w(x_\mu)}{w(x)}\right), \quad x \in M.$$  

Observe that the condition $|f_{d+1} - (\lambda_{k+1} + k^2 + 1)| < 1$ on $G_k$, $k \in \mathbb{N}$, implies that the function $\tilde{w}_1$ is well-defined (cf. the estimate of the function $B$ in the proof of Lemma 5.9). Hence $\tilde{w}_1 \in \mathcal{O}(M)$. Notice that $\tilde{w}_1(x) \neq 0$ for $x \in Y$.

We move to the main part of proof.

First we take $Y = Y_1 \subset M$ having a point in each connected component of $M$. We get a function $\tilde{w}_1 \in \mathcal{O}(M)$ such that $\tilde{w}_1(x) \neq 0$ for each $x \in Y_1$. In particular $M_1 := \{x \in M : \tilde{w}_1(x) = 0\}$ is $(d-1)$-dimensional analytic subset of $M$. Next we take $Y_2 \subset M_1$ having a point in each connected component of $\text{Reg}(M_1)$. We get $\tilde{w}_2 \in \mathcal{O}(M)$ such that $\tilde{w}_2(x) \neq 0$ for each $x \in Y_2$. Thus $M_2 := \{x \in M : \tilde{w}_1(x) = \tilde{w}_2(x) = 0\}$ is a $(d-2)$-dimensional analytic subset of $M$. We repeat the procedure and we obtain $\tilde{w}_1, \ldots, \tilde{w}_d \in \mathcal{O}(M)$ without common zeros on $M$. By Theorems 2, 3, there exist $c_1, \ldots, c_d \in \mathcal{O}(M)$ such that $c_1 \tilde{w}_1 + \ldots + c_d \tilde{w}_d = 1$ on $M$. Assume that $h_s$ is constructed with respect to the family $\mathcal{F}_s := \{uc_s : u \in \mathcal{F}\}$.

We get $f_s$, $H_{s,k}$, $Q_{s,k}$, $\omega_{s,k}$, $\lambda_{s,k}, (x_{s,j})_{j=1}^{\lambda_{s,k}}, (x_{s,\nu})_{\nu=1}^\infty$, $\Xi_{s,k}(\cdot)$, $w_s$, $\tilde{w}_s$ for $s = 1, \ldots, d, k \geq 1$.

Now we are going to construct a global system of holomorphic multivalued projections on $X \to M$ (cf. Definition 5.1). Fix arbitrary domains $U_j \subset U_{j+1} \subset X$ such that $\bigcup_{j=1}^\infty U_j = X$, $U_j \cap M \neq \emptyset$. Let $(t_j)_{j=1}^\infty \subset \mathbb{N}$ be such that

- $f_s(U_j) \subset \mathbb{D}^d(\alpha_s(t_j))$, where $\alpha_s(t_j) \in (0, +\infty)$;
- $U_j \cap M \subset Q_{s,t_j}$;
The construction of a global system of holomorphic multivalued projections has been finished.

Proof of the Lemma 4.8. Fix an arbitrary domain $U \subset X$, $U \cap M \neq \emptyset$ and $k_0 \in \mathbb{N}$ such that $f^{-1}(\mathbb{D}^d(\alpha(k_0)))$. Let $f^{d+1}_d$ be for a moment arbitrary and let $\varphi \in F$. Take a $k \geq k_0$.

First, we are going to prove that $\tilde{\varphi}_k \in O(U)$. Note that if $x \in U$ and $j \in \Xi_k(x)$, then $x_j \in H_k \cap f^{-1}(\mathbb{D}^d(\alpha(k))) = Q_k$. Hence $\{x_j : j \in \Xi_k(x)\} = \{x^1_k, \ldots, x^k_k\} = \{\omega_k(f(x))\}, x \in U$. Moreover,

$$\tilde{\varphi}_k(x) = w^{1-k}(x) = \sum_{\nu=0}^{k-1} \sum_{\nu=0}^{k-1} \tilde{S}_\nu(\omega_k(f(x))) w^\nu(x), \quad x \in U,$$

where

$$S_{\nu}(t) := \sum_{j=1}^\lambda \varphi(t_j) h(t_j),$$

$$S_{\nu}(t) := (-1)^{\lambda_k - 1 - \nu} \sum_{j=1}^\lambda \varphi(t_j) h(t_j) \sigma_{k-\nu}(w(t_1), \ldots, w(t_{j-1}), w(t_{j+1}), \ldots, w(t_\lambda)), \quad \nu = 0, \ldots, \lambda_k - 2, t = (t_1, \ldots, t_\lambda) \in Q^k_k.$$
and \( \sigma_1, \ldots, \sigma_{\lambda_k} : \mathbb{C}^{\lambda_k-1} \to \mathbb{C} \) are standard symmetric polynomials. Consequently, by Proposition 4.10 we conclude that \( \tilde{\varphi}_k \in \mathcal{O}(U) \).

Now we are going to find a function \( f^*_{d+1} \in \mathcal{O}(U) \) (independent of \( U \)) such that \( (\tilde{\varphi}_k)_{k=1}^\infty \) converges uniformly on \( U \).

We construct \( f^*_{d+1} \) via Theorem 2.6 in such a way that \( |f^*_{d+1} - k^2 \beta_k \lambda_k - 1| < 1 \) on \( G_k \), where \( \beta_k \geq \sup \{ \sup_{G_k} |\varphi| : \varphi \in \mathcal{F} \} \). Our aim is to prove that \( \tilde{\varphi}(x) - \tilde{\varphi}_k(x) \to 0 \) uniformly on \( U \) as \( l \to k \to +\infty \). Take \( l > k \geq k_0 \). For \( x \in U \) write

\[
\tilde{\varphi}(x) - \tilde{\varphi}_k(x) = \sum_{j \in \Xi_k(x) \setminus \Xi_k(x)} \varphi(x_j) h(x_j) \prod_{\mu \in \Xi_k(x), \mu \neq j} \left( 1 - \frac{w(x_\mu)}{w(x)} \right) + \sum_{j \in \Xi_k(x)} \varphi(x_j) h(x_j) \prod_{\mu \in \Xi_k(x), \mu \neq j} \left( 1 - \frac{w(x_\mu)}{w(x)} \right) \cdot \left( \prod_{\mu \in \Xi_k(x), \mu \neq j} \left( 1 - \frac{w(x_\mu)}{w(x)} \right) \right) - 1 = I_{k,l}(x) + J_{k,l}(x).
\]

We have

\[
|I_{k,l}(x)| \leq \left( \sum_{j \notin \Xi_k(x)} |\varphi(x_j) h(x_j)| \right) \prod_{\mu \in \mathbb{N}} \left( 1 + \frac{|w(x_\mu)|}{|w(x)|} \right) =: A_k(x) B(x),
\]

\[
|J_{k,l}(x)| \leq \left( \sum_{j \in \mathbb{N}} |\varphi(x_j) h(x_j)| \right) \left( \prod_{\mu \in \mathbb{N}} \left( 1 + \frac{|w(x_\mu)|}{|w(x)|} \right) \right) \cdot \left( \prod_{\mu \notin \Xi_k(x)} \left( 1 + \frac{|w(x_\mu)|}{|w(x)|} \right) \right) - 1 =: C(x) D(E_k(x) - 1).
\]

Observe that \( M \cap f^{-1}(\mathbb{D}^d(\alpha(k_0))) \subset Q_{k_0} \cup \bigcup_{s=k_0}^\infty \text{int}G_s \). Let

\[
\gamma := \max_U \text{Re}f_{d+1}, \quad \delta := \max_{H_{k_0}} (-\text{Re}f_{d+1}).
\]

Observe that if \( x \in U \) and \( x_\mu \in \text{int}G_s \), then we have

\[
\log \left( 1 + \frac{|w(x_\mu)|}{|w(x)|} \right) \leq \frac{|w(x_\mu)|}{|w(x)|} = e^{-\text{Re}f_{d+1}(x_\mu)} e^{\text{Re}f_{d+1}(x_\mu)} \leq e^{-\lambda_{s+1} - s^2 + \gamma} \leq e^{\frac{\gamma}{\lambda_{s+1} s^2}}.
\]

If \( x_\mu \in Q_{k_0} \), then

\[
\log \left( 1 + \frac{|w(x_\mu)|}{|w(x)|} \right) \leq \frac{|w(x_\mu)|}{|w(x)|} = e^{-\text{Re}f_{d+1}(x_\mu)} e^{\text{Re}f_{d+1}(x_\mu)} \leq e^{\delta + \gamma}.
\]
Thus for all \( x \in U \) we have
\[
\log B(x) = \sum_{\mu: x_\mu \in Q_{k_0}} \log \left( 1 + \frac{|w(x_\mu)|}{|w(x)|} \right) + \sum_{s=k_0}^{\infty} \sum_{\mu: x_\mu \in \text{int} G_s} \log \left( 1 + \frac{|w(x_\mu)|}{|w(x)|} \right)
\]
\[
\leq \lambda_{k_0} e^{\delta + \gamma} + \sum_{s=k_0}^{\infty} \frac{\alpha^\gamma}{s^2},
\]
and therefore the function \( B \) is uniformly bounded on \( U \).

Similarly,
\[
C(x) = \sum_{\mu: x_\mu \in Q_{k_0}} |\varphi(x_\mu) h(x_\mu)| + \sum_{s=k_0}^{\infty} \sum_{\mu: x_\mu \in \text{int} G_s} |\varphi(x_\mu) h(x_\mu)|
\]
\[
\leq M + \sum_{s=k_0}^{\infty} \sum_{\mu: x_\mu \in \text{int} G_s} \beta_s e^{-\text{Re} f_{s+1}^2(x_\mu)} \leq M + \sum_{s=k_0}^{\infty} \sum_{\mu: x_\mu \in \text{int} G_s} \beta_s e^{-s^2 \lambda_s}
\]
\[
\leq M + \sum_{s=k_0}^{\infty} \sum_{\mu: x_\mu \in \text{int} G_s} \beta_s \frac{1}{\beta_s \lambda_s s^2} \leq M + \sum_{s=k_0}^{\infty} \frac{1}{s^2},
\]
where \( M := \lambda_{k_0} \sup_{Q_{k_0}} |\varphi h| \). On the other hand,
\[
A_k(x) = \sum_{s=k_0}^{\infty} \sum_{j: x_j \in \text{int} G_s} \beta_s h(x_j) = \sum_{s=k_0}^{\infty} \sum_{j: x_j \in \text{int} G_s} \beta_s e^{-\text{Re} f_{s+1}^2(x_j)}
\]
\[
\leq \sum_{s=k_0}^{\infty} \sum_{j: x_j \in \text{int} G_s} \beta_s e^{-\beta_s \lambda_{s+1} - s^2} \leq \sum_{s=k_0}^{\infty} \sum_{j: x_j \in \text{int} G_s} \beta_s \frac{1}{\beta_s \lambda_{s+1} s^2} \leq \sum_{s=k_0}^{\infty} \frac{1}{s^2},
\]
which proves that \( A_k(x) B(x) \to 0 \) uniformly on \( U \), when \( k \to +\infty \). We have proved that the product \( \prod_{\mu \in \mathbb{N}} \left( 1 + \frac{|w(x_\mu)|}{|w(x)|} \right) \) converges uniformly on \( U \). In particular,
\[
E_k(x) = \frac{\prod_{\mu \in \mathbb{N}} \left( 1 + \frac{|w(x_\mu)|}{|w(x)|} \right)}{\prod_{\mu \in \mathbb{N}} \left( 1 + \frac{|w(x_\mu)|}{|w(x)|} \right)} \to 1, \quad \text{uniformly for } x \in U.
\]

Observe that by Remark 5.3, we have the extension operator \( L_{ij} : F \to \mathcal{O}(X) \). Now we are going to check its continuity. Note that \( F \) and \( \mathcal{O}(X) \) are endowed with the locally uniform convergence topologies.
Continuity of $L_\Pi$. Let $F \ni \varphi_t \to \varphi \in F$ locally uniformly. Fix $\varepsilon > 0$ and compact set $K \subset X$. Observe that $F' := F \cup (\varphi_t - \varphi)^*_t$ is also locally bounded. Moreover, there is $L'_{\Pi} : F' \to \mathcal{O}(X)$ extension operator such that $L_{\Pi} = L'_{\Pi}$ on $F$. Indeed, the map $f_{d+1}^*$ is good for the both families $F$ and $F'$ if we take $\beta_k = \beta'_{k} \geq 2\sup\{\sup_{Q_0} |\varphi| : \varphi \in F\}$, where $\beta'_k$ is constant in the construction of the operator $L_{\Pi}$. For $x \in K$ we have

$$|L_{\Pi}(\varphi_t) - L_{\Pi}(\varphi))(x) = |L_{\Pi}(\varphi_t - \varphi))(x)$$

$$= \left| \sum_{s=1}^{d} \frac{1}{h_s(x)} \sum_{j=1}^{\infty} (\varphi_t - \varphi)(x_{s,j})c_s(x_{s,j})h_s(x_{s,j}) \prod_{\mu \in \mathbb{N}} \left(1 - \frac{w_s(x_{s,\mu})}{w_s(x)} \right) \right|$$

$$\leq \sum_{s=1}^{d} \frac{1}{h_s(x)} \prod_{\mu \in \mathbb{N}} \left(1 + \frac{|w_s(x_{s,\mu})|}{|w_s(x)|} \right) \sum_{j=1}^{\infty} |(\varphi_t - \varphi)(x_{s,j})c_s(x_{s,j})h_s(x_{s,j})|.$$}

Let $f(K) \subset D^d(a(k_0))$ and $k_1 \geq k_0$, where $k_0, k_1 \in \mathbb{N}$. By the proof of the previous lemma we get the following estimate

$$\prod_{\mu \in \mathbb{N}} \left(1 + \frac{|w_s(x_{s,\mu})|}{|w_s(x)|} \right) \leq \lambda_{k_0} e^{\delta + \gamma} + \sum_{s=k_0}^{\infty} \frac{e^{\gamma}}{s^2}.$$}

Since $K$ is compact, we conclude that the map $x \mapsto \frac{1}{h_s(x)} \prod_{\mu \in \mathbb{N}} \left(1 + \frac{|w_s(x_{s,\mu})|}{|w_s(x)|} \right)$ is bounded on $K$. On the other hand,

$$\sum_{j=1}^{\infty} |(\varphi_t - \varphi)(x_{s,j})c_s(x_{s,j})h_s(x_{s,j})| \leq M + \sum_{s=k_1}^{\infty} \frac{1}{s^2},$$

where $M := \lambda_{k_1} \sup_{Q_{k_1}} |(\varphi_t - \varphi)c_s h_s|$. Now we observe that if $k_1$ and $t$ are sufficiently large, we obtain

$$\|L_{\Pi}(\varphi_t) - L_{\Pi}(\varphi)\|_K \leq \varepsilon.$$}

Acknowledgements. I would like to thank anonymous reviewer for valuable advices on this paper. I would also like to thank my supervisor professor Marek Jarnicki for help in writing the paper.

References


Received July 26, 2016

Jagiellonian University
Faculty of Mathematics and Computer Science
Institute of Mathematics
Lojasiewicza 6
30-348 Kraków, Poland
e-mail: kamil.drzyzga@gmail.com