

## GENERALIZED ISOPERIMETRIC FVPS VIA CAPUTO APPROACH

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**Abstract.** In this paper, we study several fractional variational problems with functionals that contain  $n$  unknown functions with their higher order Caputo derivatives and Riemann–Liouville integrals. We prove generalized fractional Euler–Lagrange equations. We also study an isoperimetric problem with multiple constraints, and we find the optimality conditions. Some examples are provided to illustrate the applications of the results.

**1. Introduction and Preliminaries.** It is known that the fractional calculus arises in various fields of applied sciences. For more details see, [18–20, 25, 26]. Moreover, fractional differential equations play a central role in engineering sciences and applied mathematics in building mathematical models of many physical phenomena. See, for instance [1, 10, 15, 16, 29, 30, 32, 33].

The calculus of variations is concerned with finding a function for which a given functional attains an extremum value. Books [9, 13, 17, 21, 22, 24] may serve as an introduction to the calculus of variations for mathematicians and scientists, and the reader can find interesting results in geometry and differential equations therein. The calculus of variations plays an important role in problems arising in classical mechanics, economics, electrical engineering, urban planning and other fields. For details, see [14, 27, 34, 35] and the references therein. Some researchers study different fractional variational problems and several important research results have been obtained in [2–8, 11, 12, 23, 31, 36].

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The present paper is devoted to building necessary conditions of Euler–Lagrange type for fractional variational problem of the form:

$$J(y_1, \dots, y_n) = \int_{x_0}^{x_1} \left( L \circ_{\alpha_k^j}^{\gamma_k} [y_1, \dots, y_n]_{\beta_k^j} \right) (x) dx \rightarrow \text{extremum},$$

where  $\gamma_k > 0$ ,  $j - 1 < \alpha_k^j, \beta_k^j < j$ ;  $j, k = 1, 2, \dots, n$ ,  $n \in \mathbb{N}^*$  with  $\mathbb{N}^* = \mathbb{N} - \{0\}$ ,  $(y_k)_{k=1,2,\dots,n} \in S := \{(y_1, \dots, y_n) \in C^n([x_0, x_1]); y_k^{(i)}(x_0) = a_k^i, y_k^{(i)}(x_1) = b_k^i, i = 0, 1, \dots, n - 1, n \in \mathbb{N}^*\}$ , and

$$\begin{aligned} \alpha_k^j [y_1, \dots, y_n]_{\beta_k^j} := & \left( x, y_1(x), \dots, y_n(x), \right. \\ & C D_{x_0}^{\alpha_1^1} y_1(x), C D_{x_0}^{\alpha_1^2} y_1(x) \dots, C D_{x_0}^{\alpha_1^n} y_1(x), \\ & \dots \\ & C D_{x_0}^{\alpha_n^1} y_n(x), C D_{x_0}^{\alpha_n^2} y_n(x) \dots, C D_{x_0}^{\alpha_n^n} y_n(x), \\ & C D_{x_1}^{\beta_1^1} y_1(x), C D_{x_1}^{\beta_1^2} y_1(x), \dots, C D_{x_1}^{\beta_1^n} y_1(x), \\ & \dots \\ & C D_{x_1}^{\beta_n^1} y_n(x), C D_{x_1}^{\beta_n^2} y_n(x), \dots, C D_{x_1}^{\beta_n^n} y_n(x), \\ & \left. x_0 I_x^{\gamma_1} y_1(x), \dots, x_0 I_x^{\gamma_n} y_n(x) \right). \end{aligned}$$

We will consider the fractional isoperimetric problem that consists in maximizing (or minimizing) the above functional subject to the given boundary conditions and  $m$  fractional integral constraints:

$$I_q(y_1, \dots, y_n) = \int_{x_0}^{x_1} \left( F_q \circ_{\alpha_k^j}^{\gamma_k} [y_1, \dots, y_n]_{\beta_k^j} \right) (x) dx = l_q,$$

where  $q = 1, 2, \dots, m$ ,  $m < n$ , and  $l_q \in \mathbb{R}$ .

Finally, we will present the optimality conditions for a functions-time pair  $\left( (y_k)_{k=1,\dots,n}, T \right) \in \left\{ (C^n([x_0, x_1]), [x_0, x_1]) : y_k^{(i)}(x_0) = a_k^i \right\}$  to be an optimal solution to:

$$J^*(y_1, \dots, y_n, T) = \int_{x_0}^T \left( L^* \circ_{\alpha_k^j} [y_1, \dots, y_n] \right) dx,$$

where

$$\begin{aligned} \alpha_k^j[y_1, \dots, y_n] := & \left( x, y_1(x), \dots, y_n(x), \right. \\ & {}_{x_0}^C D_x^{\alpha_1^1} y_1(x), {}_{x_0}^C D_x^{\alpha_1^2} y_1(x), \dots, {}_{x_0}^C D_x^{\alpha_1^n} y_1(x), \\ & \dots \\ & \left. {}_{x_0}^C D_x^{\alpha_n^1} y_n(x), {}_{x_0}^C D_x^{\alpha_n^2} y_n(x), \dots, {}_{x_0}^C D_x^{\alpha_n^n} y_n(x) \right). \end{aligned}$$

We consider the both free and fixed end-point problems.

To the best of our knowledge, there are no papers dealing with this kind of high dimensional variational problems.

Now, we present some basic definitions and lemmas that we need to prove our main results.

**DEFINITION 1.** [19,25,26] The left and right Riemann–Liouville fractional integral operators of order  $\alpha > 0$  for an integrable function  $y$  on  $[x_0, x_1]$  are defined by:

$$(1) \quad {}_{x_0} I_x^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^x (x-t)^{\alpha-1} y(t) dt,$$

and

$$(2) \quad {}_x I_{x_1}^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_x^{x_1} (t-x)^{\alpha-1} y(t) dt,$$

respectively, where  $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$ .

**DEFINITION 2.** [19,25,26] The left and right Riemann–Liouville fractional derivative operators of order  $\alpha > 0$  for a function  $y : [x_0, x_1] \rightarrow \mathbb{R}$  are defined as:

$$(3) \quad {}_{x_0}^{RL} D_x^\alpha y(x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dx} \right)^{(n)} \int_{x_0}^x (x-t)^{n-\alpha-1} y(t) dt,$$

and

$$(4) \quad {}_x^{RL} D_{x_1}^\alpha y(x) = \frac{1}{\Gamma(n-\alpha)} \left( -\frac{d}{dx} \right)^{(n)} \int_x^{x_1} (t-x)^{n-\alpha-1} y(t) dt,$$

respectively, for  $n-1 < \alpha < n$ ,  $n \in \mathbb{N}^*$ .

**DEFINITION 3.** [19,25,26] The left and right Caputo fractional derivative operators of order  $\alpha > 0$  for a function  $y : [x_0, x_1] \rightarrow \mathbb{R}$ , which is at least  $n$ -times differentiable are defined as:

$$(5) \quad {}_{x_0}^C D_x^\alpha y(x) = \frac{1}{\Gamma(n-\alpha)} \int_{x_0}^x (x-t)^{n-\alpha-1} y^{(n)}(t) dt,$$

and

$$(6) \quad {}_x^C D_{x_1}^\alpha y(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^{x_1} (t-x)^{n-\alpha-1} (-1)^n y^{(n)}(t) dt,$$

respectively, for  $n-1 < \alpha < n$ ,  $n \in \mathbb{N}^*$ .

LEMMA 4. [19, 25, 26, 28] Assume that  $n-1 < \alpha < n$ ,  $n \in \mathbb{N}^*$  and  $f$  is of class  $C^n$  on  $[x_0, x_1]$ . Then its left and right Caputo derivatives of order  $\alpha$  are continuous on  $[x_0, x_1]$ .

LEMMA 5. [13, 24] Let  $f : [x_0, x_1] \rightarrow \mathbb{R}$  be a continuous function such that:

$$(7) \quad \int_{x_0}^{x_1} f(x) \sigma(x) dx = 0$$

holds for every  $\sigma \in C^n([x_0, x_1])$ ,  $n \geq 0$ , satisfying  $\sigma(x_0) = \sigma(x_1) = 0$ . Then,  $f(x) = 0$  on  $[x_0, x_1]$ .

LEMMA 6. [28] Let  $\alpha > 0$ ,  $p, q \geq 1$ , and  $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ . If  $g \in L^p([x_0, x_1])$  and  $f \in L^q([x_0, x_1])$ , then

$$(8) \quad \int_{x_0}^{x_1} g(x) {}_{x_0}I_x^\alpha f(x) dx = \int_{x_0}^{x_1} f(x) {}_xI_{x_1}^\alpha g(x) dx.$$

LEMMA 7. [28] Let  $n-1 < \alpha < n$ ,  $f, g : [x_0, x_1] \rightarrow \mathbb{R}$ ,  $f$  be of class  $C^n$ ,  $g$  and  ${}_x D_{x_1}^\alpha g$  be continuous functions on  $[x_0, x_1]$ . Then,

$$(9) \quad \int_{x_0}^{x_1} g(x) {}_{x_0}^C D_x^\alpha f(x) dx = \int_{x_0}^{x_1} f(x) {}_x^R L D_{x_1}^\alpha g(x) dx + \sum_{j=0}^{n-1} [{}_x^R L D_{x_1}^{\alpha+j-n} g(x) {}_x^R L D_{x_1}^{n-j-1} f(x)]_{x_0}^{x_1},$$

$$(10) \quad \int_{x_0}^{x_1} g(x) {}_x^C D_{x_1}^\alpha f(x) dx = \int_{x_0}^{x_1} f(x) {}_{x_0}^R L D_x^\alpha g(x) dx + \sum_{j=0}^{n-1} [(-1)^{(n+j)} {}_{x_0}^R L D_x^{\alpha+j-n} g(x) {}_{x_0}^R L D_x^{n-j-1} f(x)]_{x_0}^{x_1}.$$

REMARK 8. If  $f(x_0) = f(x_1) = 0$ , then

$$(11) \quad \int_{x_0}^{x_1} g(x) {}_{x_0}^C D_x^\alpha f(x) dx = \int_{x_0}^{x_1} f(x) {}_x^R L D_{x_1}^\alpha f(x) dx$$

and

$$(12) \quad \int_{x_0}^{x_1} g(x) {}_x^C D_{x_1}^\alpha f(x) dx = \int_{x_0}^{x_1} f(x) {}_{x_0}^R L D_x^\alpha f(x) dx.$$

**2. Necessary Optimality Conditions.** In this section, we present necessary optimality conditions of Euler–Lagrange type for variational problems with functional involving  $n$  unknown functions with their higher order left and right Caputo fractional derivatives and fractional integrals.

Define the nonlinear operator  ${}_{\alpha_k^j}^{\gamma_k}[y_1, \dots, y_n]_{\beta_k^j}$  by:

$$(13) \quad {}_{\alpha_k^j}^{\gamma_k}[y_1, \dots, y_n]_{\beta_k^j} := \left( x, y_1(x), \dots, y_n(x), \right. \\ \left. {}_{x_0}^C D_{x_1}^{\alpha_1^1} y_1(x), \dots, {}_{x_0}^C D_{x_1}^{\alpha_1^n} y_1(x), \right. \\ \dots \\ \left. {}_{x_0}^C D_{x_1}^{\alpha_n^1} y_n(x), \dots, {}_{x_0}^C D_{x_1}^{\alpha_n^n} y_n(x), \right. \\ \left. {}_x^C D_{x_1}^{\beta_1^1} y_1(x), \dots, {}_x^C D_{x_1}^{\beta_1^n} y_1(x), \right. \\ \dots \\ \left. {}_x^C D_{x_1}^{\beta_n^1} y_n(x), \dots, {}_x^C D_{x_1}^{\beta_n^n} y_n(x), \right. \\ \left. {}_{x_0}^I \gamma_1 y_1(x), \dots, {}_{x_0}^I \gamma_n y_n(x) \right),$$

where  $\gamma_k > 0$ ,  $j - 1 < \alpha_k^j, \beta_k^j < j$ ,  $j, k = 1, 2, \dots, n$ ,  $n \in \mathbb{N}^*$ .

Define the set  $S$  by:

$$(14) \quad S := \left\{ (y_1, \dots, y_n) \in C^n([x_0, x_1]); y_k^{(i)}(x_0) = a_k^i, y_k^{(i)}(x_1) = b_k^i, \right. \\ \left. k = 1, 2, \dots, n, i = 0, 1, \dots, n - 1, n \in \mathbb{N}^* \right\},$$

Let  $J$  be a functional of the form:

$$(15) \quad J(y_1, \dots, y_n) = \int_{x_0}^{x_1} \left( L \circ {}_{\alpha_k^j}^{\gamma_k}[y_1, \dots, y_n]_{\beta_k^j} \right) (x) dx,$$

and consider the problem of finding functions  $(y_k)_{k=1,2,\dots,n} \in S$  that maximize or minimize the functional  $J$ .

For the sake of convenience, we denote by  $\partial_p L$ ,  $p = 1, \dots, 2n^2 + 2n + 1$  the partial derivative of the function  $L : [x_0, x_1] \times \mathbb{R}^{2n^2 + 2n + 1} \rightarrow \mathbb{R}$  with respect to its  $p$ -th argument. Also, we assume that:

( $H_1$ ): The function  $L$  is of class  $C^1$  with respect to all of its arguments.

( $H_2$ ):  $(\partial_p L)_{p=n+2,\dots,n^2+n+1}$  has continuous right Riemann–Liouville fractional derivative of order  $\alpha_k^j$ ,  $j, k = 1, 2, \dots, n$ , respectively.

( $H_3$ ):  $(\partial_p L)_{p=n^2+n+2,\dots,2n^2+n+1}$  has continuous left Riemann–Liouville fractional derivative of order  $\beta_k^j$ ,  $j, k = 1, 2, \dots, n$ , respectively.

**THEOREM 9.** *Assume that the functional of the form given by Eq. (15) has an extremum in  $S$  at  $(y_k)_{k=1,2,\dots,n}$ . Then for all  $x \in [x_0, x_1]$ , the functions*

$(y_k)_{k=1,2,\dots,n}$  satisfy the generalized fractional Euler–Lagrange equations:

$$(16) \quad \begin{aligned} & \partial_p L + \sum_{j=1}^n {}^{RL}D_{x_1}^{\alpha_j} \partial_{n(p-1)+j+1} L \\ & + \sum_{j=1}^n {}^{RL}D_x^{\beta_j} \partial_{n(p-1)+n^2+j+1} L + {}^xI_{x_1}^{\gamma_{p-1}} \partial_{2n^2+n+p} L \\ & = 0, \end{aligned}$$

where  $p = 2, \dots, n+1$ .

PROOF. Define  $S^*$  by:

$$(17) \quad S^* := \left\{ \sigma_k \in C^n([x_0, x_1]) : \sigma_k^{(i)}(x_0) = \sigma_k^{(i)}(x_1) = 0, i = 0, 1, \dots, n-1 \right\}.$$

For  $\epsilon_k > 0$ ,  $\sigma_k \in S^*$ , let  $y_k + \epsilon \sigma_k$  be the variation of  $y_k$ , with  $(y_k)_{k=1,2,\dots,n} \in S$  is solution for the functional in Eq. (15).

Then,

$$(18) \quad J(\epsilon_1, \dots, \epsilon_n) = \int_{x_0}^{x_1} \left( L \circ \begin{matrix} \gamma_k \\ \alpha_k \end{matrix} [y_1 + \epsilon_1 \sigma_1, \dots, y_n + \epsilon_n \sigma_n] \begin{matrix} \beta_k \\ \beta_k \end{matrix} \right) (x) dx.$$

Since  $J(\epsilon_1, \dots, \epsilon_n)$  attains an extreme value at  $(\epsilon_1, \dots, \epsilon_n) = (0, \dots, 0)$ , we get:

$$(19) \quad \left( \frac{\partial J}{\partial \epsilon_k} \right) (\epsilon_k = 0) = 0, \quad k = 1, 2, \dots, n.$$

We know that the linearity of  ${}^{C}D_{x_0}^{\alpha_k}$  and  ${}^{C}D_{x_1}^{\beta_k}$  implies

$$(20) \quad {}^{C}D_{x_0}^{\alpha_k} (y_k + \epsilon_k \sigma_k) (x) = {}^{C}D_{x_0}^{\alpha_k} y_k + \epsilon_k {}^{C}D_{x_0}^{\alpha_k} \sigma_k (x),$$

and

$$(21) \quad {}^{C}D_{x_1}^{\beta_k} (y_k + \epsilon_k \sigma_k) (x) = {}^{C}D_{x_1}^{\beta_k} y_k + \epsilon_k {}^{C}D_{x_1}^{\beta_k} \sigma_k (x).$$

Consequently, Eq. (19) implies

$$(22) \quad \begin{aligned} & \int_{x_0}^{x_1} \left( \partial_p L \sigma_{p-1} (x) + \sum_{j=1}^n \partial_{n(p-1)+j+1} L {}^{C}D_{x_0}^{\alpha_j} \sigma_{p-1} (x) \right. \\ & \left. + \sum_{j=1}^n \partial_{n(p-1)+n^2+j+1} L {}^{C}D_{x_1}^{\beta_j} \sigma_{p-1} (x) + \partial_{2n^2+n+p} L {}^xI_x^{\gamma_{p-1}} \sigma_{p-1} (x) \right) dx \\ & = 0, \quad p = 2, \dots, n+1. \end{aligned}$$

Integrating by parts as in Eqs. (8), (9) and (10), we get

$$\begin{aligned}
& \int_{x_0}^{x_1} \left( \partial_p L \sigma_{p-1}(x) + \sigma_{p-1}(x) \sum_{j=1}^n {}^{RL}D_{x_1}^{\alpha_j^{p-1}} \partial_{n(p-1)+j+1} L \right. \\
(23) \quad & \left. + \sigma_{p-1}(x) \sum_{j=1}^n {}^{RL}D_{x_0}^{\beta_j^{p-1}} \partial_{n(p-1)+n^2+j+1} L + \sigma_{p-1}(x) {}_x I_{x_1}^{\gamma_{p-1}} \partial_{2n^2+n+p} L \right) dx \\
& = 0, \quad p = 2, \dots, n+1.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \int_{x_0}^{x_1} \left( \partial_p L + \sum_{j=1}^n {}^{RL}D_{x_1}^{\alpha_j^{p-1}} \partial_{n(p-1)+j+1} L \right. \\
(24) \quad & \left. + \sum_{j=1}^n {}^{RL}D_{x_0}^{\beta_j^{p-1}} \partial_{n(p-1)+n^2+j+1} L + {}_x I_{x_1}^{\gamma_{p-1}} \partial_{2n^2+n+p} L \right) \sigma_{p-1}(x) dx \\
& = 0, \quad p = 2, \dots, n+1.
\end{aligned}$$

It follows from Lemma 5 that

$$\begin{aligned}
& \partial_p L + \sum_{j=1}^n {}^{RL}D_{x_1}^{\alpha_j^{p-1}} \partial_{n(p-1)+j+1} L \\
(25) \quad & + \sum_{j=1}^n {}^{RL}D_{x_0}^{\beta_j^{p-1}} \partial_{n(p-1)+n^2+j+1} L + {}_x I_{x_1}^{\gamma_{p-1}} \partial_{2n^2+n+p} L \\
& = 0, \quad p = 2, \dots, n+1.
\end{aligned}$$

Equation (25) is the fractional Euler–Lagrange equations for the variational problem considered. This completes the proof.  $\square$

EXAMPLE 10. Consider the following problem:

$$\begin{aligned}
(26) \quad J(y_1, y_2) = & \int_{-1}^1 \left( 2x^2 + y_1^2 + 2y_2^2 + \left( {}_{-1}^C D_x^{\frac{2}{3}} y_1 \right)^2 + \frac{1}{2} \left( {}_{-1}^C D_x^{\frac{4}{3}} y_1 \right)^2 \right. \\
& + 3 \left( {}_{-1}^C D_x^{\frac{3}{5}} y_2 \right)^2 + 2 \left( {}_{-1}^C D_x^{\frac{5}{3}} y_2 \right)^2 - \left( {}_x^C D_1^{\frac{1}{2}} y_1 \right) - \left( {}_x^C D_1^{\frac{3}{2}} y_1 \right) \\
& \left. + \left( {}_x^C D_1^{\frac{3}{5}} y_2 \right) + \left( {}_x^C D_1^{\frac{8}{5}} y_2 \right) + {}_{-1} I_x^{\frac{7}{2}} y_1 + 2 {}_{-1} I_x^{\frac{10}{3}} y_2 \right) dx,
\end{aligned}$$

with

$$\begin{aligned}
(27) \quad & y_1(-1) = 1, \quad y_1'(-1) = \sqrt{2}, \quad y_1(1) = 2\sqrt{2}, \quad y_1'(1) = -1, \\
& y_2(-1) = 0, \quad y_2'(-1) = 2, \quad y_2(1) = -1, \quad y_2'(1) = \frac{1}{3}.
\end{aligned}$$

Here,  $n = 2$ ,  $x_0 = -1$ ,  $x_1 = 1$ ,  $\alpha_1^1 = \frac{2}{3}$ ,  $\alpha_1^2 = \frac{4}{3}$ ,  $\alpha_2^1 = \frac{3}{5}$ ,  $\alpha_2^2 = \frac{5}{3}$ ,  $\beta_1^1 = \frac{1}{2}$ ,  $\beta_1^2 = \frac{3}{2}$ ,  $\beta_2^1 = \frac{3}{5}$ ,  $\beta_2^2 = \frac{8}{5}$ ,  $\gamma_1 = \frac{7}{2}$ ,  $\gamma_2 = \frac{10}{3}$ , and

$$\begin{aligned}
L &= 2x^2 + y_1^2 + 2y_2^2 \\
&+ \left( {}_{-1}^C D_x^{\frac{2}{3}} y_1 \right)^2 + \frac{1}{2} \left( {}_{-1}^C D_x^{\frac{4}{3}} y_1 \right)^2 + 3 \left( {}_{-1}^C D_x^{\frac{3}{5}} y_2 \right)^2 \\
(28) \quad &+ 2 \left( {}_{-1}^C D_x^{\frac{5}{3}} y_2 \right)^2 - \left( {}_x^C D_1^{\frac{1}{2}} y_1 \right) - \left( {}_x^C D_1^{\frac{3}{2}} y_1 \right) \\
&+ \left( {}_x^C D_1^{\frac{3}{5}} y_2 \right) + \left( {}_x^C D_1^{\frac{8}{5}} y_2 \right) + {}_{-1}I_x^{\frac{7}{2}} y_1 + 2 {}_{-1}I_x^{\frac{10}{3}} y_2.
\end{aligned}$$

The extremes for  $J$  must satisfy the fractional Euler–Lagrange equations given in Eq. (16). So, we get

$$\begin{aligned}
(29) \quad &2y_1 + 2 {}_x^R L D_1^{\frac{2}{3}} \left( {}_{-1}^C D_x^{\frac{2}{3}} y_1 \right) + {}_x^R L D_1^{\frac{4}{3}} \left( {}_{-1}^C D_x^{\frac{4}{3}} y_1 \right) \\
&- {}_{-1}^R L D_x^{\frac{1}{2}} (1) - {}_{-1}^R L D_x^{\frac{3}{2}} (1) + {}_x I_1^{\frac{7}{2}} (1) = 0,
\end{aligned}$$

and

$$\begin{aligned}
(30) \quad &4y_2 + 6 {}_x^R L D_1^{\frac{3}{5}} \left( {}_{-1}^C D_x^{\frac{3}{5}} y_2 \right) + 4 {}_x^R L D_1^{\frac{5}{3}} \left( {}_{-1}^C D_x^{\frac{5}{3}} y_2 \right) \\
&+ {}_{-1}^R L D_x^{\frac{3}{5}} (1) + {}_{-1}^R L D_x^{\frac{8}{5}} (1) + {}_x I_1^{\frac{10}{3}} (2) = 0.
\end{aligned}$$

**3. Generalized Isoperimetric Problems.** We are now interested in finding the functions  $(y_k)_{k=1,2,\dots,n} \in S$  for which the functional given by Eq. (15), subject to the following fractional isoperimetric constraints:

$$\begin{aligned}
(31) \quad I_1(y_1, \dots, y_n) &= \int_{x_0}^{x_1} \left( F_1 \circ \gamma_k^j [y_1, \dots, y_n]_{\beta_k^j} \right) (x) dx = l_1, \\
I_2(y_1, \dots, y_n) &= \int_{x_0}^{x_1} \left( F_2 \circ \gamma_k^j [y_1, \dots, y_n]_{\beta_k^j} \right) (x) dx = l_2, \\
&\dots \\
I_m(y_1, \dots, y_n) &= \int_{x_0}^{x_1} \left( F_m \circ \gamma_k^j [y_1, \dots, y_n]_{\beta_k^j} \right) (x) dx = l_m, \\
l_q &\in \mathbb{R}, \quad q = 1, 2, \dots, m, m < n.
\end{aligned}$$

has an extremum.

Similarly, for all  $q = 1, 2, \dots, m$ , we denote by  $\partial_p F_q$ ,  $p = 1, \dots, 2n^2 + 2n + 1$  the partial derivative of the function  $F_q : [x_0, x_1] \times \mathbb{R}^{2n^2 + 2n + 1} \rightarrow \mathbb{R}$  with respect to its  $p$ -th argument.

We also suppose that:



(H<sub>4</sub>): For each  $q = 1, 2, \dots, m$ , the function  $F_q$  is of class  $C^1$  with respect to all of its arguments.

(H<sub>5</sub>):  $(\partial_p F_q)_{p=n+2, \dots, n^2+n+1}$  has continuous right Riemann–Liouville fractional derivatives of order  $\alpha_k^j$ ,  $j, k = 1, 2, \dots, n$ , respectively.

(H<sub>6</sub>):  $(\partial_p F_q)_{p=n^2+n+2, \dots, 2n^2+n+1}$  has continuous left Riemann–Liouville fractional derivatives of order  $\beta_k^j$ ,  $j, k = 1, 2, \dots, n$ , respectively.

**THEOREM 11.** *Suppose that  $(y_k)_{k=1,2,\dots,n} \in S$  is an extremizer of the functional given by Eq. (15), such that  $I_q(y_1, \dots, y_n) = l_q$ ,  $q = 1, 2, \dots, m$ ,  $m < n$ . If  $(y_k)_{k=1,2,\dots,n}$  is not an extremizer of  $I_q$ , then there exist constants  $(\omega_q)_{q=1,2,\dots,m}$  satisfying*

$$(32) \quad \begin{aligned} \partial_p E + \sum_{j=1}^n {}^{RL}D_{x_1}^{\alpha_{p-1}^j} \partial_{n(p-1)+j+1} E \\ + \sum_{j=1}^n {}^{RL}D_{x_0}^{\beta_{p-1}^j} \partial_{n(p-1)+n^2+j+1} E + x {}^{I_{x_1}^{\gamma_{p-1}}} \partial_{2n^2+n+p} E = 0, \end{aligned}$$

for all  $x \in [x_0, x_1]$ , where  $p = 2, \dots, n+1$ , and  $E = L - \sum_{q=1}^m \omega_q F_q$ .

**PROOF.** Suppose that  $(y_k)_{k=1,2,\dots,n} \in S$  is a solution to the functional given in Eq. (15), such that  $I_q(y_1, \dots, y_n) = l_q$ ,  $q = 1, 2, \dots, m$ ,  $m < n$ , and  $(y_k)_{k=1,2,\dots,n}$  is not an extremizer of  $I_q$ .

Let  $\epsilon_k^t > 0$ ,  $\sigma_k^t \in S^*$ ,  $k = 1, 2, \dots, n$ ,  $t = 1, 2, \dots, m+1$ ,  $m < n$ , and  $y_k + \sum_{t=1}^{m+1} \epsilon_k^t \sigma_k^t$  be the variation of  $y_k$ . Then, we consider

$$(33) \quad \begin{aligned} J(\epsilon_1^1, \dots, \epsilon_n^1, \dots, \epsilon_1^{m+1}, \dots, \epsilon_n^{m+1}) \\ = \int_{x_0}^{x_1} \left( L \circ_{\alpha_k^j} \left[ y_1 + \sum_{t=1}^{m+1} \epsilon_1^t \sigma_1^t, \dots, y_n + \sum_{t=1}^{m+1} \epsilon_n^t \sigma_n^t \right]_{\beta_k^j} \right) (x) dx, \end{aligned}$$

and

$$(34) \quad \begin{aligned} I_q^*(\epsilon_1^1, \dots, \epsilon_n^1, \dots, \epsilon_1^{m+1}, \dots, \epsilon_n^{m+1}) \\ = \int_{x_0}^{x_1} \left( F_q \circ_{\alpha_k^j} \left[ y_1 + \sum_{t=1}^{m+1} \epsilon_1^t \sigma_1^t, \dots, y_n + \sum_{t=1}^{m+1} \epsilon_n^t \sigma_n^t \right]_{\beta_k^j} \right) (x) dx - l_q. \end{aligned}$$

The assumption that  $(y_k)_{k=1,2,\dots,n}$  is not an extremizer of  $I_q$ , means that there exists a family of functions  $(\sigma_k^t)_{k=1,2,\dots,n}^{t=2,\dots,m+1}$ , where

$$(35) \quad \left( \frac{\partial I_q^*}{\partial \epsilon_{p-1}^t} \right) \Big|_{(\epsilon_1^1 = \dots = \epsilon_1^{m+1} = 0, \dots, \epsilon_n^1 = \dots = \epsilon_n^{m+1} = 0)} \neq 0,$$

for each  $t = 2, \dots, m+1$ ,  $k = 1, \dots, n$ ,  $p = 2, \dots, n+1$ .

Also, by Eq. (34), we get  $I_q^*(0, \dots, 0) = 0$ ,  $q = 1, 2, \dots, m$ , for all  $\epsilon_k^t = 0$ ,  $k = 1, 2, \dots, n$ ,  $t = 1, 2, \dots, m + 1$ .

Since  $I_q^*(0, \dots, 0) = 0$ ,  $q = 1, 2, \dots, m$ , applying implicit function theorem, there exists a family of functions  $(\epsilon_k^t)(\cdot)$ ,  $k = 1, 2, \dots, n$ ,  $t = 2, \dots, m + 1$ , defined in a neighborhood of zero, where

$$(36) \quad I_q^*(\epsilon_k^1, \epsilon_k^2(\epsilon_k^1), \dots, \epsilon_k^{m+1}(\epsilon_k^1)) = 0.$$

By the Lagrange multiplier rule, there exist constants  $(\omega_q)_{q=1,2,\dots,m}$  such that

$$(37) \quad \nabla \left( J(0, \dots, 0) - \sum_{q=1}^m \omega_q I_q^*(0, \dots, 0) \right) = (0, \dots, 0).$$

For all  $p = 2, \dots, n + 1$ , we have

$$(38) \quad \begin{aligned} \left( \frac{\partial J}{\partial \epsilon_{p-1}^1} \right) (0, \dots, 0) &= \int_{x_0}^{x_1} \left( \partial_p L + \sum_{j=1}^n {}^{RL}D_{x_1}^{\alpha_{p-1}^j} \partial_{n(p-1)+j+1} L \right. \\ &\quad \left. + \sum_{j=1}^n {}^{RL}D_{x_0}^{\beta_{p-1}^j} \partial_{n(p-1)+n^2+j+1} L \right. \\ &\quad \left. + {}_x I_{x_1}^{\gamma_{p-1}} \partial_{2n^2+n+p} L \right) \sigma_{p-1}^1(x) dx, \end{aligned}$$

and

$$(39) \quad \begin{aligned} \left( \frac{\partial I_q^*}{\partial \epsilon_{p-1}^1} \right) (0, \dots, 0) &= \int_{x_0}^{x_1} \left( \partial_p F_q + \sum_{j=1}^n {}^{RL}D_{x_1}^{\alpha_{p-1}^j} \partial_{n(p-1)+j+1} F_q \right. \\ &\quad \left. + \sum_{j=1}^n {}^{RL}D_{x_0}^{\beta_{p-1}^j} \partial_{n(p-1)+n^2+j+1} F_q \right. \\ &\quad \left. + {}_x I_{x_1}^{\gamma_{p-1}} \partial_{2n^2+n+p} F_q \right) \sigma_{p-1}^1(x) dx. \end{aligned}$$

Taking  $E = L - \sum_{q=1}^m \omega_q F_q$ , we obtain

$$(40) \quad \begin{aligned} &\int_{x_0}^{x_1} \left( \partial_p E + \sum_{j=1}^n {}^{RL}D_{x_1}^{\alpha_{p-1}^j} \partial_{n(p-1)+j+1} E + \sum_{j=1}^n {}^{RL}D_{x_0}^{\beta_{p-1}^j} \partial_{n(p-1)+n^2+j+1} E \right. \\ &\quad \left. + {}_x I_{x_1}^{\gamma_{p-1}} \partial_{2n^2+n+p} E \right) \sigma_{p-1}^1(x) dx \\ &= 0, \quad p = 2, \dots, n + 1. \end{aligned}$$

Using Lemma 5 of the calculus of variations, we can obtain:

$$\begin{aligned} \partial_p E + \sum_{j=1}^n {}^{RL}D_{x_1}^{\alpha_j^{p-1}} \partial_{n(p-1)+j+1} E \\ + \sum_{j=1}^n {}^{RL}D_{x_0}^{\beta_j^{p-1}} \partial_{n(p-1)+n^2+j+1} E + {}_x I_{x_1}^{\gamma^{p-1}} \partial_{2n^2+n+p} E \\ = 0, \quad p = 2, \dots, n+1. \end{aligned}$$

This ends the proof.  $\square$

EXAMPLE 12. Now, we present an example of a fractional isoperimetric problem to illustrate the above theorem.

$$\begin{aligned} J(y_1, y_2, y_3) &= \int_{-1}^1 \left( L \circ \gamma_k^k [y_1, \dots, y_n]_{\beta_k^j} \right) (x) dx \\ &= \int_{-1}^1 \left( 2x^2 + y_1^2 + y_2^2 + y_3^2 \right. \\ &\quad + 2 \left( {}_{-1}^C D_x^{\frac{1}{2}} y_1 \right)^2 - \left( {}_{-1}^C D_x^{\frac{3}{2}} y_1 \right)^2 + \frac{1}{2} \left( {}_{-1}^C D_x^{\frac{7}{3}} y_1 \right)^2 \\ &\quad + 3 \left( {}_{-1}^C D_x^{\frac{1}{3}} y_2 \right)^2 + \frac{1}{2} \left( {}_{-1}^C D_x^{\frac{5}{3}} y_2 \right)^2 + \left( {}_{-1}^C D_x^{\frac{9}{4}} y_2 \right)^2 \\ &\quad - \left( {}_{-1}^C D_x^{\frac{3}{4}} y_3 \right)^2 + 3 \left( {}_{-1}^C D_x^{\frac{5}{4}} y_3 \right)^2 + \frac{1}{4} \left( {}_{-1}^C D_x^{\frac{11}{5}} y_3 \right)^2 \\ &\quad - {}_x D_1^{\frac{6}{7}} y_1 - {}_x D_1^{\frac{9}{7}} y_1 - {}_x D_1^{\frac{15}{7}} y_1 \\ &\quad - 2 {}_x D_1^{\frac{2}{5}} y_2 + 3 {}_x D_1^{\frac{7}{5}} y_2 + {}_x D_1^{\frac{14}{5}} y_2 \\ &\quad - {}_x D_1^{\frac{3}{8}} y_3 - \frac{1}{2} {}_x D_1^{\frac{11}{8}} y_3 - \frac{1}{2} {}_x D_1^{\frac{17}{8}} y_3 \\ &\quad \left. + \frac{1}{3} {}_{-1} I_x^{\frac{5}{2}} y_1 + 2 {}_{-1} I_x^{\frac{7}{2}} y_2 + {}_{-1} I_x^{\frac{9}{2}} y_3 \right) dx, \end{aligned} \tag{41}$$

subject to the boundary conditions:

$$\begin{aligned} y_1(-1) &= 0, & y_1^{(1)}(-1) &= -1, & y_1^{(2)}(-1) &= 1, \\ y_1(1) &= \sqrt{2}, & y_1^{(1)}(1) &= 2, & y_1^{(2)}(1) &= \frac{3}{7}, \\ y_2(-1) &= -2, & y_2^{(1)}(-1) &= 0, & y_2^{(2)}(-1) &= 2\sqrt{3}, \\ y_2(1) &= 1, & y_2^{(1)}(1) &= \frac{1}{2}, & y_2^{(2)}(1) &= -1, \\ y_3(-1) &= 2\sqrt{5}, & y_3^{(1)}(-1) &= -2, & y_3^{(2)}(-1) &= 2, \\ y_3(1) &= \frac{3}{4}, & y_3^{(1)}(1) &= 0, & y_3^{(2)}(1) &= 2, \end{aligned} \tag{42}$$

and the fractional isoperimetric constraints:

$$\begin{aligned}
(43) \quad I_1(y_1, y_2, y_3) &= \int_{-1}^1 \left( F_1 \circ_{\alpha_k^j}^{\gamma_k} [y_1, \dots, y_n]_{\beta_k^j} \right) (x) dx \\
&= \int_{-1}^1 \left( x - 2y_1^2 - y_2^2 + y_3^2 \right. \\
&\quad - 2 \left( {}_{-1}^C D_x^{\frac{1}{2}} y_1 \right) + \left( {}_{-1}^C D_x^{\frac{3}{2}} y_1 \right) + 4 \left( {}_{-1}^C D_x^{\frac{7}{3}} y_1 \right) \\
&\quad - \left( {}_{-1}^C D_x^{\frac{1}{3}} y_2 \right)^2 + \frac{3}{2} \left( {}_{-1}^C D_x^{\frac{5}{3}} y_2 \right) - \left( {}_{-1}^C D_x^{\frac{9}{4}} y_2 \right) \\
&\quad + 2 \left( {}_{-1}^C D_x^{\frac{3}{4}} y_3 \right) + \left( {}_{-1}^C D_x^{\frac{5}{4}} y_3 \right) - \left( {}_{-1}^C D_x^{\frac{11}{5}} y_3 \right)^2 \\
&\quad + 2 {}_x^C D_1^{\frac{6}{7}} y_1 - 2 {}_x^C D_1^{\frac{9}{7}} y_1 + {}_x^C D_1^{\frac{15}{7}} y_1 \\
&\quad - {}_x^C D_1^{\frac{2}{5}} y_2 - {}_x^C D_1^{\frac{7}{5}} y_2 + {}_x^C D_1^{\frac{14}{5}} y_2 \\
&\quad + {}_x^C D_1^{\frac{3}{8}} y_3 - 3 {}_x^C D_1^{\frac{11}{8}} y_3 - 4 {}_x^C D_1^{\frac{17}{8}} y_3 \\
&\quad \left. + \frac{1}{4} {}_{-1} I_x^{\frac{5}{2}} y_1 - 2 {}_{-1} I_x^{\frac{7}{2}} y_2 + 3 {}_{-1} I_x^{\frac{9}{2}} y_3 \right) dx \\
&= 2,
\end{aligned}$$

$$\begin{aligned}
(44) \quad I_2(y_1, y_2, y_3) &= \int_{-1}^1 \left( F_2 \circ_{\alpha_k^j}^{\gamma_k} [y_1, \dots, y_n]_{\beta_k^j} \right) (x) dx \\
&= \int_{-1}^1 \left( x^2 - y_1^2 + y_2^2 + 2y_3^2 \right. \\
&\quad + \left( {}_{-1}^C D_x^{\frac{1}{2}} y_1 \right) + 2 \left( {}_{-1}^C D_x^{\frac{3}{2}} y_1 \right) - \left( {}_{-1}^C D_x^{\frac{7}{3}} y_1 \right) \\
&\quad + \left( {}_{-1}^C D_x^{\frac{1}{3}} y_2 \right)^2 + \left( {}_{-1}^C D_x^{\frac{5}{3}} y_2 \right) + 2 \left( {}_{-1}^C D_x^{\frac{9}{4}} y_2 \right) \\
&\quad - \left( {}_{-1}^C D_x^{\frac{3}{4}} y_3 \right) - \frac{1}{2} \left( {}_{-1}^C D_x^{\frac{5}{4}} y_3 \right)^2 + \left( {}_{-1}^C D_x^{\frac{11}{5}} y_3 \right)^2 \\
&\quad + 3 {}_x^C D_1^{\frac{6}{7}} y_1 - {}_x^C D_1^{\frac{9}{7}} y_1 + \frac{1}{3} {}_x^C D_1^{\frac{15}{7}} y_1 \\
&\quad - 2 {}_x^C D_1^{\frac{2}{5}} y_2 - {}_x^C D_1^{\frac{7}{5}} y_2 + {}_x^C D_1^{\frac{14}{5}} y_2 \\
&\quad - {}_x^C D_1^{\frac{3}{8}} y_3 + {}_x^C D_1^{\frac{11}{8}} y_3 - {}_x^C D_1^{\frac{17}{8}} y_3 \\
&\quad \left. + 2 {}_{-1} I_x^{\frac{5}{2}} y_1 - {}_{-1} I_x^{\frac{7}{2}} y_2 - {}_{-1} I_x^{\frac{9}{2}} y_3 \right) dx \\
&= 3.
\end{aligned}$$

Here,  $n = 3$ ,  $m = 2$ ,  $x_0 = -1$ ,  $x_1 = 1$ ,  $\alpha_1^1 = \frac{1}{2}$ ,  $\alpha_1^2 = \frac{3}{2}$ ,  $\alpha_1^3 = \frac{7}{3}$ ,  $\alpha_2^1 = \frac{1}{3}$ ,  $\alpha_2^2 = \frac{5}{3}$ ,  $\alpha_2^3 = \frac{9}{4}$ ,  $\alpha_3^1 = \frac{3}{4}$ ,  $\alpha_3^2 = \frac{5}{4}$ ,  $\alpha_3^3 = \frac{11}{5}$ ,  $\beta_1^1 = \frac{6}{7}$ ,  $\beta_1^2 = \frac{9}{7}$ ,  $\beta_1^3 = \frac{15}{7}$ ,  $\beta_2^1 = \frac{2}{5}$ ,  $\beta_2^2 = \frac{7}{5}$ ,  $\beta_2^3 = \frac{14}{5}$ ,  $\beta_3^1 = \frac{3}{8}$ ,  $\beta_3^2 = \frac{11}{8}$ ,  $\beta_3^3 = \frac{17}{8}$ ,  $\gamma_1 = \frac{5}{2}$ ,  $\gamma_2 = \frac{7}{2}$ ,  $\gamma_3 = \frac{9}{2}$ , and

$$(45) \quad E = L - \omega_1 F_1 - \omega_2 F_2.$$

By applying Eq. (34), we get the following fractional Euler–Lagrange equations:

$$(46) \quad \begin{aligned} & 2y_1 + 4\omega_1 y_1 + 2\omega_2 y_1 + {}_x RL D_1^{\frac{1}{2}} \left( 4 {}_{-1} C D_x^{\frac{1}{2}} y_1 + 2\omega_1 - \omega_2 \right) \\ & - {}_x RL D_1^{\frac{3}{2}} \left( 2 {}_{-1} C D_x^{\frac{3}{2}} y_1 + \omega_1 + 2\omega_2 \right) + {}_x RL D_1^{\frac{7}{3}} \left( -{}_{-1} C D_x^{\frac{7}{3}} y_1 - 4\omega_1 + \omega_2 \right) \\ & - {}_{-1} RL D_x^{\frac{6}{7}} (1 + 2\omega_1 + 3\omega_2) + {}_{-1} RL D_x^{\frac{9}{7}} (-1 + 2\omega_1 + \omega_2) \\ & - {}_{-1} RL D_x^{\frac{15}{7}} \left( 1 + \omega_1 + \frac{1}{3}\omega_2 \right) + {}_x I_1^{\frac{5}{2}} \left( \frac{1}{3} - \frac{1}{4}\omega_1 - 2\omega_2 \right) = 0, \end{aligned}$$

$$(47) \quad \begin{aligned} & 2y_2 + 2\omega_1 y_2 - 2\omega_2 y_2 + {}_x RL D_1^{\frac{1}{3}} \left( 6 {}_{-1} C D_x^{\frac{1}{3}} y_2 + 2\omega_1 {}_{-1} C D_x^{\frac{1}{3}} y_2 - 2\omega_2 {}_{-1} C D_x^{\frac{1}{3}} y_2 \right) \\ & + {}_x RL D_1^{\frac{5}{3}} \left( -{}_{-1} C D_x^{\frac{5}{3}} y_2 - \frac{3}{2}\omega_1 - \omega_2 \right) + {}_x RL D_1^{\frac{9}{4}} \left( 2 {}_{-1} C D_x^{\frac{9}{4}} y_2 + \omega_1 - 2\omega_2 \right) \\ & + {}_{-1} RL D_x^{\frac{2}{5}} (-2 + \omega_1 + 2\omega_2) + {}_{-1} RL D_x^{\frac{7}{5}} (3 + \omega_1 + \omega_2) \\ & + {}_{-1} RL D_x^{\frac{14}{5}} (1 - \omega_1 - \omega_2) + {}_x I_1^{\frac{7}{2}} (2 + 2\omega_1 + \omega_2) = 0, \end{aligned}$$

and

$$(48) \quad \begin{aligned} & 2y_3 - 2\omega_1 y_3 - 4\omega_2 y_3 + {}_x RL D_1^{\frac{3}{4}} \left( -2 {}_{-1} C D_x^{\frac{3}{4}} y_3 - 2\omega_1 + \omega_2 \right) \\ & + {}_x RL D_1^{\frac{5}{4}} \left( 6 {}_{-1} C D_x^{\frac{5}{4}} y_3 - 2\omega_1 {}_{-1} C D_x^{\frac{5}{4}} y_3 + \omega_2 {}_{-1} C D_x^{\frac{5}{4}} y_3 \right) \\ & + {}_x RL D_1^{\frac{11}{5}} \left( \frac{1}{2} {}_{-1} C D_x^{\frac{11}{5}} y_3 + 2\omega_1 {}_{-1} C D_x^{\frac{11}{5}} y_3 - 2\omega_2 {}_{-1} C D_x^{\frac{11}{5}} y_3 \right) \\ & + {}_{-1} RL D_x^{\frac{3}{8}} (-1 - \omega_1 + \omega_2) + {}_{-1} RL D_x^{\frac{11}{8}} \left( -\frac{1}{2} + 3\omega_1 - \omega_2 \right) \\ & + {}_{-1} RL D_x^{\frac{17}{8}} \left( -\frac{1}{2} + 4\omega_1 + \omega_2 \right) + {}_x I_1^{\frac{9}{2}} (1 - 3\omega_1 + \omega_2) = 0. \end{aligned}$$

**4. Optimal Time Problem.** Our aim here is to find the optimal time  $T$  as well as the functions  $(y_k)_{k=1,2,\dots,n}$  that maximize or minimize the functional

of the form:

$$(49) \quad J^*(y_1, \dots, y_n, T) = \int_{x_0}^T L^*(x, y_1(x), \dots, y_n(x), \\ C D_x^{\alpha_1} y_1(x), \dots, C D_x^{\alpha_1} y_1(x), \dots, \\ C D_x^{\alpha_n} y_n(x), \dots, C D_x^{\alpha_n} y_n(x)) dx,$$

where

$$(50) \quad \left( (y_k)_{k=1, \dots, n}, T \right) \in B := \{ (C^n([x_0, x_1]), [x_0, x_1]) : y_k^{(i)}(x_0) = a_k^i, \\ i = 0, 1, \dots, n-1, n \in \mathbb{N}^* \}.$$

We consider the both free and fixed end-point problems.

**THEOREM 13.** *Assume that  $\left( (y_k)_{k=1, \dots, n}, T \right) \in B$  is an extremizer of  $J^*$  defined by Eq. (49). Then,  $\left( (y_k)_{k=1, \dots, n}, T \right)$  is a solution of the Euler–Lagrange equations:*

$$(51) \quad \partial_p L^* + \sum_{j=1}^n {}^R L D_T^{\alpha_j} \partial_{n(p-1)+j+1} L^* = 0, \quad p = 2, \dots, n+1,$$

for all  $x \in [x_0, T]$ , and satisfies the following transversality conditions:

1.

$$(52) \quad L^*(T, y_1(T), \dots, y_n(T), C D_T^{\alpha_1} y_1(T), \dots, C D_T^{\alpha_1} y_1(T), \\ \dots, C D_T^{\alpha_n} y_n(T), \dots, C D_T^{\alpha_n} y_n(T)) = 0,$$

2.

$$(53) \quad \sum_{i=0}^{n-1} \sum_{j=1}^n {}^R L D_T^{\alpha_j} \partial_{n(p-1)+j+1} L^*(x) = 0, \quad \text{at } x = T.$$

**PROOF.** Define

$$(54) \quad B^* := \{ \sigma_k \in C^n([x_0, x_1]) : \sigma_k^{(i)}(x_0) = 0, \quad k = 1, \dots, n, \\ i = 0, 1, \dots, n-1 \}.$$

Let  $\sigma_k \in B^*$  and  $y_k + \epsilon \sigma_k$  be variation of  $y_k$ , for all  $k = 1, 2, \dots, n$ , and let  $T + \epsilon \Delta T$  be variation of  $T$ , such that  $\epsilon > 0$ ,  $\Delta T \in \mathbb{R}$ .

Consider

$$(55) \quad J^*(\epsilon) = J^*(y_1 + \epsilon \sigma_1, \dots, y_n + \epsilon \sigma_n, T + \epsilon \Delta T).$$

Assume that  $\left((y_k)_{k=1,\dots,n}, T\right) \in B$  is an extremizer of  $J^*$ . Then, we get:

$$(56) \quad \left(\frac{\partial J^*}{\partial \epsilon}\right) (\epsilon = 0) = 0.$$

Applying Leibniz integral rule, Eq. (56) implies that

$$(57) \quad \int_{x_0}^T \left( \partial_p L^* \sigma_{p-1}(x) + \sum_{j=1}^n \partial_{n(p-1)+j+1} L^* {}_x^C D_x^{\alpha_j} \sigma_{p-1}(x) \right) dx \\ + \Delta T L^* \left( T, y_1(T), \dots, y_n(T), {}_x^C D_T^{\alpha_1} y_1(T), \dots, {}_x^C D_T^{\alpha_n} y_1(T), \dots, \right. \\ \left. \dots, {}_x^C D_T^{\alpha_1} y_n(T), \dots, {}_x^C D_T^{\alpha_n} y_n(T) \right) \\ = 0, \quad p = 2, \dots, n+1.$$

Integrating the second term by parts as in Eq. (9) given in Lemma 6, we get:

$$(58) \quad \int_{x_0}^T \left[ \partial_p L^* + \sum_{j=1}^n {}_x^{RL} D_T^{\alpha_j} \partial_{n(p-1)+j+1} L^* \right] \sigma_{p-1}(x) dx \\ + \sum_{i=0}^{n-1} \sum_{j=1}^n \left[ {}_x^{RL} D_T^{\alpha_j+i-n} \left( \partial_{n(p-1)+j+1} L^* \right) (x) (-1)^{(n-i-1)} \sigma_{p-1}^{(n-i-1)}(x) \right]_{x=T} \\ + \Delta T L^* \left( T, y_1(T), \dots, y_n(T), {}_x^C D_T^{\alpha_1} y_1(T), \dots, {}_x^C D_T^{\alpha_n} y_1(T), \dots, \right. \\ \left. \dots, {}_x^C D_T^{\alpha_1} y_n(T), \dots, {}_x^C D_T^{\alpha_n} y_n(T) \right) \\ = 0, \quad p = 2, \dots, n+1.$$

By fixing  $\sigma_{p-1} \equiv 0$ ,  $p = 2, \dots, n+1$ , and by the arbitrariness of  $\Delta T$ , we obtain the first transversality condition. Then, the second transversality condition is proved by choosing  $\sigma_{p-1}$  to be equal to zero on  $[x_0, T)$  and  $\sigma_{p-1}^{(n-i-1)}(T) \neq 0$ ,  $p = 2, \dots, n+1$ ,  $i = 0, 1, \dots, n-1$ . If, for all  $p = 2, \dots, n+1$ ,  $i = 0, 1, \dots, n-1$ ,  $\sigma_{p-1}^{(n-i-1)}$  is free on  $[a, T)$  and  $\sigma_{p-1}^{(n-i-1)}(T) = 0$ , we obtain Euler–Lagrange equation (51). This ends the proof.  $\square$

EXAMPLE 14. Consider the following optimal time problem:

$$(59) \quad J^*(y_1, y_2) = \int_{-1}^T \left( 2x^2 + \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + 2 \left( {}_{-1}^C D_x^{\frac{2}{3}} y_1 \right) \right. \\ \left. + 3 \left( {}_{-1}^C D_x^{\frac{5}{4}} y_1 \right) - \left( {}_{-1}^C D_x^{\frac{1}{2}} y_2 \right) + 2 \left( {}_{-1}^C D_x^{\frac{7}{5}} y_2 \right)^2 \right) dx,$$

with the boundary conditions:

$$y_1(-1) = 1, \quad y_1^{(1)}(-1) = \sqrt{2}, \quad y_2(-1) = -2, \quad y_2^{(1)}(-1) = -1.$$

We have:  $n = 2$ ,  $x_0 = -1$ ,  $x_1 = 1$ ,  $\alpha_1^1 = \frac{2}{3}$ ,  $\alpha_1^2 = \frac{5}{4}$ ,  $\alpha_2^1 = \frac{1}{2}$ ,  $\alpha_2^2 = \frac{7}{5}$ , and

$$L^* = \left( 2x^2 + \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + 2 \left( {}_{-1}^C D_x^{\frac{2}{3}} y_1 \right) - \left( {}_{-1}^C D_x^{\frac{5}{4}} y_1 \right)^2 - {}_{-1}^C D_x^{\frac{1}{2}} y_2 + 2 \left( {}_{-1}^C D_x^{\frac{7}{5}} y_2 \right)^2 \right) dx.$$

By applying Eqs. (51), (52) and (53), we obtain the following fractional Euler–Lagrange equations:

$$(60) \quad \begin{cases} y_1 + {}_x^R L D_T^{\frac{2}{3}}(2) - 2 {}_x^R L D_T^{\frac{5}{4}} \left( {}_{-1}^C D_x^{\frac{5}{4}} y_1 \right) = 0, \\ y_2 - {}_x^R L D_T^{\frac{1}{2}}(1) + 4 {}_x^R L D_T^{\frac{7}{5}} \left( {}_{-1}^C D_x^{\frac{7}{5}} y_2 \right) = 0, \end{cases}$$

and for the transversality conditions, we get:

$$(61) \quad \begin{aligned} 2T^2 + \frac{1}{2}y_1^2(T) + \frac{1}{2}y_2^2(T) + 2 \left( {}_{-1}^C D_x^{\frac{2}{3}} y_1 \right)(T) - \left( {}_{-1}^C D_x^{\frac{5}{4}} y_1 \right)^2(T) \\ - {}_{-1}^C D_x^{\frac{1}{2}} y_2(T) + 2 \left( {}_{-1}^C D_x^{\frac{7}{5}} y_2 \right)^2(T) = 0, \end{aligned}$$

$$(62) \quad \begin{aligned} \left[ {}_x I_T^{\frac{4}{3}}(2) - 2 {}_x I_T^{\frac{3}{4}} \left( {}_{-1}^C D_x^{\frac{5}{4}} y_1 \right) + {}_x I_T^{\frac{1}{3}}(2) - 2 {}_x^R L D_T^{\frac{1}{4}} \left( {}_{-1}^C D_x^{\frac{5}{4}} y_1 \right) \right]_{x=T} = 0, \\ \left[ -{}_x I_T^{\frac{3}{2}}(1) + 4 {}_x I_T^{\frac{3}{5}} \left( {}_{-1}^C D_x^{\frac{7}{5}} y_2 \right) - {}_x I_T^{\frac{1}{2}}(1) + 4 {}_x^R L D_T^{\frac{2}{5}} \left( {}_{-1}^C D_x^{\frac{7}{5}} y_2 \right) \right]_{x=T} = 0, \end{aligned}$$

where  ${}_x^R L D_T^k = {}_x I_T^{-k}$ , for  $k < 0$ .

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