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LOCAL HOMOLOGY AND SERRE CATEGORIES

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Abstract. We show some results about local homology modules when they are in a Serre subcategory of the category of R-modules. For an ideal $\mathfrak{a}$ of R, we also define the concept of the condition $C^{\mathfrak{a}}$ on a Serre category, which seems dual to the condition $C_{\mathfrak{a}}$ in Melkersson [**1**]. As a main result we show that for an Artinian R-module M and any Serre subcategory $\mathcal{S}$ of the category of R-modules and a non-negative integer s, $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{\mathfrak{a}}_s(M)) \in \mathcal{S}$ if $\mathrm{H}^{\mathfrak{a}}_i(M) \in \mathcal{S}$ for all $i > s$.

1. Introduction. Throughout this paper, R is a commutative Noetherian ring with identity, $\mathfrak{a}$ an ideal of R and M an R-module. Cuong and Nam [**4**] defined the i-th local homology $\mathrm{H}^{\mathfrak{a}}_i(M)$ of an R-module M with respect to the ideal $\boldsymbol{a}$ by

$$\mathrm{H}^{\mathfrak{a}}_i(M) = \varprojlim \operatorname{Tor}^R_i(R/\mathfrak{a}^n, M).$$

This definition slightly differs from the Greenlees and May definition [**6**]. However, they are equivalent for the Artinian modules. It should be noted that this definition of local homology modules is, in some sense, dual to the definition of local cohomology in Grothendieck [**7**]. For each $i \geq 0$, the local cohomology module $\mathrm{H}^{\mathfrak{a}}_i(M)$ with respect to $\mathfrak{a}$ is defined by

$$\mathrm{H}^i_{\mathfrak{a}}(M) = \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M).$$

We refer the reader to [**4**], [**3**] and [**7**] for the basic properties of local homology and local cohomology modules.

A subcategory $\mathcal{S}$ of the category of R-modules is called a Serre subcategory if it is closed under taking submodules, quotients and extensions. In [**7**] Grothendieck put forward the following conjecture: For any ideal $\mathfrak{a}$ of R and

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any finitely generated R-module M, the module $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M))$ is finitely generated for all $i \geq 0$. There is a similar conjecture in local homology theory: For any ideal $\mathfrak{a}$ of R and any Artinian R-module M, the module $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_i^{\mathfrak{a}}(M))$ is Artinian for all $i \geq 0$. In general, neither conjectures is true; see, e.g. [**8**]. It is well known that Noetherian and Artinian R-modules are Serre subcategories of the category of R-modules. Instead of the above conjectures, one can raise the following general question, which is a new attitude to the above conjectures.

Question. When is $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_i^{\mathfrak{a}}(M))$ in a Serre category?

The main aim of this paper is to show some properties of local homology modules concerning the above question (see Section 3). The organization of this paper is as follows:

In Section 2, we deal with the question of when local homology modules $\mathrm{H}_i^{\mathfrak{a}}(M)$ of an Artinian R-module M belong to $\mathcal{S}$. In order to answer this question, we achieve some results (Theorems 2.4, 2.5).

In Section 3, first we define the condition $C^{\mathfrak{a}}$ on a Serre category $\mathcal{S}$ as follows: If M is $\mathfrak{a}$-separated and if $\frac{M}{\mathfrak{a}M} \in \mathcal{S}$, then $M \in \mathcal{S}$. This definition is dual to the condition $C_{\mathfrak{a}}$ in Melkersson [**1**]. Next we show that if $(R, \mathfrak{m})$ is a local ring, $\mathcal{S}$ a non-zero Serre subcategory of the category of R-modules and satisfies the condition $C^{\mathfrak{a}}$ and M an Artinian R-module with $\mathrm{Ndim}\, M = d$, then $\mathrm{H}_d^{\mathfrak{a}}(M) \in \mathcal{S}$. Finally, as a main result of this section we prove the following theorem:

Theorem. *Let $(R, \mathfrak{m})$ be a local ring, $\mathcal{S}$ a non-zero Serre subcategory of the category of R-modules, and M an Artinian R-module. Assume that s is a non-nagative integer such that $\mathrm{H}_i^{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i > s$. Then $\mathrm{Hom}_R(R/\mathfrak{a}, H_s^{\mathfrak{a}}(M)) \in \mathcal{S}$.*

2. Local Homology and Serre Subcategory. Let $\mathfrak{a}$ be an ideal of R and M an R-module. The i-th local homology module $\mathrm{H}_i^{\mathfrak{a}}(M)$ of an R-module M with respect to the ideal $\mathfrak{a}$ is defined in [**4**] by

$$\mathrm{H}_i^{\mathfrak{a}}(M) = \varprojlim_t \mathrm{Tor}_i^R(R/\mathfrak{a}^t, M).$$

It is clear that $\mathrm{H}_0^{\mathfrak{a}}(M) \cong \Lambda_{\mathfrak{a}}(M)$, where

$$\Lambda_{\mathfrak{a}}(M) \cong \varprojlim_t M/\mathfrak{a}^t M$$

is the $\mathfrak{a}$-adic completion of M. Denote by $\mathrm{L}_i^{\mathfrak{a}}$ the i-th left derived functor of $\Lambda_{\mathfrak{a}}$. We have $\mathrm{H}_i^{\mathfrak{a}}(M) \cong \mathrm{L}_i^{\mathfrak{a}}(M)$ for all $i \geq 0$, provided that M is linearly compact (see [**5**] for more details). Moreover, if M is a finitely generated module, then $\mathrm{H}_i^{\mathfrak{a}}(M) = 0$ for all $i > 0$ by [**4**, 2.1].

REMARK 2.1. Let M be an Artinian R-module. Then there exists a positive integer n such that $\mathfrak{a}^t M = \mathfrak{a}^n M$ for all $t \geqslant n$ and so $\mathrm{H}_0^{\mathfrak{a}}(M) \cong \Lambda_{\mathfrak{a}}(M) \cong M/\mathfrak{a}^n M$, and we also have the short exact sequence of Artinian modules

$$0 \to \bigcap_{t>0} \mathfrak{a}^t M \to M \to \Lambda_{\mathfrak{a}}(M) \to 0.$$

We put $N := \mathfrak{a}^n M$ and we may assume that $\mathfrak{a}N = N$, and so there is an element x in $\mathfrak{a}$ such that $xN = N$. Thus the two short exact sequences of Artinian modules

$$0 \to 0 :_N x \to N \xrightarrow{x} N \to 0.$$

and

$$0 \to N \to M \to M/N \to 0,$$

give rise to the long exact sequences of local homology modules

$$\cdots \to \mathrm{H}_{i+1}^{\mathfrak{a}}(N) \to \mathrm{H}_i^{\mathfrak{a}}(0 :_N x) \to \mathrm{H}_i^{\mathfrak{a}}(N) \xrightarrow{x} \mathrm{H}_i^{\mathfrak{a}}(N) \to \cdots$$

and

$$\cdots \to \mathrm{H}_{i+1}^{\mathfrak{a}}(M/N) \to \mathrm{H}_i^{\mathfrak{a}}(N) \to \mathrm{H}_i^{\mathfrak{a}}(M) \to \mathrm{H}_i^{\mathfrak{a}}(M/N) \to \cdots .$$

By [**5**, 3.8], $\mathrm{H}_0^{\mathfrak{a}}(M/N) \cong M/N$ and $\mathrm{H}_i^{\mathfrak{a}}(M/N) = 0$ for all $i > 0$, since M/N is $\mathfrak{a}$-separated (i.e. $\bigcap_{t>0} \mathfrak{a}^t(M/N) = 0$). Thus for all $i \geq 1$, $\mathrm{H}_i^{\mathfrak{a}}(N) \cong \mathrm{H}_i^{\mathfrak{a}}(M)$ and $\mathrm{H}_0^{\mathfrak{a}}(N) = 0$.

LEMMA 2.2. [**4**, 3.7] *If M is an Artinian R-module and $\hat{R}$ is the completion of R in the $\mathfrak{a}$-adic topology, then*

$$\mathrm{H}_i^{\mathfrak{a}}(M) \cong \mathrm{H}_i^{\mathfrak{a}\hat{R}}(M)$$

for all $i \geq 0$.

Now we recall the concept of Noetherian dimension by using the terminology of Kirby [**9**].

DEFINITION 2.3. The Noetherian dimension of M denoted by $\operatorname{Ndim}_R M$, is defined inductively as follows: when $M = 0$, put $\operatorname{Ndim} M = -1$. Then by induction, for any integer $d \geq 0$, we define $\operatorname{Ndim}_R M = d$ if $\operatorname{Ndim}_R M < d$ is false, and for every ascending sequence $M_0 \subseteq M_1 \subseteq \cdots$ of submodules of M, there exists a positive integer n_0 such that $\operatorname{Ndim}(M_{n+1}/M_n) < d$ for all $n \geq n_0$. Thus M is non-zero and finitely generated if and only if $\operatorname{Ndim}_R M = 0$. Also $\operatorname{Ndim}_R M < \infty$ if M is an Artinian module (see [**9**] and [**14**]).

THEOREM 2.4. *Let $(R, \mathfrak{m})$ be a local ring and M an Artinian R-module. Assume that s is a non-negative integer such that $\mathfrak{a} \subseteq \operatorname{Rad}(\operatorname{Ann}(\mathrm{H}_i^{\mathfrak{a}}(M)))$ for all $i > s$. Then $\mathrm{H}_i^{\mathfrak{a}}(M) = 0$ for all $i > s$.*

Proof. We prove the statement by induction on $d := \operatorname{Ndim} M$. For $d = 0$, $\mathrm{H}_i^{\mathfrak{a}}(M) = 0$ for all $i > 0$ by [**5**, 4.8]. Now let $d > 0$ and assume that the claim holds for all R-modules of Noetherian dimension less than d. With the notation of Remark 2.1, we have $\mathrm{H}_i^{\mathfrak{a}}(M) \cong \mathrm{H}_i^{\mathfrak{a}}(N)$ for all $i > 0$ and there is an element $x \in \mathfrak{a}$ such that $xN = N$; there also exists a positive integer k such that $x^k \mathrm{H}_i^{\mathfrak{a}}(N) = 0$ for all $i > s$ by our assumption. The short exact sequence

$$0 \to 0 :_N x^k \to N \xrightarrow{x^k} N \to 0$$

induces a short exact sequence of local homology modules

$$0 \to \mathrm{H}_{i+1}^{\mathfrak{a}}(N) \to \mathrm{H}_i^{\mathfrak{a}}(0 :_N x^k) \to \mathrm{H}_i^{\mathfrak{a}}(N) \to 0$$

for all $i > s$. Thus $\mathfrak{a} \subseteq \operatorname{Rad}(\operatorname{Ann}(\mathrm{H}_i^{\mathfrak{a}}(0 :_N x^k)))$ for all $i > s$. It should be noted that by [**5**, 4.7] $\operatorname{Ndim}(0 :_N x^k) \leqslant d - 1$. By the inductive hypothesis $\mathrm{H}_i^{\mathfrak{a}}(0 :_N x^k) = 0$ for all $i > s$. Hence $\mathrm{H}_i^{\mathfrak{a}}(N) = 0$ for all $i > s$. □

Theorem 2.5. *Let $(R, \mathfrak{m})$ be a local ring, $\mathcal{S}$ a non-zero Serre subcategory of the category of R-modules and M an Artinian R-module. Assume that s is a non-negative integer. If $\mathrm{H}_i^{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i < s$, then $\mathrm{H}_s^{\mathfrak{a}}(M)/\mathfrak{m}\,\mathrm{H}_s^{\mathfrak{a}}(M) \in \mathcal{S}$.*

Proof. We prove the statement by induction on s. Let $s = 0$. Since M is Artinian, there exists a positive integer n such that $\mathfrak{a}^t M = \mathfrak{a}^n M$ for all $t \geq n$. So we have

$$\mathrm{H}_0^{\mathfrak{a}}(M) \cong \Lambda_{\mathfrak{a}}(M) \cong \varprojlim_t M/\mathfrak{a}^t M = M/\mathfrak{a}^n M.$$

Thus $\mathrm{H}_0^{\mathfrak{a}}(M)$ is Artinian and so $\mathrm{H}_0^{\mathfrak{a}}(M)/\mathfrak{m}\,\mathrm{H}_0^{\mathfrak{a}}(M)$ has finite length. By [**2**, 2.11], $\mathrm{H}_0^{\mathfrak{a}}(M)/\mathfrak{m}\,\mathrm{H}_0^{\mathfrak{a}}(M) \in \mathcal{S}$. Let $s > 0$. From Remark 2.1, we have $\mathrm{H}_i^{\mathfrak{a}}(M) \cong \mathrm{H}_i^{\mathfrak{a}}(N)$ for all $i > 0$, where $N := \mathfrak{a}^n M$. Thus the proof will be completed if we show that $\mathrm{H}_s^{\mathfrak{a}}(N)/\mathfrak{m}\,\mathrm{H}_s^{\mathfrak{a}}(N) \in \mathcal{S}$. Now the short exact sequence

$$0 \to 0 :_N x \to N \xrightarrow{x} N \to 0$$

induces the long exact sequence of local homology modules

$$\cdots \to \mathrm{H}_i^{\mathfrak{a}}(N) \xrightarrow{x} \mathrm{H}_i^{\mathfrak{a}}(N) \to \mathrm{H}_{i-1}^{\mathfrak{a}}(0 :_N x) \to \mathrm{H}_{i-1}^{\mathfrak{a}}(N) \xrightarrow{x} \mathrm{H}_{i-1}^{\mathfrak{a}}(N) \to \cdots.$$

It follows from the hypothesis that $\mathrm{H}_i^{\mathfrak{a}}(0 :_N x) \in \mathcal{S}$ for all $i < s - 1$. Hence $\mathrm{H}_{s-1}^{\mathfrak{a}}(0 :_N x)/\mathfrak{m}\,\mathrm{H}_{s-1}^{\mathfrak{a}}(0 :_N x) \in \mathcal{S}$. From the long exact sequence we have the following short exact sequence

$$0 \to \mathrm{H}_s^{\mathfrak{a}}(N)/x\,\mathrm{H}_s^{\mathfrak{a}}(N) \to \mathrm{H}_{s-1}^{\mathfrak{a}}(0 :_N x) \to 0 :_{\mathrm{H}_{s-1}^{\mathfrak{a}}(N)} x \to 0.$$

Now we have the following exact sequence

$$\operatorname{Tor}_1^R(R/\mathfrak{m}, 0 :_{\mathrm{H}_{s-1}^{\mathfrak{a}}(N)} x) \to \mathrm{H}_s^{\mathfrak{a}}(N)/\mathfrak{m}\,\mathrm{H}_s^{\mathfrak{a}}(N) \to \mathrm{H}_{s-1}^{\mathfrak{a}}(0 :_N x)/\mathfrak{m}\,\mathrm{H}_{s-1}^{\mathfrak{a}}(0 :_N x).$$

By [**2**, 2.1], $\operatorname{Tor}_1^R(R/\mathfrak{m}, 0 :_{\mathrm{H}_{s-1}^{\mathfrak{a}}(N)} x) \in \mathcal{S}$. Thus $\mathrm{H}_s^{\mathfrak{a}}(N)/\mathfrak{m}\,\mathrm{H}_s^{\mathfrak{a}}(N) \in \mathcal{S}$ □

3. A Condition on Serre subcategories. In this section we introduce the concept of $C^{\mathfrak{a}}$ condition on a Serre category of R-modules, which is dual to the Melkersson condition $C_{\mathfrak{a}}$. In connection with this condition we also prove some results on local homology modules. Recall that a Serre subcategory $\mathcal{S}$ of the category of R-modules satisfies the condition:

$(C_{\mathfrak{a}})$ if $M = \Gamma_{\mathfrak{a}}(M)$ and if $0 :_M \mathfrak{a}$ is in $\mathcal{S}$ then M is in $\mathcal{S}$ ([**1**]).

Definition 3.1. Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, $\mathfrak{a}$ an ideal of R and M an R-module. We say that $\mathcal{S}$ satisfies the condition:

$$(C^{\mathfrak{a}}) \qquad \text{if } \bigcap_{n>0} \mathfrak{a}^n M = 0 \text{ and if } \frac{M}{\mathfrak{a}M} \in \mathcal{S}, \text{ then } M \in \mathcal{S}.$$

Example 3.2. The following classes of modules are Serre subcategories that satisfy the condition $C^{\mathfrak{a}}$.
(i) The class of Noetherian R-modules, when R is a complete ring with respect to the $\mathfrak{a}$-adic completion [**5**, 5.1].
(ii) The class of coartinian Noetherian R-modules, when R is a complete local ring.

We recall some concepts and definitions to clarify the example. Let T be a multiplicatively closed subset of R and M an R-module. In [**11**], Melkersson and Schenzel called the module ${}_{\mathbb{T}}M = \operatorname{Hom}_R({}_{\mathbb{T}}R, M)$ the co-localization of M with respect to T and $\operatorname{Cos}_R(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) |_{\mathfrak{p}} M \neq 0\}$ the co-support of M. Next, Yassemi [**15**] defined the co-support of an R-module M, denoted by $\operatorname{Cosupp}_R(M)$, to be the set of primes $\mathfrak{p}$ such that there exists a cocyclic homomorphic image L of M with $\operatorname{Ann}(L) \subseteq \mathfrak{p}$. Yassemi showed $\operatorname{Cos}_R(M) = \operatorname{Cosupp}(M)$, when M is an Artinian R-module.
In [**12**], Nam defined the $\mathfrak{a}$-coartinian module M as follows: An R-module M is said to be $\mathfrak{a}$-coartinian if $\operatorname{Cosupp}(M) \subseteq V(\mathfrak{a})$ and $\operatorname{Tor}_i^R(R/\mathfrak{a}, M)$ is an Artinian R-module for each i. This definition is in some sense dual to the concept of $\mathfrak{a}$-cofinite modules, which was first introduced by Hartshorne (see [**8**]).

Proposition 3.3. *Let $(R, \mathfrak{m})$ be a local ring and $\mathcal{S}$ a non-zero Serre subcategory of the category of R-modules such that it satisfies $C^{\mathfrak{m}}$ condition. If M is a Noetherian R-module, then $M \in \mathcal{S}$.*

Proof. Since M is a Noetherian R-module, $\bigcap \mathfrak{m}^n M = 0$. Also $\frac{M}{\mathfrak{m}M}$ is Noetherian and $\mathfrak{m}\frac{M}{\mathfrak{m}M} = 0$. Thus $\frac{M}{\mathfrak{m}M}$ is an Artinian module and so is of finite length. Hence $\frac{M}{\mathfrak{m}M} \in \mathcal{S}$ by ([**2**], 2.11). Since $\mathcal{S}$ satisfies the $C^{\mathfrak{m}}$ condition, $M \in \mathcal{S}$. □

Theorem 3.4. *Let $(R, \mathfrak{m})$ be a local ring, $\mathcal{S}$ a non-zero Serre subcategory of the category of R-modules such that it satisfies the condition $C^{\mathfrak{a}}$ and let M be an Artinian R-module with* $\operatorname{Ndim} M = d$. *Then* $\mathrm{H}^{\mathfrak{a}}_d(M) \in \mathcal{S}$.

Proof. We prove the statement by induction on d. If $d = 0$, then M is a finitely generated, and $\mathfrak{a}$-separated R-module. By [**5**, 3.8], $\mathrm{H}_0^{\mathfrak{a}}(M) \cong \Lambda_{\mathfrak{a}}(M) \cong M$. Thus $\mathrm{H}_0^{\mathfrak{a}}(M)$ is of finite length and by [**2**, 2.11], $\mathrm{H}_0^{\mathfrak{a}}(M) \in \mathcal{S}$. Let $d > 0$; from Lemma 2.2, we have $\mathrm{H}_d^{\mathfrak{a}}(M) \cong \mathrm{H}_d^{\mathfrak{a}}(\bigcap_{t>0} \mathfrak{a}^t M)$. If $\mathrm{Ndim}(\bigcap_{t>0} \mathfrak{a}^t M) < d$, then $\mathrm{H}_d^{\mathfrak{a}}(M) = 0$ by [**5**, 4.8] and then there is nothing to prove. Let $\mathrm{Ndim}(\bigcap_{t>0} \mathfrak{a}^t M) = d$. We can replace M by $\bigcap_{t>0} \mathfrak{a}^t M$ and we may assume that there is an element $x \in \mathfrak{a}$ such that $xM = M$. Now from the short exact sequence of Artinian modules

$$0 \to 0 :_M x \to M \xrightarrow{x} M \to 0$$

we get an exact sequence of local homology modules

$$\mathrm{H}_d^{\mathfrak{a}}(M) \xrightarrow{x} \mathrm{H}_d^{\mathfrak{a}}(M) \xrightarrow{\delta} \mathrm{H}_{d-1}^{\mathfrak{a}}(0 :_M x).$$

Note by [**5**, 4.7] that $\mathrm{Ndim}(0 :_M x) \leq d-1$. It follows by the inductive hypothesis that $\mathrm{H}_{d-1}^{\mathfrak{a}}(0 :_M x) \in \mathcal{S}$. On the other hand, we have $\mathrm{H}_d^{\mathfrak{a}}(M)/x\,\mathrm{H}_d^{\mathfrak{a}}(M) \cong \mathrm{Im}\,\delta \subseteq \mathrm{H}_{d-1}^{\mathfrak{a}}(0 :_M x)$. Thus $\mathrm{H}_d^{\mathfrak{a}}(M)/x\,\mathrm{H}_d^{\mathfrak{a}}(M) \in S$. Therefore $\mathrm{H}_d^{\mathfrak{a}}(M)/\mathfrak{a}\,\mathrm{H}_d^{\mathfrak{a}}(M) \in \mathcal{S}$ and this implies that $\mathrm{H}_d^{\mathfrak{a}}(M) \in \mathcal{S}$, since $\mathrm{H}_d^{\mathfrak{a}}(M)$ is $\mathfrak{a}$-separated by [**4**, 3.3] and $\mathcal{S}$ satisfies the condition $C^{\mathfrak{a}}$. □

Theorem 3.5. *Let $(R, \mathfrak{m})$ be a local ring, $\mathcal{S}$ a non-zero Serre subcategory of the category of R-modules, and M an Artinian R-module. Assume that s is a non-negative integer such that $\mathrm{H}_i^{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i > s$. Then $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_s^{\mathfrak{a}}(M)) \in \mathcal{S}$.*

Proof. We prove the statement by induction on $d := \mathrm{Ndim}\, M$. For $d = 0$, $\mathrm{H}_i^{\mathfrak{a}}(M) = 0$ for all $i > 0$ by [**4**, 4.8] and also $\mathrm{H}_0^{\mathfrak{a}}(M) \in \mathcal{S}$ by Theorem 3.4. Now let $d > 0$ and assume that the claim holds for all R-modules of Noetherian dimension less than d. By the same argument as in the proof of Theorem 3.4, there is an element $x \in \mathfrak{a}$, such that there is an exact sequence of Artinian modules

$$0 \to 0 :_M x \to M \xrightarrow{x} M \to 0.$$

We get the following exact sequence of local homology modules

$$\cdots \to \mathrm{H}_{i+1}^{\mathfrak{a}}(M) \to \mathrm{H}_i^{\mathfrak{a}}(0 :_M x) \to \mathrm{H}_i^{\mathfrak{a}}(M) \to \cdots,$$

which implies that $\mathrm{H}_i^{\mathfrak{a}}(0 :_M x) \in \mathcal{S}$ for all $i > s$. Thus by the induction hypothesis $\mathrm{Hom}(R/\mathfrak{a}, \mathrm{H}_s^{\mathfrak{a}}(0 :_M x)) \in \mathcal{S}$. Now consider the exact sequence

$$\cdots \to \mathrm{H}_{s+1}^{\mathfrak{a}}(M) \xrightarrow{\varphi} \mathrm{H}_s^{\mathfrak{a}}(0 :_M x) \xrightarrow{\psi} \mathrm{H}_s^{\mathfrak{a}}(M) \xrightarrow{x} \mathrm{H}_s^{\mathfrak{a}}(M) \to \cdots.$$

From this exact sequence, we deduce the following two exact sequences

$$0 \to \mathrm{im}\,\varphi \to \mathrm{H}_s^{\mathfrak{a}}(0 :_M x) \to \mathrm{im}\psi \to 0$$

and

$$0 \to \operatorname{im}\psi \to \mathrm{H}_s^{\mathfrak{a}}(M) \xrightarrow{x} x\,\mathrm{H}_s^{\mathfrak{a}}(M) \to 0.$$

The above exact sequences induce the following exact sequences

$$\cdots \to \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_s^{\mathfrak{a}}(0 :_M x)) \to \mathrm{Hom}_R(R/\mathfrak{a}, \operatorname{im}\psi) \to \mathrm{Ext}_R^1(R/\mathfrak{a}, \operatorname{im}\varphi) \to \cdots$$

and

$$0 \to \mathrm{Hom}_R(R/\mathfrak{a}, \operatorname{im}\psi) \to \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_s^{\mathfrak{a}}(M)) \xrightarrow{x} \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_s^{\mathfrak{a}}(M)) \to \cdots .$$

By [**2**, 2.1] $\mathrm{Ext}_R^1(R/\mathfrak{a}, \operatorname{im}\varphi) \in \mathcal{S}$, since $\mathrm{H}_{s+1}^{\mathfrak{a}}(M) \in \mathcal{S}$. Hence $\mathrm{Hom}_R(R/\mathfrak{a}, \operatorname{im}\psi) \in \mathcal{S}$. Since $x \in \mathfrak{a}$, we have $\mathrm{Hom}_R(R/\mathfrak{a}, \operatorname{im}\psi) \cong \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_s^{\mathfrak{a}}(M))$. It follows that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_s^{\mathfrak{a}}(M)) \in \mathcal{S}$. □

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