Some global solutions to the complex Monge-Ampère equation.

by

Sławomir Kołodziej

IMUJ PREPRINT **1997/02**

Abstract. We show that perturbing in a suitable way a radial non-negative function belonging to L^p_{loc} one can solve the complex Monge-Ampère equation in $\mathbb{C}^n$ with the right hand side equal to this new function.

Introduction.

Though much is known about solvability of the Dirichlet problem for the complex Monge-Ampère equation in a strictly pseudoconvex domain in $\mathbb{C}^n$ (see [BT1], [C], [K1], [K2], [K3]) we are far from understanding which measures are Monge-Ampère measures of entire plurisubharmonic functions.

This problem has been treated in some special cases. There are fairly sharp sufficient conditions which guarantee that a measure μ having finite mass in $\mathbb{C}^n$ admits a solution of

$$(dd^c u)^n = d\mu,$$

which (after suitable normalization) has logarithmic growth ([K2], see also [CK]). The result is based on a generalized version of Yau's theorem [Y].

In [J] A. Jeune proved that perturbing (the density of) the Lebesgue measure $d\lambda$ in $\mathbb{C}^n$ by a smooth function which, together with all its derivatives, tends to 0 fast enough at infinity one can solve the Monge-Ampère equation for this new measure. The solution is smooth and differs from the potential of the Euclidean metric by a bounded function.

In [TY] G.Tian and S.-T. Yau solve the equation in a more general setting on some special non-compact complete Kähler manifolds. In this case, as well as in [J] and [Y], the attempt to solve the Monge-Ampère equation is motivated by geometrical problem of finding a Kähler metric with preassigned Ricci curvature.

In this paper we consider any radial plurisubharmonic function v such that its Monge-Ampère measure $(dd^c v)^n(z) = M(z)\, d\lambda$ belongs to $L^p_{loc}(\mathbb{C}^n), p > 1$. We wish to solve

$$(*) \qquad \begin{aligned} &u \in PSH \cap C(\mathbb{C}^n) \\ &(dd^c u)^n = (M + f)\, d\lambda, \end{aligned}$$

1991 *Mathematics Subject Classification.* primary 32F07, secondary 32F05.

Key words and phrases. plurisubharmonic function, complex Monge-Ampère operator.

for $f \geq 0, f \in L^p_{loc}$ when the growth of f at infinity is controled in terms of M. To state it precisely we need to introduce some auxiliary functions:

$$F(z) := \sup_{|w|=|z|} f(w),$$
$$H(r) := \int_{B(r)} M(z)\, d\lambda,$$
$$\Phi(r) := \int_{B(r)} F(z)\, d\lambda,$$

where $B(r)$ denotes the ball $\{|z| < r\}$.

Theorem. *If $F \in L^p_{loc}(\mathbb{C}^n), p > 1$, and*

$$\int_1^\infty \frac{\Phi(t)}{tH^{(n-1)/n}(t)}\, dt < \infty.$$

then there exists a unique u solving the equation $(*)$ and such that $\lim_{|z|\to\infty}(u - v)(z) = 0$.

For $v(z) = |z|^2$ we are in the situation of Jeune's result. Then the integral in the statement of Theorem is finite if, for instance, $F(t) \leq t^{-2-\epsilon}, \epsilon > 0, t > t_0$. This is exactly one of the assumptions on f in [J].

Note also that it is enough to take any compactly supported $f \in L^p_{loc}$ if $H(r) \geq const.(\log^+ r)^{\frac{n}{n-1}+\epsilon}$,

Proof of Theorem.

Proof. By [K1],[K2] one can solve the following Dirichlet problem:

$$(1) \qquad \begin{aligned} &u_j \in PSH(B(j)) \cap C(\bar{B}(j)) \\ &(dd^c u_j)^n = (M + f)\, d\lambda \\ &u_j = v \text{ on } \partial B(j). \end{aligned}$$

Set $U_j(z) := \max_{|w|=|z|} u_j(w),\ j = 1, 2, ...$ These are radial, continuous, plurisubharmonic functions.

Proposition 1. *The sequence $\{U_j - v\}$ is uniformly bounded.*

Proof. By [BT1]:

$$(dd^c \max(u_j \circ T_1, u_j \circ T_2, ..., u_j \circ T_k))^n \geq (dd^c v)^n,$$

where T_l are unitary transformations. If we take a countable, dense set of such transformations and apply the Convergence theorem [BT2] then we arrive at

$$(2) \qquad (dd^c U_j)^n \geq (dd^c v)^n, \quad j = 1, 2, ...$$

From the Comparison Principle [BT2] one also obtains

$$\int_{B(j)} (dd^c U_j)^n \leq \int_{B(j)} (dd^c u_j)^n \leq \int_{B(j)} (dd^c v)^n + \Phi(j). \tag{3}$$

Let us denote the density of $(dd^c U_j)^n$ by M_j. Then, following [M], one can compute U_j when M_j is given. For a radial function g we shall use the notation: $\tilde{g}(t) := g(t, 0, 0, ..., 0)$.

$$\begin{aligned}\frac{d}{dt}\tilde{U}_j(t) &= \frac{2}{t}[2n\int_0^t x^{2n-1}\tilde{M}_j(x)\,dx]^{1/n} \\ &= \frac{2}{t}[2n\int_0^t x^{2n-1}(\tilde{M}(x) + h_j(x))\,dx]^{1/n},\end{aligned}$$

where h_j is just $\tilde{M}_j - \tilde{M}$. Use this to estimate $\tilde{v}(t) - \tilde{U}_j(t)$.

$$\begin{aligned}\frac{d}{dt}((\tilde{U}_j - \tilde{v})(t)) &= \frac{2}{t}(2n)^{1/n}\{[\int_0^t x^{2n-1}(\tilde{M}(x) + h_j(x))\,dx]^{1/n} \\ &- [\int_0^t x^{2n-1}\tilde{M}(x)\,dx]^{1/n}\} \leq \frac{const.}{t}[(H(t) + \Phi(t))^{1/n} - H^{1/n}(t)] \\ &\leq \frac{const.}{t}[\frac{\Phi(t)}{nH^{(n-1)/n}(t)}].\end{aligned}$$

To get the last inequality write $(a-b)/(a^{1/n} - b^{1/n})$ as $\sum a^{k/n}b^{(n-k-1)/n}$.

Integrating both sides of the above formula and recalling that $\tilde{U}_j(j) = \tilde{v}(j)$ one can use the assumptions to find $\delta > 0$ such that

$$v - \delta \leq U_j, \quad j = 1, 2, \ldots .$$

On the other hand, from the Comparison Principle and (2) we have $U_j \leq v$, and so the proposition follows.

Corollary. *The sequence $\{u_j - v\}$ is uniformly bounded.*

Proof. Denote by $\hat{U}_j$ the (radial) solution to (1) with $(M + f)\,d\lambda$ replaced by $d\hat{\mu} = (M + F)\,d\lambda$. By the proposition above

$$v - const. \leq \hat{U}_j, \quad j = 1, 2, \ldots ,$$

and by the Comparison Principle

$$\hat{U}_j \leq u_j \leq v, \quad j = 1, 2, \ldots .$$

The proof is completed.

To prove the existence of a (locally bounded) solution to $(*)$ note that the sequence $\{u_j\}$ is (locally) decreasing. Indeed, since due to the Comparison Principle

$u_{j+1} \le v = u_j$ on $\partial B(j)$ the inequality $u_{j+1} \le u_j$ extends to $B(j)$ by another application of the Comparison Principle, as the Monge-Ampère measures of both functions are equal.

Thus, using the Convergece Theorem [BT1] and Corollary above we conclude that $u = \lim u_j$ is the desired solution. For the equality $\lim_{|z|\to\infty}(v-u)(z) = 0$ it is enough to prove that $\lim_{|z|\to\infty}(v-\hat{U})(z) = 0$, where $\hat{U} := \lim \hat{U}_j$ with $\hat{U}_j$ defined above. First, observe that

$$a := \inf(v - \hat{U}) = \lim_{|z|\to\infty}(v - \hat{U}).$$

Indeed, otherwise, for some $b > a$ the set $\{v - b < \hat{U}\}$ would have non-empty bounded component on which the Comparison Principle would be violated. Thus $\hat{U} + a \le v$, and so by the construction of $\hat{U}_j$ and the Comparison Principle applied on $B(j)$ we get $\hat{U}_j \ge \hat{U} + a$, which upon letting j to infinity forces a to be 0.

Uniqueness in the form from the statement also follows from the Comparison Principle and the fact that $\lim_{|z|\to\infty}(u-w)(z) = 0$ if w is another solution to $(*)$ with $\lim_{|z|\to\infty}(v-w)(z) = 0$.

It remains to show that u is continuous. For this we need a lemma from [K2, Lemma 2.3.1]. Let us recall the definition of the relative capacity introduced in [BT2]:

$$cap(K,\Omega) = \sup\{\int_K (dd^c u)^n; u \in PSH(\Omega), -1 < u < 0\},$$

for compact K in Ω. This set function extends to open sets as inner capacity.

Lemma. *Let Ω be a strictly pseudoconvex subset of $\mathbb{C}^n$ and let w, v belong to $PSH(\Omega)$. Assume that $\emptyset \ne \mathcal{U}_c := \{w < v - c\} \subset \Omega' \subset \Omega$ and*

$$\int_{\mathcal{U}_c} (dd^c w)^n \le F(cap(\mathcal{U}_c, \Omega)), \quad c \in (0, C), \tag{4}$$

for $F(x) = Ax(\log(1+x^{-1}))^{-(n+\epsilon)}$, $A, \epsilon > 0$. Then $C < \kappa = \kappa(A, ||v||, \epsilon, \Omega', \Omega)$ (the estimate is independent of w and v as long as $||v||$ is under control). Moreover, $\kappa \to 0$ as $cap(\mathcal{U}_0, B(R)) \to 0$ when $A, \epsilon, \Omega', \Omega$ are fixed.

If $(dd^c w)^n$ belongs to $L^p(\Omega), p > 1$, then the inequality (4) is satisfied (see [K1]).

Now, suppose u_j were not locally uniformly convergent. Then for a bounded sequence z_j of points in $\mathbb{C}^n$ and a positive constant C we would have

$$u_j(z_j) > u(z_j) + C. \tag{5}$$

Having $\lim_{|z|\to\infty}(v-u)(z) = 0$ one can fix $R > 0$ such that $v - u < C/2$ on $\mathbb{C}^n \setminus B(R)$. Then the sets

$$\mathcal{U}_j(c) := \{u_j > u + c\}, \quad c \in (C/2, C)$$

are non-empty and relatively compact in $B(R)$. Since the sequence u_j is decreasing one can apply [BT2] (see [K2, Corollary 1.2.10]) to conclude that

$$cap((\mathcal{U}_j(C/2), B(R)) \to 0$$

as $j \to \infty$, which contradicts the lemma above.

Remark 1. *Suppose we wished to solve* $(*)$ *with non-positive* f. *In that case, set* $F(z) := \sup_{|w|=|z|}(-f)(w), g := F + f$, *and* $N := M - F$. *If* (N, g) *satisfies the assumption of Theorem in place of* (M, f) *then* $(*)$ *is solvable since* $M + f = N + g$.

Remark 2. *Theorem readily lends itself to some generalizations. For instance, one can work in spaces larger than all* $L^p_{loc}, p > 1$ *(see [K1], [K2]). But the real progress would be achieved by weakening the most restrictive assumption saying, roughly, that the right hand side in the Monge-Ampère equation* $(*)$ *does not differ much from the radial measure, and expressed by the finiteness of the integral in the statement.*

References

[BT1] E. Bedford and B.A. Taylor, *The Dirichlet problem for a complex Monge-Ampère equation,* Invent. Math. **37** (1976), 1-44.

[BT2] E. Bedford and B.A. Taylor, *A new capacity for plurisubharmonic functions.* Acta Math. **149** (1982), 1-40.

[C] U. Cegrell, *Pluricomplex energy.* Research Reports 14 (1996) Umeå University.

[CK] U. Cegrell and S. Kołodziej, *The global Dirichlet problem for the complex Monge-Ampère equation.* Research Reports 2 (1995) Umeå University, to appear in J. Geom. Analysis.

[J] A. Jeune, *Prescription de la courbure de Ricci sur* $\mathbb{C}^n, n > 2$. Bull.Sc.Math., 2^e série **116** (1992), 185-226.

[K1] S. Kołodziej, *Some sufficient conditions for solvability of the Dirichlet problem for the complex Monge-Ampère equation.* Ann. Polon. Math. **LXV.1** (1996), 11-21.

[K2] S. Kołodziej, *The complex Monge-Ampère equation.* Preprint 1995/11 Instytut Matematyki Uniwersytetu Jagiellońskiego, Kraków 1995.

[K3] S. Kołodziej, *The range of the complex Monge-Ampère operator II.* Indiana U. Math. J. **44,** 3 (1995), 765-782.

[M] D.R. Monn, *Regularity of the Complex Monge-Ampère Equation for the Radially Symmetric Functions of the Unit Ball.* Math. Ann. **275** (1986), 501-511.

[TY] G.Tian and S.-T. Yau, *Complete Kähler manifolds with the zero Ricci curvature. I,* J.Amer.Math.Soc. **3** (1990), 579-609.

[Y] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation,* Comm. Pure and Appl. Math. **XXXI** (1978), 339-411.

Polish Academy of Science, Institute of Mathematics
Śniadeckich 8
00-950 Warszawa

postal address:
Jagiellonian University, Institute of Mathematics
Reymonta 4, 30-059 Kraków, Poland
e-mail: kolodzie@ im.uj.edu.pl