LINEARLY INDEPENDENT DIVISORS ON COMPLEX MANIFOLDS*)

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Abstract

Let X be a connected complex manifold and let D be a non-zero divisor on X with the support $\mid D \mid$. We show that if D is a principal divisor then the group $H_1(X\setminus \mid D \mid, \mathbb{Z})$ can not be a torsion group. In particular the group $H_1(X\setminus \mid D \mid, \mathbb{Z})$ must be infinite. As a corollary we prove that simple divisors $D_1, ..., D_r$ on a complex manifold X are linearly independent in $Cl(X)$, provided the group $H_1(X \setminus (\bigcup_{i=1}^r D_i), \mathbb{Z})$ is a torsion group.

1 Terminology.

Let X be a complex manifold. Let us recal that a section f of the sheaf $\mathcal{M}(X)$ of meromorphic functions on X is called a meromorphic function on X. Thus f is locally given as a quotient of two holomorphic functions, i.e.,

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for any $x \in X$ there is a neigborhood U_α of a point x such that, $res_{U_\alpha} f = h_\alpha / g_\alpha,\ h_\alpha, g_\alpha \in \mathcal{O}(U_\alpha)$.

By a simple divisor on X we mean an irreducible analytic (algebraic) hypersurface. By an analytic divisor we mean a formal locally finite sum $\sum n_D D$, where $n_D \in \mathbb{Z}$, and D runs over all simple analytic divisors in X. The group of all (analytic) divisors on X we denote by $Div(X)$. Similarly, a free abelian group generated by all simple algebraic divisors on X, we denote by $Div_a(X)$.

Locally any analytic (algebraic) divisor on a smooth X is given by a meromorpic (rational) function. If a divisor D is given globally by a meromorphic (rational) function then it is called principal. The subgroup generated by principal analytic (algebraic) divisors is denoted by $P(X)$ $(P_a(X))$. Finally, by a group of divisor classes on X we mean a factor group $Cl(X) := Div(X)/P(X)$ and in the algebraic case the group $Cl_a(X) := Div_a(X)/P_a(X)$. Let us recal that if X is a smooth projective manifold then $Cl(X) = Cl_a(X) = Pic(X) = H^1(X, \mathcal{O}^*) - the\ Picard\ group\ of\ X$.

Let us recall that an abelian group G is called a torsion group if every element $g \in G$ has a finite order.

2 Main Result.

The aim of this paper is to prove the following:

Theorem 1 *Let X be a connected complex manifold and let D be a non-zero divisor on X with the support $\mid D \mid$. If D is a principal divisor, then the group $H_1(X\setminus \mid D \mid, \mathbb{Z})$ can not be a torsion group. In particular the group $H_1(X\setminus \mid D \mid, \mathbb{Z})$ must be infinite.*

Before we do it, we need some auxiliary results. We start with the notion of a multi-valued holomorphic function.

Definition 2 *Let X be a connected complex manifold and let $\mathcal{O}$ be the sheaf of holomorphic functions on X. A connected component $F \subset \mathcal{O}$ we call a multi-valued holomorphic function on X. Let $\pi : F \ni \mathbf{f}_x \to x \in X$ denote the canonical projection. If π is a topological covering, then we call F a full multi-valued holomorphic function on X.*

Remark 3 If F is a full multi-valued function on X, then π has the property of lifting paths. This means, in particular that any germ $\mathbf{f}_x$ can be extended along every path in X.

Example 4 a) Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $F = \sqrt[n]{z}$. It is easy to see that F is a full n-valued holomorphic function on $\mathbb{C}^*$. Additionally, if we extend a germ $\mathbf{f}_x$ along any closed path γ then we obtain a new germ $\mathbf{f}'_x = \epsilon \mathbf{f}_x$, where $\epsilon^n = 1$.

b) Let X be a connected complex manifold and $f \in \mathcal{O}^*(X)$ be a nonvanishing function on X. There exists a full multi-valued function $\sqrt[n]{f}$. Indeed, take a germ $\mathbf{f}_a$ and any germ of $(\sqrt[n]{\mathbf{z}})_{f(a)}$ in $\mathbb{C}^*$ and consider the connected component of $\mathcal{O}$ determined by the germ $(\sqrt[n]{\mathbf{z}})_{f(a)} \circ \mathbf{f}_a$. Note that in general we can obtain in this way several different full multi valued-functions $\sqrt[n]{f}$, any two of them differ by the multiplication by a constant ϵ, where $\epsilon^n = 1$.

Lemma 5 *Let X be a connected complex manifold and $f \in \mathcal{O}^*(X)$. Assume that the group $H_1(X, \mathbb{Z})$ is a torsion group. Then for every $n \in \mathbb{N}$ the function $\sqrt[n]{f}$ is single-valued.*

Proof. Let us take a point $a \in X$ and put $G := \pi_1(X, a)$. We have a canonical epimorphism $G \to H_1(X, \mathbb{Z})$ with the kernel equal to the comutator subgroup $[G, G]$ of G.

The mapping $f : X \to \mathbb{C}^*$ induces the homomorphism $f_* : G \to \pi_1(\mathbb{C}^*, f(a))$. Since the group $\pi_1(\mathbb{C}^*, f(a)) = \mathbb{Z}$ is an abelian group, we have $[G, G] \subset ker(f_*)$. Consequently, $f_*(G)$ is a torsion subgroup of $\pi_1(\mathbb{C}^*, f(a)) = \mathbb{Z}$. This implies that $f_*(G) = 0$. Now we obtain immediately, that extension of a germ $\sqrt[n]{\mathbf{f}}_a$ along every closed path in X is trivial. This means that a germ $\sqrt[n]{\mathbf{f}}_a$ determines a single-valued function. ∎

Proof of Theorem 1. Let X be a connected complex manifold and let D be a principal divisor on X with the support $\mid D \mid$. We show that the group $H_1(X \setminus \mid D \mid, \mathbb{Z})$ is not a torsion group. Indeed, suppose conversely that the group $H_1(X \setminus \mid D \mid, \mathbb{Z})$ is a torsion group.

Let f be a meromorphic function which describes D. Hence $D = Z - B$, where Z is a divisor of zeros of f and B is a divisor of poles of f. We can assume that $Z \neq 0$ (in the other case we can consider the divisor $-D$ given by $1/f$). Take a point $a \in \mid Z \mid \setminus \mid B \mid$. There exists a connected neigborhood

U of a point a such that f is holomorphic in U. Consequently, the germ $\mathbf{f}_a$ is a holomorphic germ. In particular we can write $\mathbf{f}_a = \mathbf{g_1} \cdots \mathbf{g_s}$ as a product of irreducible elements $\mathbf{g_1}, \ldots, \mathbf{g_s} \in \mathcal{O}_a$, where $s \geq 1$.

Now, take a number $N > s$ and consider the function $F := \sqrt[N]{f}$ on $X \setminus \mid D \mid$. By Lemma 5 the multi-valued function F is in fact a single valued holomorphic function on $X \setminus \mid D \mid$. Moreover, F is bounded in $U \setminus \mid D \mid$. Consequently, by the Riemann Extension Theorem the function F extends to a holomorphic function $\tilde{F}$ on U. But $F^N = f$ on $U \setminus \mid D \mid$, hence by the identity principle we have $\tilde{F}^N = f$ in U. Hence $(\mathbf{\tilde{F}_a})^{\mathbf{N}} = \mathbf{f_a} = \mathbf{g_1} \cdots \mathbf{g_s}$. Since $N > s$ it is a contradiction. ■

Corollary 6 *Let $D_1, \ldots, D_r$, where $r \in \mathbb{N}$ or $r = \infty$, be a locally finite family of simple divisors on a complex manifold X. Let $D = \bigcup_{i=1}^r D_i$. If $H_1(X \setminus D, \mathbb{Z})$ is a torsion group, then the divisors $D_1, \ldots, D_r$ are linearly independent in the group $Cl(X)$.*

Proof. Take numbers $\{k_1, \ldots, k_s\} \subset \{1, \ldots, r\}$ and put $D' := \sum_{i=1}^s D_{k_i}$. We have an epimorphism $H_1(X \setminus \mid D \mid) \to H_1(X \setminus \mid D' \mid)$ (see e.g., [2], [3]). Consequently, the group $H_1(X \setminus \mid D' \mid)$ is also a torsion group. This means that D' cannot be a support of a principal divisor. Thus for every non-zero integers n_{k_i} we have $\sum_{i=1}^s n_{k_i} D_{k_i} \neq 0$ in $Cl(X)$.■

Corollary 7 *Let X be a smooth algebraic variety and assume that the group $H_1(X, \mathbb{Z})$ is a torsion group. If a polynomial function $f \in \mathbb{C}[X]$ nowhere vanishes on X, then it is a non-zero constant. In other words $\mathbb{C}[X]^* = \mathbb{C}^*$.*

Proof. Let $\overline{X}$ be a smooth completion of X, and let f be a non-constant polynomial function on X, which nowhere vanishes. Let $\overline{f}$ be a rational extension of f to the whole of $\overline{X}$. Take $D = (\overline{f})$. Of course, we have $D = 0$ in $Cl(\overline{X})$.

On the other hand, we have $|D| \subset \overline{X} \setminus X$, hence $X \subset \overline{X} \setminus |D|$. Since the group $H_1(X, \mathbb{Z})$ is a torsion group the same holds for the group $H_1(\overline{X} \setminus |D|, \mathbb{Z})$ (see e.g., [2], [3]). Consequently, we get by Corollary 6 that the divisor D is non-zero in $Cl(\overline{X})$. This contradiction finisches the proof.■

Example 8 Let us consider a smooth simply connected Stein surface $\Gamma = \{(x, y, z) \in \mathbb{C}^3 : xz = y(y+1)\}$ and a line $l = \{(x, y, z) \in \mathbb{C}^3 : x = 0,\ y =$

$-1\} \subset \Gamma$. We show that the line l can not be described by a single holomorphic function on Γ.

Indeed, we have an open immersion $\iota : \mathbb{C}^2 \ni (u, v) \to (u, uv, v(1 + uv)) \in \Gamma$. Moreover, $l \subset \Gamma \setminus U$ where $U := \iota(\mathbb{C}^2)$. Since $U \cong \mathbb{C}^2$, we have $H_1(U, \mathbb{Z}) = 0$. Consequently, by Corollary 6, the line l is a linearly independent divisor in $Cl(\Gamma)$. In particular the line l can not be a zero set of any holomorphic function from $\mathcal{O}(\Gamma)$.

Corollary 9 *Let $D_1, ..., D_r$ be a finite family of simple algebraic divisors on an algebraic manifold X. Let $D = \bigcup_{i=1}^r D_i$. If $H_1(X \setminus D, \mathbb{Z})$ is a torsion group, then the divisors $D_1, ..., D_r$ are linearly independent in the group $Cl_a(X)$ and consequently*

$$rank\ Cl_a(X) = r + rank\ Cl_a(X \setminus D).$$

Moreover, if $Cl_a(X \setminus D) = 0$, then

$$Cl_a(X) = \bigoplus_{i=1}^{r} \mathbb{Z} D_i.$$

Proof. By the previous Corollary the divisors $D_1, \ldots, D_r$ are linearly independent in the group $Cl_a(X)$. Moreover, we have a surjective homomorphism $Cl_a(X) \to Cl_a(X \setminus D)$ with the kernel generated by $D_1, \ldots, D_r$ (see [1], p. 133, 6.5). This finisch the proof.■

In general, the statement converse to Corollary 6 does not hold (it is enough to take a point on an elliptic curve). But it is interesting, that if X is a projective rational manifold then we have the following result:

Proposition 10 *Let X be a projective rational manifold and $D_1, ..., D_r$, $r \in \mathbb{N}$, be simple divisors on X. Take $D = \bigcup_{i=1}^r D_i$. Divisors $D_1, ..., D_r$ are linearly independent in the group $Pic(X)$, if and only if the group $H_1(X \setminus D, \mathbb{Z})$ is a finite group.*

Proof. By Corollary 6 it is enough to prove "$\Longrightarrow$". Since the manifold X is rational we have $H^i(X, \mathcal{O}) = 0$ for $i > 0$. Consequently, an exact exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

gives an isomorphism $Pic(X) \cong H^2(X, \mathbb{Z})$. In particular Chern clases $c_1(D_1)$, ..., $c_1(D_r)$ are linearly independent in $H^2(X, \mathbb{Z})$. By Poincare duality divisors $D_1, ..., D_r$ (considered as cycles) are linearly independent in $H_{2n-2}(X, \mathbb{Z})$ (where $n = \dim X$). Thus the mapping $H_{2n-2}(D) \cong \bigoplus_{i=1}^r H_{2n-2}(D_i) \to H_{2n-2}(X)$ is injective.

Conseqently, the mapping $H^{2n-2}(X) \to H^{2n-2}(D)$ has a finite cokernel. Now consider the exact sequence

$$H^{2n-2}(X) \to H^{2n-2}(D) \to H^{2n-1}(X, D) \to H^{2n-1}(X)$$

We have $H^{2n-1}(X) \cong H_1(X)$ and $H_1(X)$ is equal to zero, because X is rational. Hence $H^{2n-1}(X, D) = coker(H^{2n-2}(X) \to H^{2n-2}(D))$ is a finite group. But $H_1(X \setminus D) \cong H^{2n-1}(X, D)$ q.e.d. ∎

Example 11 Let $H_1, \ldots, H_r$ be irreducible hypersurfaces in $\mathbb{P}^n(\mathbb{C})$. Then the group $H_1(\mathbb{P}^n(\mathbb{C}) \setminus \bigcup_{i=1}^r H_i, \mathbb{Z})$ is finite for $r = 1$ and for $r > 1$ it is infinite.

Indeed, we have $Pic(\mathbb{P}^n(\mathbb{C})) = \mathbb{Z}$. Consequently, divisors $H_1, \ldots, H_r$ are linearly independent if and only if $r = 1$.

In fact we can prove more:

Proposition 12 *Let $H_1, \ldots, H_r$ be irreducible hypersurfaces in $\mathbb{P}^n(\mathbb{C})$. Then rank $H_1(\mathbb{P}^n(\mathbb{C}) \setminus \bigcup_{i=1}^r H_i, \mathbb{Z}) = r - 1$.*

Proof. Let us consider an exact sequence

$$H^{2n-2}(\mathbb{P}^n, \bigcup_{i=1}^r H_i) \to \mathbb{Z} = H^{2n-2}(\mathbb{P}^n) \to \bigoplus_{i=1}^r H^{2n-2}(H_i) = \mathbb{Z}^r$$
$$\to H^{2n-1}(\mathbb{P}^n, \bigcup_{i=1}^r H_i) \to H^{2n-1}(\mathbb{P}^n) = 0,$$

Observe that $H^{2n-2}(\mathbb{P}^n, \bigcup_{i=1}^r H_i) = 0$. Indeed, we have a sequence

$$H_{2n-2}(\bigcup_{i=1}^r H_i) \to H_{2n-2}(\mathbb{P}^n) \to H_{2n-2}(\mathbb{P}^n, \bigcup_{i=1}^r H_i) \to 0$$

and $\mathbb{Z} = Pic(\mathbb{P}^n) \cong H^2(\mathbb{P}^n) \cong H_{2n-2}(\mathbb{P}^n)$. Hence coker $(H_{2n-2}(\bigcup_{i=1}^r H_i) \to H_{2n-2}(\mathbb{P}^n))$ is a finite group. Consequently, the group $H_{2n-2}(\mathbb{P}^n, \bigcup_{i=1}^r H_i)$ is finite, too. Since the group $H_{2n-3}(\mathbb{P}^n, \bigcup_{i=1}^r H_i) \subset \mathbb{Z}^r$ is a torsion free group, we get $H^{2n-2}(\mathbb{P}^n, \bigcup_{i=1}^r H_i) = 0$. Consequently, rank $H^{2n-1}(\mathbb{P}^n, \bigcup_{i=1}^r H_i) = r - 1$, and finally by duality we get rank $H_1(\mathbb{P}^n(\mathbb{C}) \setminus \bigcup_{i=1}^r H_i, \mathbb{Z}) = r - 1$.∎

Corollary 13 *Let* $G, H \subset \mathbb{P}^n(\mathbb{C})$ *be hypersurfaces. If the variety* $\mathbb{P}^n \setminus G$ *is homeomorphic to the variety* $\mathbb{P}^n \setminus H$*, then* G, H *have the same number of irreducible components.*

Example 14 Let us recall that "an exotic $\mathbb{C}^n$" is a smooth affine variety homeomorphic to $\mathbb{C}^n$ but not isomorphic to $\mathbb{C}^n$.

We show that among Zariski open subsets of $\mathbb{P}^n(\mathbb{C})$ there is no exotic $\mathbb{C}^n$'s. We have the following fact:

Let $U \subset \mathbb{P}^n(\mathbb{C})$ *be a Zariski open affine subset of* $\mathbb{P}^n(\mathbb{C})$ *which is topologically contractible. Then* U *is isomorphic to* $\mathbb{C}^n$.

Proof. Since U is affine, a complement $\mathbb{P}^n(\mathbb{C}) \setminus U := H$ is a hypersurface. By Proposition 11 the hypersurface H is irreducible. Let $d = deg\ H$. From the exact sequence (see [1], p. 133, 6.5):

$$0 \to \mathbb{Z}H \to Pic(\mathbb{P}^n(\mathbb{C})) \to Cl(U) \to 0,$$

and the fact that $Pic(\mathbb{P}^n(\mathbb{C})) = \mathbb{Z}$ we get $Cl_a(U) = \mathbb{Z}/d\mathbb{Z}$. But from [4] we know that U is a factorial variety, hence $Cl_a(U) = 0$. Consequently, $d = 1$ and H is a hyperplane. Hence $U = \mathbb{C}^n$.■

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