Generalization of Sharkovskii's theorem to multidimensional perturbations of 1-dim maps

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Abstract

In paper we present a generalization of Sharkovski theorem for multidimensional perturbations of one-dimesional maps. We show that if unperturbed one-dimensional map has a periodic point, then all sufficiently close multidimesional perturbations of this map have periodic points of all periods which are allowed by Sharkovski theorem.

Keywords: fixed point index, chaos, Sharkovskii's theorem

0 Introduction

This paper is a second part to the paper [?]. In this paper we present a generalization of the famous Sharkovskii's Theorem [?](see also [?],[?],[?], [?]) concerning the appearance of periodic points for the self-maps on the real line.

Let $f : X \to X$ be a map. We say that $x \in X$ is a periodic point of period n if $f^n(x) = x$ and $f^k(x) \neq x$ for $k = 1, 2, \ldots, n-1$.

We begin with recalling Sharkovskii's Theorem.

Theorem 1 *Let be the ordering of positive integers:*

$$3 \lhd 5 \lhd 7 \lhd \ldots \lhd 2\cdot 3 \lhd 2\cdot 5 \lhd 2\cdot \lhd 2\cdot 7 \ldots \lhd 2^2\cdot 3 \lhd 2^2\cdot 5 \ldots \lhd 2^3\cdot 3 \lhd 2^3\cdot 5 \ldots \lhd 2^3 \lhd 2^2 \lhd 2 \lhd 1$$

Let $f : I \to \mathbb{R}$ be a continuous map of an interval into the real line. If $n \lhd m$ and f has a periodic point of period n then f also has a periodic point of period m.

The ordering described in Theorem ?? is called *the Sharkovskii's ordering.* We reserve the symbol '$\lhd$' for this order.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. Let V be a real Banach space and let a continuous decomposition $V = \mathbb{R} \oplus W$ be given. According to this decomposition we will often represent elements $v \in V$ as pairs $v = (x, w)$, where $x \in \mathbb{R}$ and $w \in W$. Let $F : [0,1] \times V \to V$ be a continuous and compact map. In the sequel we will often use the notation F_λ for the partial map with fixed λ, so $F_\lambda(v) := F(\lambda, v)$ for $v \in V$. Suppose that $F_0(x, w) = (f(x), 0)$. We say that the maps F_λ are multidimensional perturbations of f. Let us recall that a continuous map is called *compact*, iff it maps bounded sets into relatively compact sets.

The main theorem proved in this paper is stated below

Theorem 2 *Let f, F be as above. If $\dim V = \infty$ then we additionally assume that F is uniformly continuous and compact.*

If f has a point of period n then for every $r > 0$ there exists $\lambda_0 > 0$, such that for $\lambda < \lambda_0$ and every $m \in \mathbb{N}$ such that $n \lhd m$ and $n \neq m$ F_λ has periodic point of period m in $\mathbb{R} \oplus B_W(r)$, where $B_W(r)$ is an open ball in W of radius r.

In [?] this theorem was proved for n odd.

In our theorem for infinite dimensional case we have assumed uniform continuity and compactness. We believe that compactness alone is sufficient, but we are unable to prove it, yet.

1 1-dim maps with even periods

The aim of this section is to investigate a form of 1-dim maps, for which all periodic points which are not fixed point have an even period. The results of this section follow essentially from the ramification of arguments presented in [?] or [?] for odd periods, but they are probably not stated there explicitly as they are not required for the proof of Sharkovskii's theorem in one-dimensional case.

In this section we assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous map. Let $x_0 < x_1 < \ldots < x_{n-1}$, $n > 0$ be a periodic trajectory. Like in section 3 in [?] we set $I = [x_0, x_{n-1}]$ and we define $I_i = [x_{i-1}, x_i]$. We will use notions and notation introduced in section 3 from [?].

Theorem 3 *Assume that n is even and there is in I no periodic point of period odd greater than one. Then*

$$I_{n/2} \overset{f-}{\to} I_{n/2} \tag{1}$$

$$f^{2j+1}([x_0, x_{n/2-1}]) \subset (x_{n/2-1}, \infty), \quad \textit{for } j \in \mathbb{N} \tag{2}$$

$$f^{2j}([x_0, x_{n/2-1}]) \subset (-\infty, x_{n/2}), \quad \textit{for } j \in \mathbb{N} \tag{3}$$

$$f^{2j+1}\{x_0, x_1, \ldots, x_{n/2-1}\} = \{x_{n/2}, x_{n/2+1}, \ldots, x_{n-1}\}, \quad \textit{for } j \in \mathbb{N} \tag{4}$$

$$f^{2j}\{x_0, x_1, \ldots, x_{n/2-1}\} = \{x_0, x_1, \ldots, x_{n/2-1}\}, \quad \textit{for } j \in \mathbb{N} \tag{5}$$

Proof

Let $i_0 := \min\{i = 0, 1, \ldots, n-1 | f(x_i) < x_i\}$. Obviously $i_0 > 0$. From the definition of i_0 follows immediately that

$$I_{i_0} \xrightarrow{f-} I_{i_0} \tag{6}$$

We define inductively the sets J_l by

$$J_0 = \{i_0\} \tag{7}$$

$$i \in J_{l+1} \quad \text{iff} \quad \exists j \in J_l \quad I_j \xrightarrow{f} I_i \tag{8}$$

We define $|J_l| = \bigcup_{i \in J_l} I_i$.

We show that for some $l \leq n$ $J_l = \{0, 1, \ldots, n-1\}$.

To see this we replace f with a map $\tilde{f}$, which agrees with f on points x_i and is affine on every I_i. It is easy to see that

$$I_i \xrightarrow{f} I_j \quad \text{iff} \quad I_i \xrightarrow{\tilde{f}} I_j$$

But for $\tilde{f}$ holds

$$I_i \xrightarrow{\tilde{f}} I_j \quad \text{iff} \quad I_j \subset \tilde{f}(I_i)$$

From this and (??) follows immediately that $J_l \subset J_{l+1}$, $|J_l| = \tilde{f}^l(I_{i_0})$. Entire trajectory of x_{i_0} must be contained in J_n, hence from connectedness of $|J_l|$ follows that $|J_n| = I$ and $J_n = \{0, 1, \ldots, n-1\}$.

From this follows that for every $j = 1, \ldots, n-1$ exist $i_1, i_2, \ldots, i_l$ such that

$$I_{i_0} \xrightarrow{f} I_{i_1} \xrightarrow{f} \ldots \xrightarrow{f} I_j \tag{9}$$

We show now that there does not exists any $j \neq i_0$ such that $I_j \xrightarrow{f} I_{i_0}$. To prove this assume the contrary. Then from (??) and follows (??) that we can construct a closed loop

$$I_{i_0} \xrightarrow{f} I_{i_1} \xrightarrow{f} \ldots \xrightarrow{f} I_j \xrightarrow{f} I_{i_0} \xrightarrow{f} I_{i_0} \xrightarrow{f} \ldots \xrightarrow{f} I_{i_0} \tag{10}$$

which have prime number p of arrows. From this loop we can deduce the existence of point $x \in I_{i_0}$ such that $f^p(x) = x$ and $f^s(x) \in I_j$. As x is different from x_i for every i then it follows that period of x is equal to p. But this in condradiction with assumption that no odd period exists.

Nonexistence of $I_j \xrightarrow{f} I_{i_0}$, for $i \neq i_0$ means that for $i \leq i_0 - 1$ and $i \geq i_0 + 1$ both ends of the segment I_i are mapped on the same side of the middle point of the segment I_{i_0}. So the set $\{x_0, x_1, \ldots, x_{i_0-1}\}$ must be mapped onto the set $\{x_{i_0}, x_{i_0+1}, \ldots, x_{n-1}\}$ and vice versa. From this it follows that $i_0 = n/2$, so conditions (??–??) are satiesfied.

Suppose that (??) is not true for some j. Hence there exists $y \leq x_{n/2-1}$, such that $y \in f^{2j+1}([x_0, x_{n/2-1}])$. From (??) follows that $x_{n/2} \in f^{2j+1}([x_0, x_{n/2-1}])$. By connectedness of $f^{2j+1}([x_0, x_{n/2-1}])$ we obtain

$$I_{n/2} \subset f^{2j+1}([x_0, x_{n/2-1}]) \tag{11}$$

Hence there exists $K \subset [x_0, x_{n/2-1}]$ such that

$$K \xrightarrow{f^{2j+1}} I_{n/2} \tag{12}$$

From $|J_n| = I$ it follows that there exists segment $L \subset I_{n/2}$ such that

$$L \xrightarrow{f^n} K \tag{13}$$

Now we can construct loop of arrows as follows

$$L \xrightarrow{f^n} K \xrightarrow{f^{2j+1}} I_{n/2} \xrightarrow{f} \ldots I_{n/2} \xrightarrow{f} L \tag{14}$$

such that the sum of the powers of f appearning above arrows is equal to some prime number p. In this way from lemma 1.2 from [?] we obtain a periodic point of odd period p. This is in contradiction with our asumptions. So (??) is true. The proof of (??) is analogous.

□

Suppose now that $n = 2^q n'$, where $q > 0$, $n' > 0$. Suppose that f do not have on I any point of period of the form $2^{q-1}m$ with m odd and $m > 1$. Then from the above theorem it follows that $x_0, x_1, \ldots, x_{n/2-2}$ are points from $2^{q-1}n'$ periodic trajectory for the map f^2 and if $q-1 > 0$ then f^2 has no periodic point in $[x_0, x_{n/2-1}]$ of period of the form $2^{q-2}m$, where m is odd and $m > 1$. If $q-1 > 0$ then we can continue this construction and we see that $x_0, \ldots, x_{n/2^2-1}$ are points from $2^{q-2}m$ periodic trajectory for the map f^{2^2} and if $q-2 > 0$ then the map f^{2^2} has no periodic points in $[x_0, x_{2^2}-1]$ of the form $2^{q-3}m$ where m is odd and $m > 1$. We can continue this induction q times. In this way we obtain the following

Theorem 4 *Suppose that $n = 2^q n'$, where $q > 0$, $n' > 0$. Suppose that f do not have on I any point of period of the form $2^{q-1}m$ with m odd and $m > 1$. Then*

$$I_{n/2^{i+1}} \xrightarrow{f^{2^i}} I_{n/2^{i+1}}, \quad \textit{for } i = 0, \ldots, q-1$$

$$f^{j \cdot 2^i + s}([x_0, x_{\frac{n}{2^i}-1}]) \subset (x_{\frac{n}{2^i}-1}, \infty), \quad \textit{for } i = 1, \ldots, q,\ j = 0, 1, 2 \ldots \textit{ and } 0 < s < 2^i$$

$$f^{j \cdot 2^i}([x_0, x_{\frac{n}{2^i}-1}]) \subset (-\infty, x_{\frac{n}{2^i}}), \quad \textit{for } i = 1, \ldots, q\ ,\ j = 0, 1, 2 \ldots$$

$$f^{2^i}(\{x_0, x_1, \ldots, x_{\frac{n}{2^i}-1}\}) = \{x_0, x_1, \ldots, x_{\frac{n}{2^i}-1}\}, \quad \textit{for } i = 1, \ldots, q$$

2 Proof of main theorem

Before we present a proof let us explain, why we cannot not follow directly the proof of 1-dim Sharkovskii's theorem. The situation is as follows: we have the Sharkovskii's theorem for odd periods and we consider now period of the form $n2^q$, where $n > 1$ is odd. In onedimensional case one consider now the map $g := f^{2^q}$, obviously g have a point of odd period and in this way one obtains points x_m which are fixed points for the maps f^{m2^q}, where $n \lhd m$, but one have no need to bother himself with the problem: what is the period of x_m, because from Sharkovskii's ordering follows that if the period is not $m2^q$, there must exist a periodic point of period $m2^q$.

In our case we need to show that the fixed points for $F_\lambda^{m2^q}$ we find in the proof have exactly period equal to $m2^q$.

Proof

For n odd and $n > 1$ this theorem was proved in [?].

Let $x_0 < x_1 < \ldots < x_{n-1}$, $n > 0$ be a periodic trajectory. As usual we set $I = [x_0, x_{n-1}]$ and we define $I_i = [x_{i-1}, x_i]$. We can assume that $n = 2^q n'$, where n' is odd, and in I there no periodic point of period $2^q m$, where m is odd and $m > 1$.

Let us fix $r > 0$.

Assume first that $n' > 1$. We define $J := [x_0, x_{n'}]$. From theorem ?? it follows that the set $\{x_0, x_1, \ldots, x_{n'}\}$ is the periodic trajectory of the map f^{2^q} . From the theorem ?? for odd periods applied to f^{2^q} and the homotopy $F_\lambda^{2^q}$ follows that there exists $\lambda_1 > 0$ such that for every $m \neq n'$ and $n' \lhd m$ there exists an m-periodic point $x_m \in J \times B_W(r)$ for a map $F_\lambda^{2^q}$, such that

$$F_\lambda^{j \cdot 2^q}(x_m) \in \text{ int } J \times B_W(r), \quad \text{for } j = 1, 2, \ldots, m \tag{15}$$

$$F_\lambda^{j \cdot 2^q}(x_m) \neq x_m, \quad \text{for } j = 1, \ldots, m-1 \tag{16}$$

$$F_\lambda^{m \cdot 2^q}(x_m) = x_m \tag{17}$$

It remains to show that the period of x_m with regard to the map F_λ is exactly $m2^q$ for λ sufficiently small.

From the uniform continuity of F and theorem ?? it follows that there exists $\lambda_2 > 0$, such that

$$F_\lambda^s(J \times \overline{B_W(r)}) \cap J \times \overline{B_W(r)} = \emptyset, \quad \text{for } s = 1, 2, \ldots, 2^q - 1 \tag{18}$$

We set $\lambda_0 = \min\{\lambda_1, \lambda_2\}$. Now is easy to see that for $\lambda < \lambda_0$ the period of x_m is exactly $m2^q$. Suppose that there exists k another period for x_m, such that $k < m2^q$. k must be a number of the form $k = m'2^{q'}$. From condition (??) it follows that $q' = q$. From (??) it follows that $m' = m$. This finishes the proof of existence of all periods of the form $2^l m$, where $l = q$ and $m > n'$ or $l > q$, when $n' > 1$.

We release now the assumption $n' > 1$. It remains to show that numbers of the form 2^i for $i = 0, 1, \ldots, q-1$ can be realised as periods of some points in J for sufficiently small λ. It follows from theorem ?? that for $i = 0, 1, \ldots, q-1$ holds

$$I_{n/2^{i+1}} \xrightarrow{f^{2^i}} I_{n/2^{i+1}} \tag{19}$$

$$f^s(I_{n/2^{i+1}}) \subset f^s([x_0, x_{n/2^i - 1}]) \subset (x_{n/2^i-1}, \infty), \quad \text{for } s = 1, \ldots, 2^i - 1 \tag{20}$$

Obviously

$$I_{n/2^{i+1}} \cap (x_{n/2^i-1}, \infty) = \emptyset, \quad \text{for } s = 1, \ldots, 2^i - 1. \tag{21}$$

From uniform continuity of F and conditions (??) (??) it follows that there exists $\lambda_3 > 0$ such that for $\lambda < \lambda_3$

$$F^s_\lambda(I_{n/2^{i+1}} \times \overline{B_W(r)}) \cap I_{n/2^{i+1}} \times \overline{B_W(r)} = \emptyset, \quad \text{for } s = 1, \ldots, 2^i - 1 \tag{22}$$

From the proof of the theorem 29 from [?] and condition (??) it follows that there exists $\lambda_4 > 0$, such that for $\lambda < \lambda_4$ there exists y_i fixed point for the map $F^{2^i}_\lambda$. If $\lambda < \min\{\lambda_3, \lambda_4\}$ then from (??) it follows immediately that y_i has period 2^i.

This finishes the proof of theorem ??. □

References

[BGMY] L. Block , J. Guckenheimer, M. Misiurewicz, L.-S. Young *Periodic points of one-dimensional maps.* Springer Lecture Notes in Mathematics, Vol. 819, 18–34, Springer-Verlag: New York, Heidelberg, Berlin

[De] R. Devaney *An Introduction to Chaotic Dynamical Systems.* 1989. Addison-Wesley Publishing Company

[LY] T. Y. Li and J. A. Yorke *Period three implies chaos.* 985–992, Amer. Math. Monthly,82, 1975

[Sh] A. N. Sharkovskii *Coexistence of cycles of a continuous map of a line into itself. (Russian)* 61–71, Ukr. Math. Z., 16, 1964, ((English) Int. J. of Bifurcations and Chaos, 1263–1274, 5, 1995)

[St] P. Stefan *A theorem of Sarkovskii on the existence of periodic orbits of contnuous endomorphisms of real line* , Comm. Math. Phys. 54, 237–248, 1977

[Z] P. Zgliczyński *Odd period implies chaos for multidimensional perturbations of 1-D maps.*, submitted

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